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A historical law of large numbers for the Marcus-Lushnikov process*

Stéphanie Jacquot†

Abstract

The Marcus-Lushnikov process is a finite stochastic particle system, in which each particle is entirely characterized by its mass. Each pair of particles with masses x and y merges into a single particle at a given rate $K(x,y)$. Under certain assumptions, this process converges to the solution to the Smoluchowski coagulation equation, as the number of particles increases to infinity. The Marcus-Lushnikov process gives at each time the distribution of masses of the particles present in the system, but does not retain the history of formation of the particles. In this paper, we set up a historical analogue of the Marcus-Lushnikov process (built according to the rules of construction of the usual Markov-Lushnikov process) each time giving what we call the historical tree of a particle. The historical tree of a particle present in the Marcus-Lushnikov process at a given time t encodes information about the times and masses of the coagulation events that have formed that particle. We prove a law of large numbers for the empirical distribution of such historical trees. The limit is a natural measure on trees which is constructed from a solution to the Smoluchowski coagulation equation.

Key words: historical trees; Marcus-Lushnikov process on trees; limit measure on trees; Smoluchowski coagulation equation; tightness; coupling.

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†University of Cambridge, Statistical Laboratory, Centre for Mathematical Sciences Wilberforce Road, Cambridge, CB3 0WB, UK e-mail: smj45@cam.ac.uk

1 Presentation of the problem

1.1 Introduction

Let $K : (0,\infty) \times (0,\infty) \to [0,\infty)$ be a symmetric measurable function. Let $\mathcal{M}\left((0,\infty),\mathbb{Z}^+\right)$ be the set of finite integer-valued measures on $(0,\infty)$. $\mathcal{M}\left((0,\infty),\mathbb{Z}^+\right)$ contains elements of the form

$$x = \sum_{i=1}^{n} m_i \delta_{y_i}$$

for $n \in \mathbb{N}$ where $y_1,\ldots,y_n > 0$ are distinct and for $i \in \{1,\ldots,n\}, m_i \in \mathbb{N}$. The Marcus-Lushnikov process with coagulation kernel K is the continuous time Markov chain on $\mathcal{M}\left((0,\infty),\mathbb{Z}^+\right)$ with non-zero transition rates given by

$$q(x,x') = \begin{cases} m_i m_j K(y_i,y_j) & \text{if } i < j \\ \frac{1}{2} m_i(m_i - 1)K(y_i,y_i) & \text{if } i = j \end{cases}$$

for $x' = x + \delta_{y_i+y_j} - \delta_{y_i} - \delta_{y_j}$.

Let us give a way of constructing a Marcus-Lushnikov process $(X_t)_{t\geq 0}$. Let $[m] = \{1,\ldots,m\}$ and let $y_1,\ldots,y_m > 0$ be the masses (not necessarily distinct) associated to each particle in $[m]$. Set

$$X_0 = \sum_{i=1}^{m} \delta_{y_i}.$$

For each $i < j$ take an independent random variable T_{ij} such that T_{ij} is exponential with parameter $K(y_i,y_j)$, and define

$$T = \min_{i<j} T_{ij}.$$

Set $X_t = X_0$ for $t < T$ and

$$X_T = X_t - (\delta_{y_i} + \delta_{y_j} - \delta_{y_i+y_j}) \text{ if } T = T_{ij};$$

then begin the construction afresh from X_T. Each pair of clusters $i < j$ coagulates at rate $K(y_i,y_j)$. Let $(X_t^N)_{t\geq 0}$ be Marcus-Lushnikov with kernel K/N and set

$$\mu_t^N = N^{-1} X_t^N.$$

Our aim in this paper is to set up a historical analogue of the process μ_t^N and to prove that it converges to a limit measure that can be constructed from the weak limit of μ_t^N. This weak limit μ_t satisfies the Smoluchowski equation [1],[2](to be made precise below).

Before defining precisely this new process let us explain why it is interesting to know about the history of formation of a cluster.

The Marcus-Lushnikov process [3] describes the stochastic Markov evolution of a finite system of coalescing particles. It gives at each time the distribution in masses of the particles present in the system but does not retain any other information that the particles might contain. In other words, we lose in part the information contained in the particles that is their history. Why is it interesting to know about the history? For instance, consider a system of N particles with associated masses

$y_1, \ldots, y_N > 0$. Assume that these particles can only be of three types say either A, B or C. Allow them to coagulate according to the rules of coagulation of the Marcus-Lushnikov process. Then, the usual Marcus-Lushnikov will give us at each time the masses of the particles present in the system but will not be able to tell us for each particle present at this time how many particles of type A, B or C this particle contains along with the order of formation. Our historical measure will encode the history of formation of each particle present in the system at a given time t.

We are now going to review the work of [1] and [2] about the convergence of μ_t^N to the solution to Smoluchowski's equation as we will use this tool to prove our main result (stated in 1.3).

1.2 Related work

Take $\varphi : (0,\infty) \to (0,\infty)$ to be a continuous sublinear function. Assume that the coagulation kernel satisfies:

$$K(x,y) \le \varphi(x)\varphi(y). \tag{1.1}$$

For μ a measure on $(0,\infty)$ such that

$$\int_0^\infty \int_0^\infty K(x,y)\mu(dx)\mu(dy) < \infty,$$

we define $L(\mu)$ as follows:

$$\langle f, L(\mu)\rangle = \frac{1}{2}\int_0^\infty \int_0^\infty \big(f(x+y) - f(x) - f(y)\big)K(x,y)\mu(dx)\mu(dy)$$

for all f bounded and measurable. Given a measure μ_0 on $(0,\infty)$, we consider the following measure-valued form of the Smoluchowski coagulation equation,

$$\mu_t = \mu_0 + \int_0^t L(\mu_s)ds. \tag{1.2}$$

We admit as a local solution any map:

$$t \to \mu_t : [0,T) \to \mathscr{M}((0,\infty))$$

where $T \in (0,\infty]$ and $\mathscr{M}((0,\infty))$ is the set of Borel measures on $(0,\infty)$, such that

1. for all Borel sets $B \subseteq (0,\infty)$,
$$t \to \mu_t(B) : [0,T) \to (0,\infty)$$
is measurable,
2. for all $t < T$,
$$\sup_{s\le t}\langle \varphi, \mu_s\rangle < \infty,$$
3. for all bounded measurable functions f, for all $t < T$,
$$\int_0^\infty f(x)\mu_t(dx) = \int_0^\infty f(x)\mu_0(dx)$$
$$+ \frac{1}{2}\int_0^t \int_0^\infty \int_0^\infty \big(f(x+y) - f(x) - f(y)\big)K(x,y)\mu_s(dx)\mu_s(dy)ds.$$

In the case $T = \infty$ we call a local solution a solution. Assume that

$$\langle \varphi, \mu_0 \rangle = \int_0^\infty \varphi(x)\mu_0(dx) < \infty, \ \langle \varphi^2, \mu_0 \rangle = \int_0^\infty \varphi^2(x)\mu_0(dx) < \infty. \tag{1.3}$$

Then [2] tells us that there exists a unique maximal solution to (1.2) denoted $(\mu_t)_{t<T}$ for some $T > 0$. Take $(\mu_t^N)_{t\geq 0}$ (as in Section 1.1) such that

$$\int_0^\infty f(x)\varphi(x)\mu_0^N(dx) \to \int_0^\infty f(x)\varphi(x)\mu_0(dx)$$

as $N \to \infty$ for all f bounded and continuous on $(0,\infty)$. Then, for all $t < T$, for all f continuous and bounded on $(0,\infty)$,

$$\sup_{s\leq t}\left|\int_0^\infty f(x)\varphi(x)\mu_s^N(dx) - \int_0^\infty f(x)\varphi(x)\mu_s(dx)\right| \to 0$$

as $N \to \infty$ in probability. Our aim is to prove a similar result to the one just above for our historical analogue of the process μ_t^N that we shall now define.

1.3 Our main result

Basic notations for trees

We define $\mathbb{T} = \mathbb{T}(\{1\})$ to be the smallest set with $1 \in \mathbb{T}$ and such that $\{\tau_1, \tau_2\} \in \mathbb{T}$ whenever $\tau_1, \tau_2 \in \mathbb{T}$. We refer to elements of $\mathbb{T}$ as trees. They are finite binary trees with leaves labeled by 1. Let $n : \mathbb{T} \to \mathbb{N}$ be the counting function, defined as follows:

$$n(1) = 1$$

and

$$n(\{\tau_1, \tau_2\}) = n(\tau_1) + n(\tau_2)$$

for all $\tau_1, \tau_2 \in \mathbb{T}$.

The historical measures

Fix $t < T$. Our principal object of interest is a process of empirical particle measures $\tilde{\mu}_t^N$ on the space of historical trees $A[0,T)$ which we shall now define. The space $A[0,t)$ is given by

$$A[0,t) = \bigcup_{\tau\in\mathbb{T}} A_\tau[0,t)$$

where $A_1[0,t) = (0,\infty)$ and for $\tau = \{\tau_1, \tau_2\} \in \mathbb{T}$,

$$A_\tau[0,t) = \big\{(s, \{\xi_1, \xi_2\}) : s \in [0,t), \xi_1 \in A_{\tau_1}[0,s), \xi_2 \in A_{\tau_2}[0,s)\big\}.$$

Note that $A[0,t) \subseteq A[0,T)$. We equip $A[0,T)$ with its Borel σ-algebra (we explain in Section 2.2 how to equip $A[0,T)$ with a topology). We define on $A[0,T)$ the mass function $m : A[0,T) \to (0,\infty)$. For $\xi \in A_1[0,T) = (0,\infty)$, we set

$$m(\xi) = \xi.$$

Recursively for $\tau = \{\tau_1, \tau_2\} \in \mathbb{T}$, $\xi = (s, \{\xi_1, \xi_2\}) \in A_\tau[0,T)$, we set

$$m(\xi) = m(\xi_1) + m(\xi_2).$$

Each $\xi \in A[0,T)$ has a set of mass labels $\lambda(\xi) \subset (0,\infty)$ determined by

$$\lambda(\xi) = \{\xi\} \text{ for } \xi \in A_1[0,T)$$

and recursively for $\tau = \{\tau_1, \tau_2\} \in \mathbb{T}$, $\xi = (s, \{\xi_1, \xi_2\}) \in A_\tau[0,T)$,

$$\lambda(\xi) = \lambda(\xi_1) \cup \lambda(\xi_2).$$

Let $\mathscr{M}\left(A[0,T), \mathbb{Z}^+\right)$ be the set of finite integer-valued measures on $A[0,T)$. Set

$$S^* = \left\{(t,\mu) \in [0,T) \times \mathscr{M}\left(A[0,T), \mathbb{Z}^+\right) : \ \mathrm{supp}(\mu) \subseteq A[0,t)\right\}$$

We define $(t, \tilde{X}_t)_{t<T}$ to be the Markov process with coagulation kernel K and values in S^* with generator $\mathscr{G}$ given by for $\Phi : S^* \to \mathbb{R}$ and $(t,\mu) \in S^*$ with $\mu = \sum_{i=1}^m \delta_{\xi^i}$,

$$\mathscr{G}\Phi\left((t,\mu)\right) = \frac{1}{2} \sum_{\substack{i,j=1,\ldots,m \\ i \neq j}} K(m(\xi^i), m(\xi^j)) \left[\Phi\left(\left(t, \mu - \delta_{\xi^i} - \delta_{\xi^j} + \delta_{(t,\{\xi^i,\xi^j\})}\right)\right) - \Phi\left((t,\mu)\right)\right] + \frac{\partial \Phi}{\partial t}.$$

Let $(t, \tilde{X}_t^N)_{t<T}$ be the same Markov process as above but with coagulation kernel K/N. We define our empirical measure $(t, \tilde{\mu}_t^N)_{t<T}$ such that

$$\tilde{\mu}_t^N = N^{-1} \tilde{X}_t^N.$$

Then, $(t, \tilde{\mu}_t^N)_{t<T}$ is a Markov process on

$$S_N^* = \left\{(t,\mu) : (t, N\mu) \in S^*\right\}$$

and its generator $\mathscr{A}^N$ is given by for $\Phi : S_N^* \to \mathbb{R}$ and $(t,\mu) \in S_N^*$ with $\mu = \frac{1}{N}\sum_{i=1}^m \delta_{\xi^i}$,

$$\begin{aligned}
&\mathscr{A}^N \Phi\left((t,\mu)\right) \\
&= \frac{1}{2} \sum_{\substack{i,j=1,\ldots,m \\ i \neq j}} \frac{K(m(\xi^i), m(\xi^j))}{N} \left[\Phi\left(\left(t, \mu - \frac{1}{N}\delta_{\xi^i} - \frac{1}{N}\delta_{\xi^j} + \frac{1}{N}\delta_{(t,\{\xi^i,\xi^j\})}\right)\right) - \Phi\left((t,\mu)\right)\right] + \frac{\partial \Phi}{\partial t}.
\end{aligned}$$

Observe that this empirical measure $\tilde{\mu}_t^N$ and our usual Marcus Lushnikov process μ_t^N (defined in Section 1.1) are related through the following equality,

$$\mu_t^N = \tilde{\mu}_t^N \circ m^{-1}.$$

Indeed just replacing the ξ^i by their masses in the generator of $(t, \tilde{\mu}_t^N)$ we obtain the generator of (t, μ_t^N). We are interested in taking the limit of this empirical measure as $N \to \infty$. For each $t < T$, we define the limit measure on $A[0,t)$ as follows. For $\xi \in A_1[0,t)$, we set

$$\tilde{\mu}_t(d\xi) = \exp\left(-\int_0^t \int_0^\infty K\left(m(\xi), y\right) \mu_r(dy)dr\right) \mu_0(d\xi)$$

where $(\mu_r)_{r<T}$ is the deterministic solution to (1.2). Recursively for $\tau = \{\tau_1, \tau_2\} \in \mathbb{T}$, $\xi = (s, \{\xi_1, \xi_2\}) \in A_\tau[0,t)$ with $s < t < T$, we define

$$\tilde{\mu}_t(d\xi) = \epsilon(\tau) K(m(\xi_1), m(\xi_2)) \tilde{\mu}_s(d\xi_1) \tilde{\mu}_s(d\xi_2) \exp\left(-\int_s^t \int_0^\infty K\left(m(\xi), y\right) \mu_r(dy)\mathrm{d}r\right) ds \quad (1.4)$$

where $\epsilon(\tau) = 1$ if $\tau_1 \neq \tau_2$ and $\epsilon(\tau) = \frac{1}{2}$ if $\tau_1 = \tau_2$. Observe that for $\tau = \{\tau_1, \tau_2\} \in \mathbb{T}$ and $\xi = (s, \{\xi_1, \xi_2\}) \in A_\tau[0,t)$, $\tilde{\mu}_t(d\xi)$ is symmetric with respect to ξ_1, ξ_2. Hence, it does not depend of the order of τ_1, τ_2 in τ, that we could have written $\{\tau_2, \tau_1\}$.

Our main result

Denote by d some metric on $\mathscr{M}(A[0,T))$, the set of finite measures on $A[0,T)$, which is compatible with the topology of weak convergence, that is to say, $d(\mu_n, \mu) \to 0$ if and only if $\langle f, \mu_n \rangle \to \langle f, \mu \rangle$ for all bounded continuous functions $f : A[0,T) \to \mathbb{R}$. We choose d so that $d(\mu, \mu') \leq \|\mu - \mu'\|$ for all $\mu, \mu' \in \mathscr{M}(A[0,T))$ where $\|.\|$ is the total variation. When the class of functions f is restricted to those of bounded support, we get a weaker topology, also metrisable, and we denote by d_0 some compatible metric, with $d_0 \leq d$. Denote by $S(K) \subset (0,\infty) \times (0,\infty)$ the set of discontinuity points of K and by μ_0^{*N} the Nth convolution power of μ_0 and assume that,

$$\mu_0^{*N} \bigotimes \mu_0^{*N'} (S(K)) = 0 \text{ for all } N, N' \geq 1. \quad (1.5)$$

Our aim in this paper is to prove the following result[1].

Theorem 1.1. *Let $K : (0,\infty) \times (0,\infty) \to [0,\infty)$ be a symmetric measurable function and let μ_0 be a measure on $(0,\infty)$. Assume that*

$$\mu_0^{*N} \bigotimes \mu_0^{*N'} (S(K)) = 0 \text{ for all } N, N' \geq 1.$$

Assume that, for some continuous sublinear function $\varphi : (0,\infty) \to (0,\infty)$, (1.1) and (1.3) are satisfied. Take $(\mu_t^N)_{t \geq 0}$ (as defined in Section 1.1) such that

$$d(\varphi \mu_0^N, \varphi \mu_0) \to 0 \text{ as } N \to \infty.$$

Let $(\mu_t)_{t<T}$ be the solution to (1.2) with $T > 0$. Let $(\tilde{\mu}_t^N)_{t<T}$ be the empirical historical measure and let $(\tilde{\mu}_t)_{t<T}$ be the measure defined by (1.4). Then, for all $t < T$,

$$\sup_{s \leq t} d(\varphi(m)\tilde{\mu}_s^N, \varphi(m)\tilde{\mu}_s) \to 0$$

as $N \to \infty$ in probability.

[1]Note that in the case where K is continuous, we have $S(K) = \emptyset$ and so (1.5) is automatically satisfied. Also, when K is supported on $\mathbb{N} \times \mathbb{N}$, (1.5) just means that $\mathrm{supp}\mu_0 \subseteq \mathbb{N}$.

Outline of the proof

1. In section 3.1, we define a historical version of the Smoluchowski equation (this equation is given by (3.1)) and we prove that the measure $(\tilde{\mu}_t)_{t<T}$ defined by (1.4) satisfies (3.1) and that this equation has a unique solution.

2. Under the assumption that
$$\varphi(x)^{-1}\varphi(y)^{-1}K(x,y) \to 0 \text{ as } (x,y) \to \infty, \ (*)$$
we prove in section 3.2 that $\tilde{\mu}_t^N$ converges weakly in probability to a limit which satisfies the historical version of the Smoluchowski equation (3.1). This section is not actually part of the proof of Theorem 1.1, but we include it partly as a warm-up for the more intricate arguments used later.

3. To prove Theorem 1.1, in section 3.3, we use a coupling argument on compact sets to prove that without the assumption $(*)$, $\tilde{\mu}_t^N$ converges weakly in probability to a limit which satisfies the historical version of the Smoluchowski equation (3.1). By uniqueness of the solution of (3.1) we deduce that this limit is exactly the measure defined in (1.4).

The next section is dedicated to equip the space $A[0,T)$ with a topology and a metric. We will need such a structure on the space $A[0,T)$ in order to equip it with a Borel σ-algebra but also to use some tightness arguments later on in the proof of Theorem 1.1.

2 The space of trees $A[0,T)$

2.1 Ordering elements of $A[0,T)$

The set $\mathbb{T}$ is countable so we can equip it with a total order $<_{\mathbb{T}}$. Hence when we write $\{\tau_1,\tau_2\} \in \mathbb{T}$, the convention is $\tau_1 <_{\mathbb{T}} \tau_2$. We shall find it convenient to order elements of $A[0,T)$ especially when defining a topology and a metric on this space. We define a total order $<$ on $A[0,T)$ as follows. For $\xi,\xi' \in A[0,T)$ with $\xi \neq \xi'$,we can find $\tau,\tau' \in \mathbb{T}$ such that $\xi \in A_\tau[0,T)$ and $\xi' \in A_{\tau'}[0,T)$. If $\tau <_{\mathbb{T}} \tau'$, then we say that $\xi < \xi'$ else $\tau = \tau'$ and we need to find an ordering for the elements of $A_\tau[0,T)$. We are going to define it recursively. For $\tau = 1$, say $\xi < \xi'$ if $\xi < \xi'$ for the usual order in $(0,\infty)$. Recursively for $\tau = \{\tau_1,\tau_2\}$ with $\tau_1 \leq_{\mathbb{T}} \tau_2, \xi = (s,\{\xi_1,\xi_2\}), \xi' = (s',\{\xi_1',\xi_2'\}) \in A_\tau[0,T)$ with $\xi_1 < \xi_2$, $\xi_1' < \xi_2'$ and $\xi \neq \xi'$, if $s < s'$ then $\xi < \xi'$ else $s = s'$, then if $\xi_2 < \xi_2'$ then $\xi < \xi'$ else $\xi_2 = \xi_2'$ and if $\xi_1 < \xi_1'$ (note they cannot be equal as $\xi \neq \xi'$) then $\xi < \xi'$. Hence $(A[0,T),<)$ is a total ordered space.

2.2 A topology on $A[0,T)$

For each $\tau \in \mathbb{T}$, we are going to define a bijection between $A_\tau[0,T)$ and a subspace $B_\tau[0,T) \subset (0,\infty)^{2n(\tau)-1}$. Hence, the topology on $A_\tau[0,T)$ will be naturally induced from the topology on $(0,\infty)^{2n(\tau)-1}$. Define
$$B[0,T) = \bigcup_{\tau\in\mathbb{T}} B_\tau[0,T)$$

where $B_1[0,T) = (0,\infty)$ and for $\tau = \{\tau_1, \tau_2\} \in \mathbb{T}$ with $\tau_1 <_{\mathbb{T}} \tau_2$,

$$B_\tau[0,T) = \left\{(s, \xi_1, \xi_2) : s \in [0,T), \xi_1 \in B_{\tau_1}[0,s), \xi_2 \in B_{\tau_2}[0,s)\right\}.$$

Let us first give the elements of $B[0,T)$ an order $<$. For $\xi, \xi' \in B[0,T)$ with $\xi \neq \xi'$,we can find $\tau, \tau' \in \mathbb{T}$ such that $\xi \in B_\tau[0,T)$ and $\xi' \in B_{\tau'}[0,T)$. If $\tau <_{\mathbb{T}} \tau'$, then we say that $\xi < \xi'$ else $\tau = \tau'$ and we need to find an ordering for the elements of $B_\tau[0,T)$. We are going to define it recursively. For $\tau = 1$, say $\xi < \xi'$ if $\xi < \xi'$ for the usual order in $(0,\infty)$. Recursively for $\tau = \{\tau_1, \tau_2\}$ with $\tau_1 \leq_{\mathbb{T}} \tau_2, \xi = (s, \xi_1, \xi_2), \xi' = (s', \xi_1', \xi_2') \in B_\tau[0,T)$ with $\xi_1 < \xi_2$, $\xi_1' < \xi_2'$ and $\xi \neq \xi'$, if $s < s'$ then $\xi < \xi'$ else $s = s'$, then if $\xi_2 < \xi_2'$ then $\xi < \xi'$ else $\xi_2 = \xi_2'$ and if $\xi_1 < \xi_1'$ (note they cannot be equal as $\xi \neq \xi'$) then $\xi < \xi'$. Hence $(B[0,T),<)$ is a total ordered space. For $\tau \in \mathbb{T}$, define $\Psi_\tau : (A_\tau[0,T),<) \to (B_\tau[0,T),<)$ to be the map on forgetting the structure of the tree. For $\tau = 1$ define

$$\Psi_1 : (A_1[0,T),<) \to (B_1[0,T),<) : \xi \to \xi.$$

Recursively for $\tau = \{\tau_1, \tau_2\} \in \mathbb{T}$, with $\tau_1 \leq_{\mathbb{T}} \tau_2$ define

$$\Psi_\tau : (A_\tau[0,T),<) \to (B_\tau[0,T),<) : \xi = (s, \{\xi_1, \xi_2\}) \to \left(s, \Psi_{\tau_1}(\xi_1), \Psi_{\tau_2}(\xi_2)\right).$$

E.g for $\tau = \{1\}$, $\xi = (s, \{y_1, y_2\}) \in A_{\{1\}}[0,T)$, we have $\Psi_\tau(\xi) = (s, y_1, y_2)$. Observe that given $x_\tau \in B_\tau[0,T)$, there exists a unique $\xi \in A_\tau[0,T)$ such that $\Psi_\tau(\xi) = x_\tau$ so Ψ_τ is a bijection. Now define

$$\Psi : (A[0,T),<) \to (B[0,T),<) : \xi \to \sum_{\tau \in \mathbb{T}} \Psi_\tau(\xi).$$

This is a bijection so the topology on $A[0,T)$ is naturally induced from the topology on $B[0,T)$. Hence for each $f : A[0,T) \to \mathbb{R}$ there exists a unique $g : B[0,T) \to \mathbb{R}$ such that $f = g \circ \Psi$. We say that f is of bounded support if g is.

2.3 A metric on $(A[0,T),<)$

We are going to define a metric d' on $A[0,T)$. Let $\xi, \xi' \in A[0,T)$ with $\xi \neq \xi'$, then $\xi \in A_\tau[0,T)$ and $\xi' \in A_{\tau'}[0,T)$ for some $\tau, \tau' \in \mathbb{T}$. If $\tau \neq \tau'$ then we set

$$d'(\xi, \xi') = \infty,$$

else $\tau = \tau'$ and then we define d' to be : for $\xi, \xi' \in A_1[0,T)$

$$d'(\xi, \xi') = |\xi - \xi'|,$$

and recursively for $\tau = \{\tau_1, \tau_2\} \in \mathbb{T}, \xi = (s, \{\xi_1, \xi_2\}), \xi' = (s', \{\xi_1', \xi_2'\}) \in A_\tau[0,T)$ with $\xi_1 < \xi_2, \xi_1' < \xi_2'$, set

$$d'(\xi, \xi') = |s - s'| + d'(\xi_1, \xi_1') + d'(\xi_2, \xi_2').$$

One can easily check that d' is a metric. Hence $\left(A[0,T), d'\right)$ is a metric space. Thus we can define open sets and compact sets on this space with the usual way.

3 Proof of Theorem 1.1

3.1 A historical version of the Smoluchowski equation

We here introduce a measure-valued form of the Smoluchowski equation on the space of trees and prove that given an initial measure μ_0 on $A[0,T)$, this equation has a unique solution.

Set up

For $t < T$, for μ a measure on $A[0,T)$ supported on $A[0,t)$ we define $\mathscr{A}(t,\mu)$ as follows

$$\langle f, \mathscr{A}(t,\mu)\rangle = \frac{1}{2}\int_{A[0,T)}\int_{A[0,T)} \{f\left((t,\{\xi,\xi'\})\right) - f(\xi) - f(\xi')\} K\left(m(\xi), m(\xi')\right)\mu(d\xi)\mu(d\xi')$$

for all $f : A[0,T) \to \mathbb{R}$ bounded and measurable. Observe that the first term in the integral above is well-defined on the support of $\mu \otimes \mu$. Given a measure $\tilde{\mu}_0$ on $A_1[0,T) = (0,\infty) \subseteq A[0,T)$ we consider the following measure-valued form of the Smoluchowski equation,

$$\tilde{\mu}_t = \tilde{\mu}_0 + \int_0^t \mathscr{A}(s,\tilde{\mu}_s)ds. \tag{3.1}$$

Notion of solutions

We admit as a local solution any map:

$$t \to \tilde{\mu}_t : [0,T) \to \mathscr{M}\left(A[0,T)\right)$$

where $T \in (0,\infty]$ and $\mathscr{M}\left(A[0,T)\right)$ is the set of finite Borel measures on $A[0,T)$, such that

1. $\mathrm{supp}(\tilde{\mu}_t) \subset A[0,t)$ for all $t \in [0,T)$
2. for all measurable sets $B \subseteq A[0,T)$,
$$t \to \tilde{\mu}_t(B) : [0,T) \to (0,\infty)$$
is measurable.
3. for all $t < T$,
$$\sup_{s\le t}\langle \varphi \circ m, \tilde{\mu}_s\rangle < \infty.$$
4. for all bounded measurable functions f, for all $t < T$,
$$\int_{A[0,T)} f(\xi)\tilde{\mu}_t(d\xi) = \int_{A[0,T)} f(\xi)\tilde{\mu}_0(d\xi)$$
$$+ \frac{1}{2}\int_0^t \int_{A[0,T)}\int_{A[0,T)} \{f\left((s,\{\xi,\xi'\})\right) - f(\xi) - f(\xi')\} K\left(m(\xi), m(\xi')\right)\tilde{\mu}_s(d\xi)\tilde{\mu}_s(d\xi')ds.$$

In the case $T = \infty$ we call a local solution a solution.

Existence and uniqueness of solutions

Proposition 3.1. *The measure* $(\tilde{\mu}_t)_{t<T}$ *defined by (1.4) is the unique solution to (3.1).*

Before proving the proposition above we need to introduce some more material.

Lemma 3.2. *Suppose* $(\tilde{\mu}_t)_{t<T}$ *is a solution to (3.1). Let* $(t,\xi) \to f_t(\xi) = f(t,\xi) : [0,T) \times A[0,T) \to \mathbb{R}$ *be a bounded and measurable function, having a bounded partial derivative* $\partial f/\partial t$*. Then, for all* $t < T$,

$$\frac{d}{dt}\langle f_t, \tilde{\mu}_t\rangle = \langle \partial f/\partial t, \tilde{\mu}_t\rangle + \langle f_t, \mathscr{A}(t,\tilde{\mu}_t)\rangle$$

Proof: Fix $t < T$, and set $\lfloor s \rfloor_n = (n/t)^{-1}\lfloor ns/t\rfloor$ and $\lceil s \rceil_n = (n/t)^{-1}\lceil ns/t\rceil$. Then,

$$\langle f_t, \tilde{\mu}_t\rangle = \langle f_0, \tilde{\mu}_0\rangle + \int_0^t \langle \partial f/\partial s, \tilde{\mu}_{\lfloor s \rfloor_n}\rangle ds + \int_0^t \langle f_{\lceil s \rceil_n}, \mathscr{A}(s,\tilde{\mu}_s)\rangle ds$$

and the proposition follows on letting $n \to \infty$, by the bounded convergence theorem. Here, when we write $\frac{d}{dt}\langle f_t, \tilde{\mu}_t\rangle$ we are thinking of it as its integral reformulation. □

For $t < T$, for $\xi \in A[0,T)$, set

$$\theta_t(\xi) = \exp\left(\int_0^t \int_0^\infty K\left(m(\xi), y\right) \mu_s(dy) ds\right)$$

where $\left(\mu_t\right)_{t<T}$ is the solution to (1.2). For all $f : A[0,T) \to \mathbb{R}$ bounded and continuous, define $G_t : \mathscr{M}\left(A[0,t)\right) \times \mathscr{M}\left(A[0,t)\right) \to \mathscr{M}\left(A[0,T)\right)$ by

$$\langle f, G_t(\mu,\nu)\rangle = \frac{1}{2}\int_{A[0,T)\times A[0,T)} \left(f\theta_t\right)\left((t,\{\xi,\xi'\})\right) K\left(m(\xi),m(\xi')\right) \theta_t(\xi)^{-1}\theta_t(\xi')^{-1}\mu(d\xi)\nu(d\xi').$$

Now suppose that $\left(\tilde{\mu}_t\right)_{t<T}$ solves (3.1) and set $\hat{\mu}_t = \theta_t\tilde{\mu}_t$. Then, $\tilde{\mu}_t \circ m^{-1}$ solves (1.2) whose solution is unique and so $\tilde{\mu}_t \circ m^{-1} = \mu_t$. By Lemma 3.2, for all f bounded and measurable,

$$\frac{d}{dt}\langle f, \hat{\mu}_t\rangle = \langle f\,\partial\theta/\partial t, \tilde{\mu}_t\rangle + \langle f\theta_t, \mathscr{A}(t,\tilde{\mu}_t)\rangle = \langle f, G_t(\hat{\mu}_t, \hat{\mu}_t)\rangle. \tag{3.2}$$

Thus, if $\tilde{\mu}_t$ is a solution to (3.1), then (3.2) holds for $\hat{\mu}_t$. Now we will need the following lemma:

Lemma 3.3. *The solution to (3.2) is unique.*

Proof:

Let $\left(\tilde{\mu}_t\right)_{t<T}$ and $\left(\tilde{\nu}_t\right)_{t<T}$ be solutions to (3.2) such that $\tilde{\mu}_0 = \tilde{\nu}_0$. One can write

$$\tilde{\mu}_t = \sum_{\tau\in\mathbb{T}} \tilde{\mu}_t^\tau,\ \tilde{\nu}_t = \sum_{\tau\in\mathbb{T}} \tilde{\nu}_t^\tau$$

where $\tilde{\mu}_t^\tau, \tilde{\nu}_t^\tau$ are supported on $A_\tau[0,t)$. For f supported on $A_1[0,T)$ we have

$$\frac{d}{dt}\langle f, \tilde{\mu}_t^1\rangle = \langle G_t(\tilde{\mu}_t^1, \tilde{\mu}_t^1)\rangle = 0$$

so $\langle f,\tilde{\mu}_t^1\rangle = \langle f,\tilde{\mu}_0\rangle$. Similarly, $\langle f,\tilde{\nu}_t^1\rangle = \langle f,\tilde{\nu}_0\rangle$, and thus $\tilde{\mu}_t^1 = \tilde{\nu}_t^1$.

Induction hypothesis: Suppose that for all $\tau \in \mathbb{T}$ with $n(\tau) \leq n-1$, we have $\tilde{\mu}_t^\tau = \tilde{\nu}_t^\tau$. Now take $\tau = \{\tau_1,\tau_2\}$ with $n(\tau) = n$. Then, for f supported on $A_\tau[0,T)$,

$$\begin{cases} \frac{d}{dt}\langle f,\tilde{\mu}_t^\tau\rangle = \langle f, G_t(\tilde{\mu}_t^{\tau_1},\tilde{\mu}_t^{\tau_2})\rangle, \\ \frac{d}{dt}\langle f,\tilde{\nu}_t^\tau\rangle = \langle f, G_t(\tilde{\nu}_t^{\tau_1},\tilde{\nu}_t^{\tau_2})\rangle. \end{cases}$$

so by the induction hypothesis,

$$\frac{d}{dt}\langle f,\tilde{\mu}_t^\tau\rangle = \frac{d}{dt}\langle f,\tilde{\nu}_t^\tau\rangle$$

so

$$\langle f,\tilde{\mu}_t^\tau\rangle = \langle f,\tilde{\nu}_t^\tau\rangle.$$

Thus, $\tilde{\mu}_t^\tau = \tilde{\nu}_t^\tau$ for all $\tau \in \mathbb{T}$ and so $\tilde{\mu}_t = \tilde{\nu}_t$.

$\square$

Proof of Proposition 3.1:

Uniqueness

Suppose that $\tilde{\mu}_t$ and $\tilde{\nu}_t$ satisfy (3.1) with $\tilde{\mu}_0 = \tilde{\nu}_0$. Then, $\hat{\mu}_t = \theta_t\tilde{\mu}_t$. and $\hat{\nu}_t = \theta_t\tilde{\nu}_t$ solves (3.2) whose solution is unique and so $\hat{\mu}_t = \hat{\nu}_t$. Thus, $\tilde{\mu}_t = \tilde{\nu}_t$.

The measure defined by (1.4) satisfies (3.1)

Let $(\tilde{\mu}_t)_{t<T}$ be the measure defined by (1.4). It is enough to show that $\hat{\mu}_t = \theta_t\tilde{\mu}_t$ satisfies (3.2). Indeed, suppose that $(\tilde{\nu}_t)_{t<T}$ satisfies (3.1); then $\hat{\nu}_t = \theta_t\tilde{\nu}_t$ satisfies (3.2), and by uniqueness of the solutions of (3.2) we know that $\tilde{\mu}_t = \tilde{\nu}_t$, and so $(\tilde{\mu}_t)_{t<T}$ would satisfy (3.1). Take $\tau = \{\tau_1,\tau_2\} \in \mathbb{T}$ and take $f : A[0,T) \to \mathbb{R}$ bounded and measurable with support in $A_\tau[0,T)$. Hence,

$$\langle f,\hat{\mu}_t\rangle = \langle f\theta_t,\tilde{\mu}_t^\tau\rangle.$$

Now,

$$\begin{aligned}\frac{d}{dt}\langle f,\hat{\mu}_t\rangle =& \frac{d}{dt}\left[\int_0^t\int_{A[0,T)^2}\epsilon(\tau)f\left((s,\{\xi_1,\xi_2\})\right)\tilde{\mu}_s^{\tau_1}(d\xi_1)\tilde{\mu}_s^{\tau_2}(d\xi_2)K\left(m(\xi_1),m(\xi_2)\right)\right.\\ &\left.\exp\left(-\int_0^s\int_0^\infty K\left(m(\xi_1)+m(\xi_2),y\right)\mu_r(dy)dr\right)ds\right]\\ =&\ \epsilon(\tau)\int_{A[0,T)^2} f\left((t,\{\xi_1,\xi_2\})\right)\tilde{\mu}_t^{\tau_1}(d\xi_1)\tilde{\mu}_t^{\tau_2}(d\xi_2)K\left(m(\xi_1),m(\xi_2)\right)\\ &\exp\left(-\int_0^t\int_0^\infty K\left(m(\xi_1)+m(\xi_2),y\right)\mu_r(dy)dr\right).\end{aligned}$$

If $\tau_1 = \tau_2$ then $\epsilon(\tau) = 1/2$ and for $\xi = (s,\{\xi_1,\xi_2\}) \in A_\tau[0,T)$,

$$\tilde{\mu}_t^{\tau_1}(d\xi_1)\tilde{\mu}_t^{\tau_2}(d\xi_2) = \tilde{\mu}_t(d\xi_1)\tilde{\mu}_t(d\xi_2).$$

If $\tau_1 \neq \tau_2$ then $\epsilon(\tau) = 1$ and for $\xi = (s, \{\xi_1, \xi_2\}) \in A_\tau[0, T)$,

$$\tilde{\mu}_t^{\tau_1}(d\xi_1)\tilde{\mu}_t^{\tau_2}(d\xi_2) = \frac{1}{2}\tilde{\mu}_t(d\xi_1)\tilde{\mu}_t(d\xi_2).$$

so in both cases we obtain,

$$\frac{d}{dt}\langle f, \hat{\mu}_t\rangle = \langle f, G_t(\hat{\mu}_t, \hat{\mu}_t)\rangle.$$

Hence, the result above is true for all f bounded and measurable with support on $A_\tau[0, T)$. This can be extended to all $f : A[0, T) \to \mathbb{R}$ bounded and measurable using the fact that,

$$f = \sum_{\tau \in \mathbb{T}} f 1_{A_\tau[0,T)}.$$

Hence, $(\tilde{\mu}_t)_{t<T}$ satisfies (3.1) as required.

$\square$

Thus, in order to prove Theorem 1.1, it is enough to prove that for any subsequential limit law of the sequence $(\tilde{\mu}_t^N)_{t<T}$, the equation (3.1) holds almost surely. Then, by uniqueness of the solutions to (3.1), it will be the measure (1.4). The next section is dedicated to prove that under the assumption

$$\varphi(x)^{-1}\varphi(y)^{-1}K(x, y) \to 0 \text{ as } (x, y) \to \infty,$$

the weak limit of $(\tilde{\mu}_t^N)_{t<T}$ satisfies (3.1) which as we explained before can be seen as a preparation for the more complicated arguments used later on in the proof of Theorem 1.1.

3.2 The weak limit of the empirical measure satisfies Smoluchowski's equation

The following proposition will show that any weak limit of $(\tilde{\mu}^N)_{N\geq 0}$ satisfies Smoluchowski's equation (3.1).

Proposition 3.4. *Let $K : (0, \infty) \times (0, \infty) \to [0, \infty)$ be a symmetric measurable function and let μ_0 be a measure on $(0, \infty)$. Assume that for some continuous sublinear function $\varphi : (0, \infty) \to (0, \infty)$, (1.1) and (1.3) are satisfied. Moreover suppose that*

$$\varphi^{-1}(x)\varphi^{-1}(y)K(x, y) \to 0 \text{ as } (x, y) \to \infty.$$

Take $(\mu_t^N)_{t\geq 0}$ (as defined in Section 1.1) such that

$$d_0(\varphi\mu_0^N, \varphi\mu_0) \to 0 \text{ as } N \to \infty.$$

Let $(\mu_t)_{t<T}$ be the solution to (1.2) with $T > 0$ and let $(\tilde{\mu}_t^N)_{t<T}$ be the empirical historical measure (as defined in section 1.3). Then,

(a) The sequence of laws $\varphi\tilde{\mu}^N$ on $D_{(\mathscr{M}(A[0,T)),d_0)}[0, \infty)$ is tight.

(b) Any weak limit point $\varphi\tilde{\mu}$ almost surely satisfies

$$\langle f, \tilde{\mu}_t\rangle = \langle f, \tilde{\mu}_0\rangle + \int_0^t \langle f, \mathscr{A}(s, \tilde{\mu}_s)\rangle$$

for all $f : A[0, T) \to \mathbb{R}$ bounded and measurable.

In order to prove Proposition 3.4, we will need the following lemma.

Lemma 3.5. *Let $(\tilde{\mu}_t^N)_{t<T}$ be a sequence of random measures on $\mathcal{M}(A[0,T))$ and let $(\tilde{\mu}_t)_{t<T}$ be a family of deterministic measures on $\mathcal{M}(A[0,T))$ such that for all $\tau \in \mathbb{T}$, for all $f : A_\tau[0,T) \to \mathbb{R}$ bounded and continuous (resp. bounded and continuous of bounded support),*

$$\sup_{s\leq t}\left|\langle \varphi(m)f, \tilde{\mu}_s^N\rangle - \langle \varphi(m)f, \tilde{\mu}_s\rangle\right| \to 0$$

as $N \to \infty$ in probability. Then,

$$\sup_{s\leq t} d\left(\varphi(m)\tilde{\mu}_s^N, \varphi(m)\tilde{\mu}_s\right) \to 0, \quad \left(\textit{resp. } \sup_{s\leq t} d_0\left(\varphi(m)\tilde{\mu}_s^N, \varphi(m)\tilde{\mu}_s\right) \to 0\right)$$

as $N \to \infty$ in probability.

We are first reviewing the work of [2]. Let $\mathbb{T}(0,\infty)$ be the space of trees on $(0,\infty)$. It is given by

$$\mathbb{T}(0,\infty) = \bigcup_{\tau\in\mathbb{T}} \mathbb{T}_\tau(0,\infty)$$

where $\mathbb{T}_1(0,\infty) = (0,\infty)$ and for $\tau = \{\tau_1, \tau_2\} \in \mathbb{T}$,

$$\mathbb{T}_\tau(0,\infty) = \left\{y = \{y_1, y_2\} : y_1 \in \mathbb{T}_{\tau_1}(0,\infty), y_2 \in \mathbb{T}_{\tau_2}(0,\infty)\right\}.$$

On this space, we can define a mass function $m' : \mathbb{T}(0,\infty) \to (0,\infty)$ and a counting function $n' : \mathbb{T}(0,\infty) \to \mathbb{N}$. For $t < T$, let

$$y_t : A[0,t) \to \mathbb{T}(0,\infty)$$

be the map on forgetting times and define $\left(\overline{\mu}_t^N\right)_{t<T}$ by

$$\overline{\mu}_t^N = \tilde{\mu}_t^N \circ y_t^{-1}.$$

Then $\left(\overline{\mu}_t^N\right)_{t<T}$ is a Marcus Lushnikov process on the space of trees $\mathbb{T}(0,\infty)$ with kernel $K(m(\cdot), m(\cdot))/N$. Define

$$\tilde{\varphi} : \mathbb{T}(0,\infty) \to (0,\infty) \text{ by } \tilde{\varphi} = \varphi \circ m'.$$

Then [2] tells us that given $\epsilon > 0$, we can find $n_0 > 0$ such that for all N,

$$\mathbb{P}\left(\sup_{s\leq t}|\langle \varphi \circ m' 1_{\{y\in\mathbb{T}(0,\infty):n'(y)\geq n_0\}}, \overline{\mu}_s^N\rangle| > \epsilon\right) < \epsilon.$$

Now, for $t < T$,

$$\overline{\mu}_t^N\left(\{y \in \mathbb{T}(0,\infty) : n'(y) \geq n_0\}\right) = \tilde{\mu}_t^N\left(\{\xi \in A[0,T) : n'\left(y_t(\xi)\right) \geq n_0\}\right).$$

Hence, for all $t < T$,

$$\mathbb{P}\left(\sup_{s\leq t}|\langle \varphi \circ m 1_{\{\xi\in A[0,T):n'\left(y_s(\xi)\right)\geq n_0\}}, \tilde{\mu}_s^N\rangle| > \epsilon\right) < \epsilon. \tag{3.4}$$

Proof of Lemma 3.5: We are going to apply the tightness argument (3.4) to prove the result of the lemma. We will prove it only for the case f bounded and continuous, the case f bounded, continuous and of bounded support being similar. Take $f \in C_b(A[0,T))$. For $n \in \mathbb{N}$ consider

$$A_n = \bigcup_{\substack{\tau \in \mathbb{T} \\ \text{with } n(\tau) \leq n}} A_\tau[0,T).$$

Since

$$A[0,T) = \bigcup_{\tau \in \mathbb{T}} A_\tau[0,T)$$

it is clear that

$$A[0,T) = \bigcup_{n \geq 1} A_n.$$

Moreover, $(A_n)_{n\geq 1}$ forms an increasing sequence. Set $f_n = f 1_{A_n}$. Then, $f_n \in C_b(A_n)$, and $f_n \to f$ as $n \to \infty$. Now, $|\varphi(m) f_n| \leq \varphi(m) \|f\|_\infty$. Hence by the dominated convergence theorem,

$$\tilde{\mu}_t(\varphi(m) f_n) \to \tilde{\mu}_t(\varphi(m) f)$$

as $n \to \infty$. We can write

$$f_n = \sum_{\substack{\tau \in \mathbb{T} \\ \text{with } n(\tau) \leq n}} f_n 1_{A_\tau[0,T)}.$$

Thus, for all $t < T$,

$$\langle f_n, \tilde{\mu}_t^N \rangle = \sum_{\substack{\tau \in \mathbb{T} \\ \text{with } n(\tau) \leq n}} \langle f_n 1_{A_\tau[0,T)}, \tilde{\mu}_t^N \rangle.$$

Now, since $f_n 1_{A_\tau[0,T)} \in C_b(A_\tau[0,T))$, for all $t < T$ we have

$$\sup_{s \leq t} \left| \langle \varphi(m) f_n 1_{A_\tau[0,T)}, \tilde{\mu}_s^N \rangle - \langle \varphi(m) f_n 1_{A_\tau[0,T)}, \tilde{\mu}_s \rangle \right| \to 0$$

as $N \to \infty$ in probability. Thus, since the sum over $\{\tau \in \mathbb{T} \text{ with } n(\tau) \leq n\}$ is finite, we obtain, for all $t < T$,

$$\sup_{s \leq t} \left| \langle \varphi(m) f_n, \tilde{\mu}_s^N \rangle - \langle \varphi(m) f_n, \tilde{\mu}_s \rangle \right| \to 0$$

as $N \to \infty$, in probability. Let us fix $\epsilon > 0$ and consider, for all $t < T$,

$$\begin{aligned}
\sup_{s \leq t} \left| \langle \varphi(m) f, \tilde{\mu}_s^N \rangle - \langle \varphi(m) f, \tilde{\mu}_s \rangle \right| &\leq \sup_{s \leq t} \left| \langle \varphi(m) f, \tilde{\mu}_s^N \rangle - \langle \varphi(m) f_n, \tilde{\mu}_s^N \rangle \right| \\
&\quad + \sup_{s \leq t} \left| \langle \varphi(m) f_n, \tilde{\mu}_s^N \rangle - \langle \varphi(m) f_n, \tilde{\mu}_s \rangle \right| \\
&\quad + \sup_{s \leq t} \left| \langle \varphi(m) f_n, \tilde{\mu}_s \rangle - \langle \varphi(m) f, \tilde{\mu}_s \rangle \right|
\end{aligned}$$

By (3.4), we know that for all $t < T$, we can find $n_0 > 0$, such that for all N,

$$\mathbb{P}\left(\sup_{s \leq t} |\langle \varphi \circ m 1_{\{\xi \in A[0,T) : n'(y_s(\xi)) \geq n_0\}}, \tilde{\mu}_s^N \rangle| > \frac{\epsilon}{\|f\|_\infty} \right) < \frac{\epsilon}{3}.$$

Now, $f - f_n = f 1_{A_n^c}$. Thus, for $t < T$,

$$|\langle \varphi(m)f, \tilde{\mu}_t^N \rangle - \langle \varphi(m)f_n, \tilde{\mu}_t^N \rangle| = |\langle \varphi(m)(f - f_n), \tilde{\mu}_t^N \rangle| \leq \|f\|_\infty |\langle \varphi(m)1_{A_n^c}, \tilde{\mu}_t^N \rangle|$$

For all $n \geq n_0, A_n^c \subseteq A_{n_0}^c$. Hence

$$\begin{aligned}\mathbb{P}\left(\sup_{s \leq t}\left|\langle \varphi(m)f, \tilde{\mu}_s^N \rangle - \langle \varphi(m)f_n, \tilde{\mu}_s^N \rangle\right| > \epsilon\right) &< \mathbb{P}\left(\|f\|_\infty \sup_{s \leq t}|\langle \varphi(m)1_{A_n^c}, \tilde{\mu}_s^N \rangle| > \epsilon\right) \\ &< \frac{\epsilon}{3}\end{aligned}$$

for all $n \geq n_0$.

Moreover, we can choose n_1 so that, for all $n \geq n_1$,

$$\sup_{s \leq t}\left|\langle \varphi(m)f_n, \tilde{\mu}_s \rangle - \langle \varphi(m)f, \tilde{\mu}_s \rangle\right| < \frac{\epsilon}{3}$$

Hence, we can find N_0 such that for all $N > N_0$,

$$\mathbb{P}\left(\sup_{s \leq t}|\langle \varphi(m)f, \tilde{\mu}_s^N \rangle - \langle \varphi(m)f, \tilde{\mu}_s \rangle| > \epsilon\right) < \epsilon,$$

as required.

□

Now we have the tools required to prove Proposition 3.4.

Proof of Proposition 3.4:

First we prove (a). For an integer-valued measure μ on $A[0,T)$, denote by $\mu^{(1)}$ the integer-valued measure on $A[0,T) \times A[0,T)$ given by

$$\mu^{(1)}(A \times A') = \mu(A)\mu(A') - \mu(A \cap A').$$

Similarly, when $N\mu$ is an integer-valued measure, set

$$\mu^{(N)}(A \times A') = \mu(A)\mu(A') - N^{-1}\mu(A \cap A').$$

For $f : A[0,T) \to \mathbb{R}$ bounded and measurable, define

$$\Phi_f : S_N^* \to \mathbb{R} : (t,\mu) \to \int_{A[0,T)} f(\xi)\mu(d\xi).$$

Then for $(t,\mu) \in S_N^*$ with $t < T$ and $\mu = \frac{1}{N}\sum_{i=1}^m \delta_{\xi^i}$, one can easily check that

$$\mathscr{A}^N \Phi_f\left((t,\mu)\right) = \frac{1}{2}\int_{A[0,T)}\int_{A[0,T)} K\left(m(\xi), m(\xi')\right)\left[f\left((t,\{\xi,\xi'\})\right) - f(\xi) - f(\xi')\right]\mu^{(N)}(d\xi, d\xi').$$

For convenience, we write $\mathscr{A}^N\Phi_f\left((t,\mu)\right)=\mathscr{A}_t^N(\mu)(f)$. Then,

$$M_t^{f,N}=\langle f,\tilde{\mu}_t^N\rangle-\langle f,\tilde{\mu}_0^N\rangle-\int_0^t\mathscr{A}_s^N(\tilde{\mu}_s^N)(f)ds \tag{3.5}$$

is a martingale having previsible process,

$$\langle M^{f,N}\rangle_t=\int_0^t\mathscr{Q}_s^N(\tilde{\mu}_s^N)(f)ds$$

where for all $s\le t$ and $(s,\mu)\in S_N^*$,

$$\mathscr{Q}_s^N(\mu)(f)=\frac{1}{N}\int_{A[0,T)}\int_{A[0,T)}K\left(m(\xi),m(\xi')\right)\left[f\left((s,\{\xi,\xi'\})\right)-f(\xi)-f(\xi')\right]^2\mu^{(N)}(d\xi,d\xi').$$

Fix $\epsilon>0$. Since $\langle\varphi,\mu_0^N\rangle\to\langle\varphi,\mu_0\rangle$ as $N\to\infty$, we can find $N_0>0$ such that for all $N\ge N_0$,

$$\langle\varphi,\mu_0^N\rangle<\langle\varphi,\mu_0\rangle+\epsilon.$$

Let $M_0=\max_{N\le N_0}\langle\varphi,\mu_0^N\rangle$ and set $\Lambda=\max(M_0,\langle\varphi,\mu_0\rangle+\epsilon)$. Thus, for all N,

$$\langle\varphi,\mu_0^N\rangle\le\Lambda.$$

Since φ is subadditive, then for all N and all $s<T$,

$$\langle\varphi,\mu_s^N\rangle\le\langle\varphi,\mu_0^N\rangle\le\Lambda.$$

Now, $\mu_t^N=\tilde{\mu}_t^N\circ m^{-1}$, so

$$\langle\varphi(m),\tilde{\mu}_s^N\rangle=\langle\varphi,\mu_s^N\rangle.$$

Thus, for all N and all $s<T$,

$$\langle\varphi(m),\tilde{\mu}_s^N\rangle\le\Lambda.$$

Hence by (1.1), we get the following bounds,

$$\begin{aligned}\|\mathscr{A}_s^N(\tilde{\mu}_s^N)(f)\|&\le 2\|f\|\Lambda^2\\ \|\mathscr{Q}_s^N(\tilde{\mu}_s^N)(f)\|&\le\frac{4}{N}\|f\|^2\Lambda^2.\end{aligned} \tag{3.6}$$

Let $\mathscr{S}_{\mathscr{M}(A[0,T))}$ be the Borel σ-algebra of $D_{\mathscr{M}(A[0,T))}[0,\infty)$. For $A\in\mathscr{S}_{\mathscr{M}(A[0,T))}$, define

$$\mathscr{L}_N(A)=\mathbb{P}\left((\varphi(m)\wedge 1)\tilde{\mu}^N\in A\right).$$

We want to show the family of probability measures $(\mathscr{L}_N)_N$ on $D_{\mathscr{M}\left(A[0,T),d_0\right)}[0,\infty)$ are tight. In order to prove that we will apply Jakubowski's criterion of tightness [6]. Observe first that since $\left(\mathscr{M}\left(A[0,T)\right),d_0\right)$ is a metric space, it is a fortiori completely regular and with metrisable compact sets. For $f:A[0,T)\to\mathbb{R}$ bounded and continuous such that $\frac{f}{\varphi(m)\wedge 1}$ is bounded, define

$$G_f:\mathscr{M}\left(A[0,T)\right)\to\mathbb{R}:\mu\to\langle\frac{f}{\varphi(m)\wedge 1},\mu\rangle.$$

Let

$$\mathbb{F} = \left\{ G_f : f \in C_b(A[0,T)) \text{ with } \frac{f}{\varphi(m)\wedge 1} \text{ bounded} \right\}.$$

This is a family of continuous functions on $\mathcal{M}\left(A[0,T), d_0\right)$. One can easily check that $\mathbb{F}$ separates points and that it is closed under addition. For $f : A[0,T) \to \mathbb{R}$ bounded and continuous such that $\frac{f}{\varphi(m)\wedge 1}$ is bounded, define $\tilde{f} : D_{\mathcal{M}(A[0,T))}[0,\infty) \to D_{\mathbb{R}}[0,\infty)$ by

$$\left[\tilde{f}(x)\right](t) = G_f\left(x(t)\right) = \langle \frac{f}{\varphi(m)\wedge 1}, x(t)\rangle,$$

then Jakubowski's criterion says that the family of probability measures $(\mathcal{L}_N)_N$ on $D_{\mathcal{M}(A[0,T))}[0,T)$ is tight if and only if

1. for all $\epsilon > 0$, for all $t < T$, there exists $K_\epsilon \subset \mathcal{M}(A[0,T))$ compact such that

$$\mathbb{P}\left((\varphi(m)\wedge 1)\tilde{\mu}_s^N \in K_\epsilon \text{ for all } 0 \le s \le t\right) > 1-\epsilon.$$

2. for each $\Psi_f \in \mathbb{F}$, the family of probability measures $\left(\mathcal{L}_N \circ \tilde{f}^{-1}\right)_N$ on $D_{\mathbb{R}}[0,\infty)$ is tight.

1. For $t < T$, note the bound,

$$\|(\varphi(m)\wedge 1)\tilde{\mu}_s^N\| \le \langle \varphi(m), \tilde{\mu}_s^N\rangle \le \Lambda.$$

Given $\epsilon > 0$, let $K_\epsilon = \{\mu \in \mathcal{M}(A[0,T)) : \|\mu\| \le \Lambda\}$. Then K_ϵ is compact in $\mathcal{M}(A[0,T))$ and

$$\mathbb{P}\left((\varphi(m)\wedge 1)\tilde{\mu}_s^N \in K_\epsilon \text{ for all } 0 \le s < T\right) = 1 > 1-\epsilon.$$

2. Assume that $|f| \le \varphi(m)\wedge 1$. For $A \in \mathcal{S}_{\mathbb{R}}$ (the σ-algebra of $D_{\mathbb{R}}[0,\infty)$),

$$\begin{aligned}
\mathcal{L}_N \circ \tilde{f}^{-1}(A) &= \mathbb{P}\left((\varphi(m)\wedge 1)\tilde{\mu}^N \in \tilde{f}^{-1}(A)\right) \\
&= \mathbb{P}\left((\varphi(m)\wedge 1)\tilde{\mu}^N \in \left\{x \in D_{\mathcal{M}(A[0,T))}[0,\infty) : \langle \frac{f}{\varphi(m)\wedge 1}, x\rangle \in A\right\}\right) \\
&= \mathbb{P}\left(\langle f, \tilde{\mu}^N\rangle \in A\right).
\end{aligned}$$

So we need to prove that the laws of the sequence $\langle f, \tilde{\mu}^N\rangle$ on $D_{\mathbb{R}}[0,\infty)$ are tight. Observe that

$$\left|\langle f, \tilde{\mu}_s^N\rangle\right| \le \Lambda$$

for all N and all $s \le t < T$, so we have compact containment. Moreover, by Doob's L^2-inequality, for all $s < t$,

$$\mathbb{E}\sup_{s\le r\le t} |M_r^{f,N} - M_s^{f,N}|^2 \le 4\mathbb{E}\int_s^t \mathcal{Q}_r^N \Phi(\tilde{\mu}_s^N)dr \le \frac{16}{N}\Lambda^2(t-s),$$

so

$$\mathbb{E}\sup_{s\le r\le t} |\langle f, \tilde{\mu}_r^N - \tilde{\mu}_s^N\rangle|^2 \le C[(t-s)^2 + N^{-1}(t-s)],$$

where $C = 16\Lambda^2$. Since the space $(\mathbb{R}, |.|)$ is complete and separable, we can apply [7], Corollary 7.4 to see that the laws of the sequence $\langle f, \tilde{\mu}^N\rangle$ on $D_{\mathbb{R}}[0,T)$ are tight. Hence we can apply Jakubowski's criterion [6] to see that the laws of the sequence $(\varphi(m)\wedge 1)\tilde{\mu}^N$ on $D_{\mathscr{M}(A[0,T))}[0,T)$ are tight. By consideration of subsequences and a theorem of Skorokhod (see, e.g, [5], Chapter IV), it suffices from this point on to consider the case where $(\varphi(m)\wedge 1)\tilde{\mu}^N$ converges almost surely in $D_{\mathscr{M}(A[0,T))}[0,\infty)$ with limit $(\varphi(m)\wedge 1)\tilde{\mu}$. Note that for all $s<T$,

$$\|\tilde{\mu}_s^N - \tilde{\mu}_{s^-}^N\| \le \frac{3}{N}$$

so $\tilde{\mu}^N \in C\left([0,\infty),\mathscr{M}\left(A[0,T)\right)\right)$. Fix $\delta>0$. The function $\varphi^\delta=\varphi 1_{(0,\delta]}$ is subadditive so

$$\langle \varphi^\delta,\mu_s^N\rangle \le \langle \varphi^\delta,\mu_0^N\rangle \le \langle \varphi^\delta,\mu_0\rangle + |\langle \varphi^\delta,\mu_0^N-\mu_0\rangle|$$

On $(0,\infty)$, $\varphi^\delta(m)=\varphi^\delta$. Also, since $\mu^N=\tilde{\mu}^N\circ m^{-1}$, we have for all $s<T$,

$$\langle \varphi(m),\tilde{\mu}_s^N\rangle = \langle \varphi,\mu_s^N\rangle.$$

Thus, for all N and all $s<T$,

$$\langle \varphi^\delta(m),\tilde{\mu}_s^N\rangle \le \langle \varphi^\delta,\mu_0\rangle + |\langle \varphi^\delta,\mu_0^N-\mu_0\rangle|.$$

Given $\epsilon>0$, we can find $\delta>0$ such that

$$\sup_N \sup_{s\le t}\langle \varphi^\delta(m),\tilde{\mu}_s^N\rangle < \epsilon.$$

Let $f: A_\tau[0,T)\to\mathbb{R}$ be bounded and continuous of bounded support. Let $g: B_\tau[0,T)\to\mathbb{R}$ such that $f=g\circ\Psi_\tau$ where Ψ_τ is defined in Section 2.2. We can write $g=g_1+g_2$ where g_1 is continuous of compact support say B and g_2 is supported on

$$B_\tau\left([0,T),(0,\delta)\right) = B_\tau[0,T)\cap\{\xi\in A_\tau[0,T):\ \text{for all } y\in\lambda(\xi), y\le\delta\}$$

with $\|g_2\|\le\|f\|$. Then $f=f_1+f_2$ where $f_1=g_1\circ\Psi_\tau$ is supported on $\Psi_\tau^{-1}(B)$ compact and $f_2=g_2\circ\Psi_\tau$ is supported on $\Psi_\tau^{-1}\left(B_\tau\left([0,T),(0,\delta)\right)\right)$. Then, for $t<T$,

$$\begin{aligned}&\limsup_{N\to\infty}\sup_{s\le t}\langle \varphi(m)f,\tilde{\mu}_s^N-\tilde{\mu}_s\rangle\\ &\le \limsup_{N\to\infty}\sup_{s\le t}\langle \varphi(m)f_1,\tilde{\mu}_s^N-\tilde{\mu}_s\rangle + 2\|f\|\sup_N\sup_{s\le t}\langle\varphi^\delta(m),\tilde{\mu}_s^N\rangle \le 2\epsilon\|f\|.\end{aligned}$$

Since f and ϵ were chosen arbitrary, this shows that

$$\sup_{s\le t}\left|\langle \varphi(m)f,\tilde{\mu}_s^N\rangle - \langle\varphi(m)f,\tilde{\mu}_s\rangle\right| \to 0 \text{ a.s.}$$

for all $\tau\in\mathbb{T}$, all $f\in C_b(A_\tau[0,T))$ with bounded support. Hence we can apply Lemma 3.4 and obtain that

$$\sup_{s\le t} d_0(\varphi(m)\tilde{\mu}_s^N,\varphi(m)\tilde{\mu}_s)\to 0 \text{ a.s.} \tag{3.7}$$

Now let us prove (b). We are going to show that for any weak limit point $\varphi\tilde{\mu}$, almost surely, $(\tilde{\mu}_t)_{t<T}$ satisfies

$$\langle f,\tilde{\mu}_t\rangle = \langle f,\mu_0\rangle + \int_0^t \langle f,\mathscr{A}(s,\tilde{\mu}_s)\rangle ds \tag{3.8}$$

for all $f : A[0,T) \to \mathbb{R}$ bounded and continuous. For convenience we will write, for $(s,\mu) \in [0,T) \times \mathscr{M}(A[0,T))$,

$$\mathscr{A}(s,\mu) = \mathscr{A}_s(\mu).$$

Let $\tau \in \mathbb{T}$ and take $f : A_\tau[0,T) \to \mathbb{R}$ to be continuous and supported on

$$\mathscr{B} = \{\xi \in A[0,T) : m(\xi) \in B, B \subset (0,\infty) \text{ compact}\}$$

Then, for $t < T$, as $N \to \infty$,

$$\mathbb{E} \sup_{0 \le r \le t} |M_r^{f,N}|^2 \le \frac{1}{N} 16 t \Lambda^2 \|f\|^2 \to 0$$

and for $s \le t$,

$$\begin{aligned}
\left| \left(\mathscr{A}_s^N - \mathscr{A}_s \right) (\tilde{\mu}_s^N)(f) \right| &= \frac{1}{2N} \left| \int_{A[0,T)} \{f((s,\{\xi,\xi\})) - 2f(\xi)\} K(m(\xi), m(\xi)) \tilde{\mu}_s^N(d\xi) \right| \\
&\le \frac{3}{2N} \|f\| \int_{\mathscr{B}} \varphi(m(\xi))^2 \tilde{\mu}_s^N(d\xi) \\
&\le \frac{3}{2N} \|f\| \|\varphi(m(\xi)) 1_{\mathscr{B}}\| \langle \varphi, \mu_0^N \rangle \to 0.
\end{aligned}$$

Hence it will suffice to show that, as $N \to \infty$,

$$\sup_{s \le t} \left| \mathscr{A}_s(\tilde{\mu}_s^N)(f) - \mathscr{A}_s(\tilde{\mu}_s)(f) \right| \to 0 \text{ a.s.} \tag{3.9}$$

Given $\delta > 0$ and $C < \infty$, we can write $K = K_1 + K_2$, where K_1 is continuous of compact support and $0 \le K_2 \le K$ and K_2 is supported on

$$\{(x,y) : x \le \delta\} \cup \{(x,y) : y \le \delta\} \cup \{(x,y) : \left|(x,y)\right| \ge C\}.$$

Then, with obvious notations,

$$\sup_{s \le t} \left| \mathscr{A}_s^1(\tilde{\mu}_s^N)(f) - \mathscr{A}_s^1(\tilde{\mu}_s)(f) \right| \to 0 \text{ a.s.}$$

Let

$$\begin{aligned}
\mathscr{K}_1 &= \{\xi = (r,\{\xi_1,\xi_2\}) \in A_\tau[0,T) : m(\xi_1) \le \delta\} \\
\mathscr{K}_2 &= \{\xi = (r,\{\xi_1,\xi_2\}) \in A_\tau[0,T) : m(\xi_2) \le \delta\} \\
\mathscr{K}_3 &= \{\xi = (r,\{\xi_1,\xi_2\}) \in A_\tau[0,T) : m(\xi_1) + m(\xi_2) \ge C\}
\end{aligned}$$

and write $\mathscr{K} = \mathscr{K}_1 \cup \mathscr{K}_2 \cup \mathscr{K}_3$. Then,

$$\left| \mathscr{A}_s^2(\tilde{\mu}_s^N)(f) - \mathscr{A}_s^2(\tilde{\mu}_s)(f) \right| \le \left| \left(\mathscr{A}_s^2(\tilde{\mu}_s^N)(f) \right) \right| + \left| \left(\mathscr{A}_s^2(\tilde{\mu}_s)(f) \right) \right|.$$

Now,

$$
\begin{aligned}
\left|\mathscr{A}_s^2(\tilde{\mu}_s^N)(f)\right| &\leq \frac{3}{2}\|f\|\int_{\mathscr{K}} K\left(m(\xi_1),m(\xi_2)\right)\tilde{\mu}_s^N(d\xi_1)\tilde{\mu}_s^N(d\xi_2) \\
&\leq \frac{3}{2}\|f\|\int_{\mathscr{K}} \varphi\left(m(\xi_1)\right)\varphi\left(m(\xi_1)\right)\tilde{\mu}_s^N(d\xi_1)\tilde{\mu}_s^N(d\xi_2) \\
&\leq \frac{3}{2}\|f\|\left\{2\langle\varphi(m),\tilde{\mu}_s^N\rangle\langle\varphi^\delta(m),\tilde{\mu}_s^N\rangle+\beta(C)\langle\varphi(m),\tilde{\mu}_s^N\rangle^2\right\} \\
&\leq \frac{3}{2}\|f\|\left\{2\langle\varphi(m),\mu_0^N\rangle\langle\varphi^\delta(m),\mu_0^N\rangle+\beta(C)\langle\varphi(m),\tilde{\mu}_0^N\rangle^2\right\}
\end{aligned}
$$

where $\beta(C)=\sup_{|(x,y)|\geq C}\varphi^{-1}(x)\varphi^{-1}(y)K(x,y)$. Similarly,

$$
\left|\mathscr{A}_s^2(\tilde{\mu}_s)(f)\right| \leq \frac{3}{2}\|f\|\left\{2\langle\varphi(m),\mu_0^N\rangle\langle\varphi^\delta(m),\mu_0^N\rangle+\beta(C)\langle\varphi(m),\tilde{\mu}_0^N\rangle^2\right\}.
$$

Given $\epsilon>0$ we can choose δ and C so that

$$
\begin{aligned}
&\langle\varphi^\delta,\mu_0^N\rangle\leq\frac{1}{3}\epsilon\Lambda^{-1} \text{ for all } N \\
&\langle\varphi^\delta,\mu_0\rangle\leq\frac{1}{3}\epsilon\Lambda^{-1}, \\
&\beta(C)\leq\frac{1}{3}\epsilon\Lambda^{-2}.
\end{aligned}
$$

Then,

$$
\left|\mathscr{A}_s^2(\tilde{\mu}_s^N)(f)\right|<\epsilon,\left|\mathscr{A}_s^2(\tilde{\mu}_s)(f)\right|<\epsilon
$$

for all N and $s\leq t<T$. Hence, for all $t<T$,

$$
\limsup_{n\to\infty}\sup_{s\leq t}\left|\mathscr{A}_s^2(\tilde{\mu}_s^N)(f)-\mathscr{A}_s^2(\tilde{\mu}_s)(f)\right|\leq 2\epsilon
$$

but ϵ was arbitrary so (3.9) is proved. Hence, we can let $N\to\infty$ in (3.5) to obtain (3.8) for all $f\in C_b\left(A_\tau[0,T)\right)$ of compact support. By using the bounds (3.6), the fact that $\langle\varphi(m),\tilde{\mu}_t\rangle\leq\Lambda$, and a straightforward limit argument, we can extend this equation to all bounded measurable functions f supported on $A_\tau[0,T)$. Moreover, for $f:A[0,T)\to\mathbb{R}$ bounded and measurable we can write

$$
f=\sum_{\tau\in\mathbb{T}}f1_{A_\tau[0,T)}
$$

and so (3.8) can be extended to all f bounded and measurable on $A[0,T)$. In particular, almost surely, $\tilde{\mu}$ is a solution of Smoluchowski's equation, in the sense of (3.1).

□

Proposition 3.4 (that we have just proved) needed the extra assumption that the kernel satisfies,

$$
\varphi^{-1}(x)\varphi^{-1}(y)K(x,y)\to 0 \text{ as } (x,y)\to\infty
$$

which makes the tightness argument easier to write and will help the reader to understand the more complicated tightness argument used in the proof of Theorem 1.1. We are now ready to prove Theorem 1.1.

3.3 A coupled family of Markov processes

3.3.1 The process

In this paragraph we follow [2][2]. Set $E = (0,\infty)$ and fix $B \subseteq E$. We are going to couple $(X_t, X_t^B, \Lambda_t^B)_{t\geq 0}$. For each $B \subseteq E$, $(X_t^B)_{t\geq 0}$ and X_t will take values in the finite integer-valued measures on E, whereas $(\Lambda_t^B)_{t\geq 0}$ will be a non-decreasing process in $[0,\infty)$. Let us suppose given $(X_0, X_0^B, \Lambda_0^B)$ such that the following inequalities hold:

$$X_0^B \leq X_0, \ \langle \varphi, X_0^B \rangle + \Lambda_0^B \geq \langle \varphi, X_0 \rangle.$$

We can write

$$X_0 = \sum_{i=1}^{m} \delta_{y_i}, \ X_0^B = \sum_{i \in I(B)} \delta_{y_i}$$

where $y_1, \cdots, y_m \in E$ and $I(B) \subseteq \{1, \cdots, m\}$. Set

$$\nu^B = \Lambda_0^B - \sum_{i \notin I(B)} \varphi(y_i).$$

For $i < j$, take independent exponential random variables T_{ij} of parameter $K(y_i, y_j)$. Set $T_{ij} = T_{ji}$. Also, for $i \neq j$, take independent random variables S_{ij} of parameter $\varphi(y_i)\varphi(y_j) - K(y_i, y_j)$. For each i let S_i^B be an exponential random variable having parameter $\varphi(y_i)\nu^B$. We construct the S_i^B such that $(S_i^B : i \in \{1, \cdots, m\})$ form a family of independent random variables. Set

$$T_i^B = \min_{j \notin I(B)} \left(T_{ij} \wedge S_{ij} \right) \wedge S_i^B.$$

Hence, T_i^B is an exponential random variable of parameter

$$\sum_{j \notin I(B)} \varphi(y_i)\varphi(y_j) + \varphi(y_i)\nu^B = \varphi(y_i)\Lambda_0^B.$$

The random variables $\left(T_{ij}, T_i^B : i, j \in I(B), i < j\right)$ form an independent family, whereas for $i \in I(B), j \notin I(B)$ we have

$$T_i^B \leq T_{ij}.$$

Now set

$$T = \left(\min_{i<j} T_{ij} \right) \wedge \left(\min_i T_i^B \right).$$

[2] James Norris has pointed out to us that the analogous construction proposed in [2] contains an error, which was noticed by Wolfgang Wagner. In order to obtain coupled Markov processes $\left(X_t^B, \Lambda_t^B\right)_{t\geq 0}$ for sets $B \subseteq (0,\infty)$ as claimed in [2,pp 100-101], the random variables S_i^B must satisfy $S_i^\emptyset = S_i^B$ with probability $\nu_B/\nu_\emptyset$ for all B and i. This is possible if one restricts to a given increasing family of sets $\left(B_n : n \in \mathbb{N}\right)$ which suffices for the use made of the coupling [2], but is not so general.

We set $(X_t, X_t^B, \Lambda_t^B) = (X_0, X_0^B, \Lambda_0^B)$ for $t < T$ and set

$$(X_T, X_T^B, \Lambda_T^B) = \begin{cases} (X_0 - \delta_{y_i} - \delta_{y_j} + \delta_{y_i+y_j}, X_0^B - \delta_{y_i} - \delta_{y_j} + \delta_{y_i+y_j}, \Lambda_0^B), \\ \qquad\qquad \text{if } T = T_{ij}, i, j \in I(B), y_i + y_j \in B, \\ \left(X_0 - \delta_{y_i} - \delta_{y_j} + \delta_{y_i+y_j}, X_0^B - \delta_{y_i} - \delta_{y_j}, \Lambda_0^B + \varphi(y_i + y_j)\right), \\ \qquad\qquad \text{if } T = T_{ij}, i, j \in I(B), y_i + y_j \notin B, \\ \left(X_0, X_0^B - \delta_{y_i}, \Lambda_0^B + \varphi(y_i)\right) \text{ if } T = T_i^B, i \in I(B), \\ (X_0, X_0^B, \Lambda_0^B) \text{ otherwise.} \end{cases}$$

It is straightforward to check that X_T^B is supported on B and,

$$X_T^B \le X_T,\ \langle \varphi, X_T^B \rangle + \Lambda_T^B \ge \langle \varphi, X_T \rangle.$$

We now repeat the above construction independently from time T again and again to obtain a family of Markov processes $\left(X_t, X_t^B, \Lambda_t^B\right)_{t\ge 0}$ such that X_t^B is supported on B and

$$X_t^B \le X_t,\ \langle \varphi, X_t^B \rangle + \Lambda_t^B \ge \langle \varphi, X_t \rangle.$$

for all t. Also observe that $\Lambda_t^E = 0$ for all t and $X_t = X_t^E$ is simply the Marcus-Lushnikov process with coagulation kernel K that we described at the beginning of the paper. For $B \subseteq E$, let $\left(X_t^N, X_t^{N,B}, \Lambda_t^{N,B}\right)_{t\ge 0}$ be the Markov process described above but with kernel K/N. Set

$$\mu_t^N = N^{-1} X_t^N\ , \mu_t^{N,B} = N^{-1} X_t^{N,B},\ \lambda_t^{N,B} = N^{-1}\Lambda_t^{N,B}.$$

Then, for all $t \ge 0$, the following inequality holds:

$$\mu_t^{N,B} \le \mu_t^N,\ \langle \varphi, \mu_t^{N,B} \rangle + \lambda_t^{N,B} \ge \langle \varphi, \mu_t^N \rangle.$$

We are now going to extend what we have written above to a coupling on the space of historical trees $A[0,T)$. Define

$$\mathscr{B} = \left\{ \tilde{B} = \{\xi \in A[0,T), m(\xi) \in B\} \text{ where } B \subset (0,\infty) \text{ is compact } \right\}.$$

For $\tilde{B} \in \mathscr{B}$ we are going to couple $\left(\tilde{X}_t, \tilde{X}_t^{\tilde{B}}, \tilde{\Lambda}_t^{\tilde{B}}\right)_{t\ge 0}$ on $\mathscr{M}\left(A[0,T), \mathbb{Z}^+\right)^2 \times [0,\infty)$. One can write

$$\tilde{B} = \{\xi \in A[0,T), m(\xi) \in B\}$$

where $B \subset (0,\infty)$ is compact. Given $(X_0, X_0^B, \Lambda_0^B)$ such that,

$$X_0^B \le X_0,\ \langle \varphi, X_0^B \rangle + \Lambda_0^B \ge \langle \varphi, X_0 \rangle,$$

consider the coupling $\left(X_t, X_t^B, \Lambda_t^B\right)_{t\ge 0}$ constructed above. The process $\left(\tilde{X}_t, \tilde{X}_t^{\tilde{B}}, \tilde{\Lambda}_t^{\tilde{B}}\right)_{t\ge 0}$ is constant on time intervals where $\left(X_t, X_t^B, \Lambda_t^B\right)_{t\ge 0}$ is constant and the jumps are defined as follows. Suppose given $(\tilde{X}_{t^-}, \tilde{X}_{t^-}^{\tilde{B}}, \tilde{\Lambda}_{t^-}^{\tilde{B}})$, then we set

1. $(\tilde{X}_t, \tilde{X}_t^{\tilde{B}}, \tilde{\Lambda}_t^{\tilde{B}}) = (\tilde{X}_{t^-} - \delta_{\xi_i} - \delta_{\xi_j} + \delta_{(t,\{\xi_i,\xi_j\})}, \tilde{X}_{t^-}^{\tilde{B}} - \delta_{\xi_i} - \delta_{\xi_j} + \delta_{(t,\{\xi_i,\xi_j\})}, \tilde{\Lambda}_{t^-}^{\tilde{B}})$

   if $(X_t, X_t^B, \Lambda_t^B) = (X_{t^-} - \delta_{y_i} - \delta_{y_j} + \delta_{y_i+y_j}, X_{t^-}^B - \delta_{y_i} - \delta_{y_j} + \delta_{y_i+y_j}, \Lambda_{t^-}^B)$,

2. $(\tilde{X}_t, \tilde{X}_t^{\tilde{B}}, \tilde{\Lambda}_t^{\tilde{B}}) = (\tilde{X}_{t^-} - \delta_{\xi_i} - \delta_{\xi_j} + \delta_{(t,\{\xi_i,\xi_j\})}, \tilde{X}_{t^-}^{\tilde{B}} - \delta_{\xi_i} - \delta_{\xi_j}, \tilde{\Lambda}_{t^-}^{\tilde{B}} + \varphi\left(m((t,\{\xi_i,\xi_j\}))\right))$

   if $(X_t, X_t^B, \Lambda_t^B) = (X_{t^-} - \delta_{y_i} - \delta_{y_j} + \delta_{y_i+y_j}, X_{t^-}^B - \delta_{y_i} - \delta_{y_j}, \Lambda_{t^-}^B + \varphi\left(y_i + y_j\right))$,

3. $(\tilde{X}_t, \tilde{X}_t^{\tilde{B}}, \tilde{\Lambda}_t^{\tilde{B}}) = (\tilde{X}_{t^-}, \tilde{X}_{t^-}^{\tilde{B}} - \delta_{\xi_i}, \tilde{\Lambda}_{t^-}^{\tilde{B}} + \varphi\left(m(\xi_i)\right))$

   if $(X_t, X_t^B, \Lambda_t^B) = (X_{t^-}, X_{t^-}^B - \delta_{y_i}, \Lambda_{t^-}^B + \varphi\left(y_i\right))$,

4. $(\tilde{X}_{t^-}, \tilde{X}_{t^-}^{\tilde{B}}, \tilde{\Lambda}_{t^-}^{\tilde{B}})$ otherwise,

where we have used the notations $y_i = m(\xi_i)$ and $m\left((T,\{\xi_i,\xi_j\})\right) = y_i + y_j$.

We then obtain a family of Markov processes $\left(\tilde{X}_t, \tilde{X}_t^{\tilde{B}}, \tilde{\Lambda}_t^{\tilde{B}}\right)_{t\geq 0}$ such that $\tilde{X}_t^{\tilde{B}}$ is supported on $\tilde{B}$ and

$$\tilde{X}_t^{\tilde{B}} \leq \tilde{X}_t,\ \langle \varphi(m), \tilde{X}_t^{\tilde{B}} \rangle + \tilde{\Lambda}_t^{\tilde{B}} \geq \langle \varphi, \tilde{X}_t \rangle$$

for all t. For $\tilde{B} \subset \mathscr{B}$, let $(\tilde{X}_t^N, \tilde{X}_t^{N,\tilde{B}}, \tilde{\Lambda}_t^{N,\tilde{B}})$ be the Markov process described above but with kernel K/N. Set,

$$\tilde{\mu}_t^N = N^{-1}\tilde{X}_t^N\ , \tilde{\mu}_t^{N,\tilde{B}} = N^{-1}\tilde{X}_t^{N,\tilde{B}},\ \tilde{\lambda}_t^{N,\tilde{B}} = N^{-1}\tilde{\Lambda}_t^{N,\tilde{B}}.$$

Then, for all $t \geq 0$, the following inequality holds:

$$\tilde{\mu}_t^{N,\tilde{B}} \leq \tilde{\mu}_t^N,\ \langle \varphi, \tilde{\mu}_t^{N,\tilde{B}} \rangle + \tilde{\lambda}_t^{N,\tilde{B}} \geq \langle \varphi, \tilde{\mu}_t^N \rangle.$$

Recall that, when $N\mu$ is an integer-valued measure on $A[0,T)$, we denote by $\mu^{(N)}$ the measure on $A[0,T) \times A[0,T)$ characterized by

$$\mu^{(N)}(A \times A') = \mu(A)\mu(A') - N^{-1}\mu(A \cap A').$$

Define

$$\tilde{S}^* = \left\{(t,\mu,\lambda) \in [0,T) \times \mathscr{M}\left(A[0,T), \mathbb{Z}^+\right) \times [0,\infty) :\ \operatorname{supp}\mu \subseteq A[0,t)\right\}$$

and for $N > 0$, let

$$\tilde{S}_N^* = \left\{(t,\mu,\lambda) : (t, N\mu, \lambda) \in \tilde{S}^*\right\}.$$

For $\tilde{B} \in \mathscr{B}$, we denote by $\mathscr{M}_{\tilde{B}}$ the space of finite signed measures supported on $\tilde{B}$. For $(t,\mu,\lambda) \in \tilde{S}_N^*$, we define $\tilde{L}^{N,\tilde{B}}(t,\mu,\lambda) \in \mathscr{M}_{\tilde{B}}$ by

$$\begin{aligned}\langle (f,a), \tilde{L}^{N,\tilde{B}}(t,\mu,\lambda) \rangle =& \frac{1}{2}\int_{A[0,T)\times A[0,T)} \left\{f\left(t,\{\xi,\xi'\}\right)\right) 1_{\{(t,\{\xi,\xi'\})\in\tilde{B}\}} + a\varphi\left(m(\xi)+m(\xi')\right) 1_{\{(t,\{\xi,\xi'\})\notin\tilde{B}\}} \\ &\quad -f(\xi) - f(\xi')\big\} K\left(m(\xi), m(\xi')\right) \mu^{(N)}(d\xi, d\xi') \\ &+ \lambda \int_{A[0,T)} \left\{a\varphi\left(m(\xi)\right) - f(\xi)\right\} \varphi\left(m(\xi)\right)\mu(d\xi).\end{aligned}$$

for all $a \in \mathbb{R}$ and all f. Observe that the expression above is obtained in taking

$$\Psi : \tilde{S}_N^* \to \mathbb{R} : (t, \mu, \lambda) \to \langle f, \mu \rangle + a\lambda$$

in the generator of the Markov process $(t, \tilde{\mu}_t^{N,\tilde{B}}, \tilde{\lambda}_t^{N,\tilde{B}})$. To ease the notation we will write

$$\langle (f, a), \tilde{L}^{N,\tilde{B}}(t, \mu, \lambda) \rangle = \tilde{L}_t^{N,\tilde{B}}(\mu, \lambda)(f, a).$$

Then,

$$M_t^{f,a,N,\tilde{B}} = \langle f, \tilde{\mu}_t^{N,\tilde{B}} \rangle + a\tilde{\lambda}_t^{N,\tilde{B}} - \langle f, \tilde{\mu}_0^{N,\tilde{B}} \rangle - a\tilde{\lambda}_0^{N,\tilde{B}} - \int_0^t \tilde{L}_s^{N,\tilde{B}} \left(\tilde{\mu}_s^{N,\tilde{B}}, \tilde{\lambda}_s^{N,\tilde{B}} \right) (f, a) ds \tag{3.10}$$

is a martingale having previsible increasing process

$$\langle M^{f,a,N,\tilde{B}} \rangle_t = \int_0^t \mathscr{V}_s^{N,\tilde{B}} \left(\tilde{\mu}_s^{N,\tilde{B}}, \tilde{\lambda}_s^{N,\tilde{B}} \right) (f, a) ds$$

where for $(t, \mu, \lambda) \in \tilde{S}_N^*$,

$$\begin{aligned} \mathscr{V}_t^{N,B}(\mu, \lambda)(f, a) = &\frac{1}{2N} \int_{A[0,T) \times A[0,T)} \left\{ f\left(t, \{\xi, \xi'\}\right) 1_{\{(t,\{\xi,\xi'\}) \in \tilde{B}\}} + a\varphi\left(m(\xi) + m(\xi')\right) 1_{\{(t,\{\xi,\xi'\}) \notin \tilde{B}\}} \right. \\ &\left. - f(\xi) - f(\xi') \right\}^2 K\left(m(\xi), m(\xi')\right) \mu^{(N)}(d\xi, d\xi') \\ &+ \frac{\lambda}{N} \int_{A[0,T)} \{a\varphi(m(\xi)) - f(\xi)\}^2 \varphi(m(\xi))^2 \mu(d\xi). \end{aligned}$$

3.3.2 Related work and assumptions

We are first reviewing the work of [2]. For $B \subset (0, \infty)$ compact, we denote by $\mathscr{M}_B$ the space of finite signed measures supported on B. For $(\mu, \lambda) \in \mathscr{M}_B \times \mathbb{R}$ we define $L^B(\mu, \lambda) \in \mathscr{M}_B$ by

$$\begin{aligned} \langle (f, a), L^B(\mu, \lambda) \rangle = &\frac{1}{2} \int_0^\infty \int_0^\infty \left\{ f(x+y) 1_{x+y \in B} - a\varphi(x+y) 1_{x+y \notin B} - f(x) - f(y) \right\} \\ &\times K(x, y) \mu(dx) \mu(dy) \\ &+ \lambda \int_0^\infty \left\{ a\varphi(x) - f(x) \right\} \varphi(x) \mu(dx) \end{aligned}$$

for all f and all a. Fix μ_0 a measure on $(0, \infty)$ such that $\langle \varphi, \mu_0 \rangle < \infty$. Set

$$\mu_0^B = 1_B \mu_0, \ \lambda_0^B = \langle \varphi 1_{B^c}, \mu_0 \rangle.$$

Then, for each $B \subset (0, \infty)$ compact, the equation

$$(\mu_t^B, \lambda_t^B) = (\mu_0^B, \lambda_0^B) + \int_0^t L^B(\mu_s^B, \lambda_s^B) ds \tag{3.11}$$

has a unique solution which is the continuous map

$$[0,\infty)\to\mathcal{M}_B^+\times\mathbb{R}^+ : t\mapsto(\mu_t^B,\lambda_t^B).$$

Moreover, the following inequality holds

$$\mu_t^B\le\mu_t,\ \langle\varphi,\mu_t^B\rangle+\lambda_t^B\ge\langle\varphi,\mu_t\rangle$$

where

$$\mu_t=\lim_{B\uparrow E}\mu_t^B,\ \lambda_t=\lim_{B\uparrow E}\lambda_t^B.$$

Then, $\langle\varphi,\mu_t\rangle\le\langle\varphi,\mu_0\rangle<\infty$. Moreover, assume that $\langle\varphi^2,\mu_0\rangle<\infty$. Then, $(\mu_t)_{t<T}$ is the solution to (1.2). We are now going to introduce a similar equation to (3.11) for sets $\tilde{B}\in\mathcal{B}$. Take $\tilde{B}\in\mathcal{B}$. One can write

$$\tilde{B}=\{\xi\in A[0,T):m(\xi)\in B\}$$

where $B\subset(0,\infty)$ is compact. Let $\mathcal{M}_{\tilde{B}}$ be the space of finite signed measures supported on $\tilde{B}$. For $(t,\mu,\lambda)\in[0,T)\times\mathcal{M}_{\tilde{B}}\times[0,\infty)$, we define $\tilde{L}^{\tilde{B}}(t,\mu,\lambda)\in\mathcal{M}_{\tilde{B}}$ by

$$\begin{aligned}\langle(f,a),\tilde{L}^{\tilde{B}}(t,\mu,\lambda)\rangle=&\frac{1}{2}\int_{A[0,T)}\int_{A[0,T)}\left\{f\left((t,\{\xi,\xi'\})\right)1_{(t,\{\xi,\xi'\})\in\tilde{B}}-a\varphi\left((t,\{\xi,\xi'\})\right)1_{(t,\{\xi,\xi'\})\notin\tilde{B}}\right.\\&\left.-f(\xi)-f(\xi')\right\}K\left(m(\xi),m(\xi')\right)\mu(d\xi)\mu(d\xi')\\&+\lambda\int_{A[0,T)}\left\{a\varphi\left(m(\xi)\right)-f(\xi)\right\}\varphi\left(m(\xi)\right)\mu(d\xi)\end{aligned}$$

for all f and all a. For each $\tilde{B}\in\mathcal{B}$ consider the equation

$$(\tilde{\mu}_t^{\tilde{B}},\tilde{\lambda}_t^{\tilde{B}})=(\tilde{\mu}_0^{\tilde{B}},\tilde{\lambda}_0^{\tilde{B}})+\int_0^t\tilde{L}^{\tilde{B}}(s,\tilde{\mu}_s^{\tilde{B}},\tilde{\lambda}_s^{\tilde{B}})ds. \tag{3.12}$$

Then, $(\mu_t^B,\lambda_t^B)=(\tilde{\mu}_t^{\tilde{B}}\circ m^{-1},\tilde{\lambda}_t^{\tilde{B}})$ solves (3.11) and so by uniqueness is equal to the unique solution (3.11). One can prove that for each $\tilde{B}\in\mathcal{B}$, (3.12) has a unique solution using the same argument as when we prove that (3.1) had a unique solution. Define,

$$\tilde{\mu}_t=\lim_{\tilde{B}\uparrow A[0,T)}\tilde{\mu}_t^{\tilde{B}},\ \lambda_t=\lim_{\tilde{B}\uparrow A[0,T)}\lambda_t^{\tilde{B}}.$$

It is shown in [2] that $\lambda_t=0$ and $\tilde{\mu}_t$ is the solution to (3.1).

3.3.3 Proof of Theorem 1.1

We first prove Theorem 1.1 in the case where $supp\mu_0\subseteq\mathbb{N}$ as the proof in the discrete case does not require the complicated tightness arguments that we need in the continuous case.

a) Case where $supp\mu_0\subseteq\mathbb{N}$

We have the following proposition.

Proposition 3.6. *Assume that (1.1) and (1.3) are satisfied. Suppose that μ_0 is supported on $\mathbb{N}$. Let*

$$\tilde{B} = \{\xi \in A[0,T) : m(\xi) \in B\} \in \mathscr{B},$$

such that $B \subset \mathbb{N}$ finite. Then, there exists a constant $C = C(K, \varphi, \mu_0, B) < \infty$ such that for all $N \geq 1$, for all $t < T$, for all $0 \leq \delta \leq t$, if $\delta_0 = \|\tilde{\mu}_0^{N,\tilde{B}} - \tilde{\mu}_0^{\tilde{B}}\| + |\tilde{\lambda}_0^{N,\tilde{B}} - \tilde{\lambda}_0^{B}| \leq 1$, then

$$\mathbb{P}\left(\sup_{s\leq t}\left\{\|\tilde{\mu}_s^{N,\tilde{B}} - \tilde{\mu}_s^{\tilde{B}}\| + |\tilde{\lambda}_s^{N,\tilde{B}} - \tilde{\lambda}_s^{\tilde{B}}|\right\} > (\delta_0 + \delta)e^{Ct}\right) \leq Ce^{-\frac{N\delta^2}{Ct}}.$$

Proof of Proposition 3.6: : Let $\mu_t^B = \tilde{\mu}_t^{N,\tilde{B}} \circ m^{-1}$. Since,

$$\|\tilde{\mu}_0^{N,\tilde{B}} - \tilde{\mu}_0^{\tilde{B}}\| = \|\mu_0^{N,B} - \mu_0^B\|,$$

then

$$\|\mu_0^{N,B} - \mu_0^B\| + |\lambda_0^{N,B} - \lambda_0^B| \leq 1,$$

so we can apply [2], Proposition 4.3 to get that for all $0 \leq \delta \leq t$,

$$\mathbb{P}\left(\sup_{s\leq t}\left\{\|\mu_s^{N,B} - \mu_s^B\| + |\lambda_s^{N,B} - \lambda_s^B|\right\} > (\delta_0 + \delta)e^{Ct}\right) \leq Ce^{-\frac{N\delta^2}{Ct}}.$$

Hence, since $\|\tilde{\mu}_t^{N,\tilde{B}} - \tilde{\mu}_t^{\tilde{B}}\| = \|\mu_t^{N,B} - \mu_t^B\|$ we get the formula of the proposition, as required.

□

Let us now prove our main result that is Theorem 1.1.

Proof of Theorem 1.1: Fix $\delta > 0$ and $t < T$. We can find $B \subset (0,\infty)$ compact such that $\lambda_t^B < \delta/2$. Now,

$$d(\varphi(m)\tilde{\mu}_0^N, \varphi(m)\tilde{\mu}_0) \to 0,$$

so

$$d(\tilde{\mu}_0^{N,\tilde{B}}, \tilde{\mu}_0^{\tilde{B}}) \to 0,\ |\tilde{\lambda}_0^{N,\tilde{B}} - \lambda_0^{\tilde{B}}| \to 0$$

then by Proposition 3.6,

$$\sup_{s\leq t} d(\tilde{\mu}_s^{N,\tilde{B}}, \tilde{\mu}_s^{\tilde{B}}) \to 0,\ \sup_{s\leq t}|\tilde{\lambda}_s^{N,\tilde{B}} - \tilde{\lambda}_s^{\tilde{B}}| \to 0$$

in probability as $N \to \infty$. Thus

$$\sup_{s\leq t} d(\varphi(m)\tilde{\mu}_s^{N,\tilde{B}}, \varphi(m)\tilde{\mu}_s^{\tilde{B}}) \to 0$$

in probability as $N \to \infty$. For $s \leq t$,

$$\|\varphi(m)(\tilde{\mu}_s - \tilde{\mu}_s^{\tilde{B}})\| = \|\varphi(\mu_s - \mu_s^B)\| \leq \lambda_s^B \leq \lambda_t^B < \delta/2$$

and

$$\begin{aligned}\|\varphi(m)(\tilde{\mu}_s^N-\tilde{\mu}_s^{N,\tilde{B}})\| = \|\varphi(\mu_s^N-\mu_s^{N,B})\| &\le \lambda_s^{N,B} \le \lambda_t^{N,B}\\ &\le \lambda_t^B + |\lambda_t^{N,B}-\lambda_t^B|\\ &\le \frac{\delta}{2} + |\tilde{\lambda}_t^{N,\tilde{B}}-\tilde{\lambda}_t^{\tilde{B}}|.\end{aligned}$$

Now,

$$\begin{aligned}d(\varphi(m)\tilde{\mu}_s^{N,\tilde{B}},\varphi(m)\tilde{\mu}_s) &\le \|\varphi(m)(\tilde{\mu}_s^N-\tilde{\mu}_s^{N,\tilde{B}})\| + d(\varphi(m)\tilde{\mu}_s^{N,\tilde{B}},\varphi(m)\tilde{\mu}_s^{\tilde{B}}) + \|\varphi(m)(\tilde{\mu}_s-\tilde{\mu}_s^{\tilde{B}})\|\\ &\le \delta + d(\varphi(m)\tilde{\mu}_s^{N,\tilde{B}},\varphi(m)\tilde{\mu}_s^{\tilde{B}}) + |\tilde{\lambda}_t^{N,\tilde{B}}-\tilde{\lambda}_t^{\tilde{B}}|,\end{aligned}$$

so

$$\mathbb{P}\left(\sup_{s\le t}\left|d(\varphi(m)\tilde{\mu}_s^{N,\tilde{B}_n},\varphi(m)\tilde{\mu}_s)\right| > \delta\right) \to 0$$

as required.

□

b) General case

For the next proposition, we will assume that for each

$$\tilde{B} = \{\xi \in A[0,T) : m(\xi)\in B\} \in \mathscr{B},$$

the compact set B satisfies

$$\mu_0^{*N}(\partial B) = 0 \text{ for all } N \ge 0. \tag{3.13}$$

Proposition 3.7. *Assume that conditions (1.1),(1.3),(1.5) and (3.13) are satisfied. Take*

$$\tilde{B} = \{\xi \in A[0,T) : m(\xi)\in B\} \in \mathscr{B},$$

such that B satisfies (3.13). Suppose that

$$d(\tilde{\mu}_0^{N,\tilde{B}},\tilde{\mu}_0^{\tilde{B}}) \to 0,\ |\tilde{\lambda}_0^{N,\tilde{B}}-\tilde{\lambda}_0^{\tilde{B}}| \to 0$$

as $N\to\infty$. Then, for all $t<T$,

$$\sup_{s\le t} d(\tilde{\mu}_s^{N,\tilde{B}},\tilde{\mu}_s^{\tilde{B}}) \to 0,\ \sup_{s\le t}|\tilde{\lambda}_s^{N,\tilde{B}}-\tilde{\lambda}_s^{\tilde{B}}| \to 0$$

in probability as $N\to\infty$.

Proof of Proposition 3.6: Let $\Lambda = \sup_N\langle\varphi(m),\mu_0^N\rangle$, and note that $\Lambda<\infty$. We can find $C = C(B,\Lambda,\varphi)<\infty$, such that

$$\begin{aligned}&\left|\tilde{L}^{\tilde{B}}(t,\tilde{\mu}_t^{N,\tilde{B}},\tilde{\lambda}_t^{N,\tilde{B}})(f,a)\right| \le C(\|f\|+|a|),\\ &\left|(\tilde{L}^{\tilde{B}}-\tilde{L}^{N,\tilde{B}})(t,\tilde{\mu}_t^{N,\tilde{B}},\tilde{\lambda}_t^{N,\tilde{B}})(f,a)\right| \le \frac{C}{N}(\|f\|+|a|),\\ &\left|\tilde{v}_t^{N,\tilde{B}}(\tilde{\mu}_t^{N,\tilde{B}},\tilde{\lambda}_t^{N,\tilde{B}})(f,a)\right| \le C(\|f\|+|a|)^2.\end{aligned}$$

Hence by the same argument than in Proposition 3.4, the laws of the sequence $(\tilde{\mu}^{N,\tilde{B}}, \tilde{\lambda}^{N,\tilde{B}})$ are tight on $D\left([0,\infty), \mathcal{M}_{\tilde{B}} \times \mathbb{R}\right)$. Similarly, the laws of the sequence $(\tilde{\mu}^{N,\tilde{B}}, \tilde{\lambda}^{N,\tilde{B}}, I^N, J^N)$ are tight on $D\left([0,\infty), \mathcal{M}_{\tilde{B}} \times \mathbb{R} \times \mathcal{M}_{\tilde{B}\times\tilde{B}} \times \mathcal{M}_{\tilde{B}\times\tilde{B}}\right)$, where

$$
\begin{aligned}
I_t^N(d\xi, d\xi') &= K\left(m(\xi), m(\xi')\right) 1_{(t,\{\xi,\xi'\})\in\tilde{B}} \tilde{\mu}_t^{N,\tilde{B}}(d\xi)\tilde{\mu}_t^{N,\tilde{B}}(d\xi') \\
J_t^N(d\xi, d\xi') &= K\left(m(\xi), m(\xi')\right) 1_{(t,\{\xi,\xi'\})\notin\tilde{B}} \tilde{\mu}_t^{N,\tilde{B}}(d\xi)\tilde{\mu}_t^{N,\tilde{B}}(d\xi').
\end{aligned}
$$

We denote by (X, λ, I, J) some weak limit point of this sequence, which by passing to a subsequence and the usual argument of Skorokhod can be regarded as a pointwise limit on $D\left([0,\infty), \mathcal{M}_{\tilde{B}} \times \mathbb{R} \times \mathcal{M}_{\tilde{B}\times\tilde{B}} \times \mathcal{M}_{\tilde{B}\times\tilde{B}}\right)$. Then, there exist measurable functions,

$$
I, J : \Omega \times [0,\infty) \times \tilde{B} \times \tilde{B} \to [0,\infty)
$$

symmetric on $\tilde{B} \times \tilde{B}$, such that

$$
\begin{aligned}
I_t(d\xi, d\xi') &= I(t,\xi,\xi')\tilde{\mu}_t(d\xi)\tilde{\mu}_t(d\xi') \\
J_t(d\xi, d\xi') &= J(t,\xi,\xi')\tilde{\mu}_t(d\xi)\tilde{\mu}_t(d\xi')
\end{aligned}
$$

in $\mathcal{M}_{\tilde{B}\times\tilde{B}}$ and such that

$$
\begin{aligned}
I(t,\xi,\xi') &= K\left(m(\xi), m(\xi')\right) 1_{(t,\{\xi,\xi'\})\in\tilde{B}} \\
J(t,\xi,\xi') &= K\left(m(\xi), m(\xi')\right) 1_{(t,\{\xi,\xi'\})\notin\tilde{B}}
\end{aligned}
$$

whenever $\left(m(\xi), m(\xi')\right) \notin S(K)$ and $(t, \{\xi,\xi'\}) \notin \partial\tilde{B}$. Moreover, we can pass to the limit in (3.10) to obtain, for all $f : A[0,T) \to \mathbb{R}$ continuous, for all $a \in \mathbb{R}$ and all $t < T$, almost surely,

$$
\begin{aligned}
\langle (f,a), (t, \tilde{\mu}_t, \tilde{\lambda}_t)\rangle = \langle (f,a), (\tilde{\mu}_0, \tilde{\lambda}_0)\rangle &+ \frac{1}{2}\int_0^t \int_{A[0,T)} \int_{A[0,T)} \left\{f\left((s,\{\xi,\xi'\})\right) - f(\xi) - f(\xi')\right\} \\
&\qquad \times I(s,\xi,\xi')\tilde{\mu}_s(d\xi)\tilde{\mu}_s(d\xi')ds \\
&+ \frac{1}{2}\int_0^t \int_{A[0,T)} \int_{A[0,T)} \left\{a\varphi\left(m(\xi) + m(\xi')\right) - f(\xi) - f(\xi')\right\} \\
&\qquad \times J(s,\xi,\xi')\tilde{\mu}_s(d\xi)\tilde{\mu}_s(d\xi')ds \\
&+ \int_0^t \tilde{\Lambda}_s \int_{A[0,T)} \left\{a\varphi\left(m(\xi)\right) - f(\xi)\right\} \varphi\left(m(\xi)\right) \tilde{\mu}_s(d\xi)ds.
\end{aligned}
\tag{3.14}
$$

Observe that, in the iteration scheme

$$
\mu_t^0 = \mu_0,\ \mu_t^{N+1} << \mu_0 + \int_0^t (\mu_s^N + \mu_s^N * \mu_s^N)ds
$$

for all $N \geq 0$, so by induction we have

$$
\mu_t^N << \gamma_0 = \sum_{k=1}^{\infty} \mu_0^{*k}
$$

where μ_0^{*k} is the k-fold convolution of μ_0. For $\tilde{B} \in \mathscr{B}$, if $(t, \tilde{\mu}_t^{\tilde{B}}, \tilde{\lambda}_t^{\tilde{B}})$ is the unique solution to (3.12), then $\tilde{\mu}_t^{\tilde{B}} << \gamma_0$. Hence, the equation above forces $\tilde{\mu}_t \bigotimes \tilde{\mu}_t$ to be absolutely continuous with respect to

$$\left(\sum_{k=1}^{\infty}(\mu_0^{*k})\right)^{\bigotimes 2}$$

for all $t < T$ almost surely. Hence by the assumptions (1.5) and (3.13), we can replace $I(t,\xi,\xi')$ by $K\left(m(\xi), m(\xi')\right) 1_{(t,\{\xi,\xi'\})\in\tilde{B}}$ and $J(t,\xi,\xi')$ by $K\left(m(\xi), m(\xi')\right) 1_{(t,\{\xi,\xi'\})\notin\tilde{B}}$ in (3.14). But this is now (3.12) which has a unique solution $(t, \tilde{\mu}_t^{\tilde{B}}, \tilde{\lambda}_t^{\tilde{B}})_{t<T}$. The proposition is proved.

□

The proof of Theorem 1.1 from Proposition 3.7 is the same than in the discrete case a) except that we choose B so that it satisfies the condition (3.13).

Remark A different proof of Theorem 1.1 can be found (http://arxiv.org/abs/0907.5305). This alternative proof is combinatorically more complex but may offer different insights, as it handle the tree-valued process more directly.

3.4 Some simulations

Here are some simulations using Visual Basic of the Marcus-Lushnikov process on trees. The graphics below represent trees that have been simulated following the Marcus-Lushnikov process on trees with different kernel K and an initial number of particles N. In these simulations, all the initial particles have mass 1. These pictures show for each kernel the sort of limit-trees we can expect to find in the limit measure. Note that we have rearranged the positions of the particles in the trees below in order to reduce the number of crossing between the different branches of the tree (which would have made the picture difficult to read).

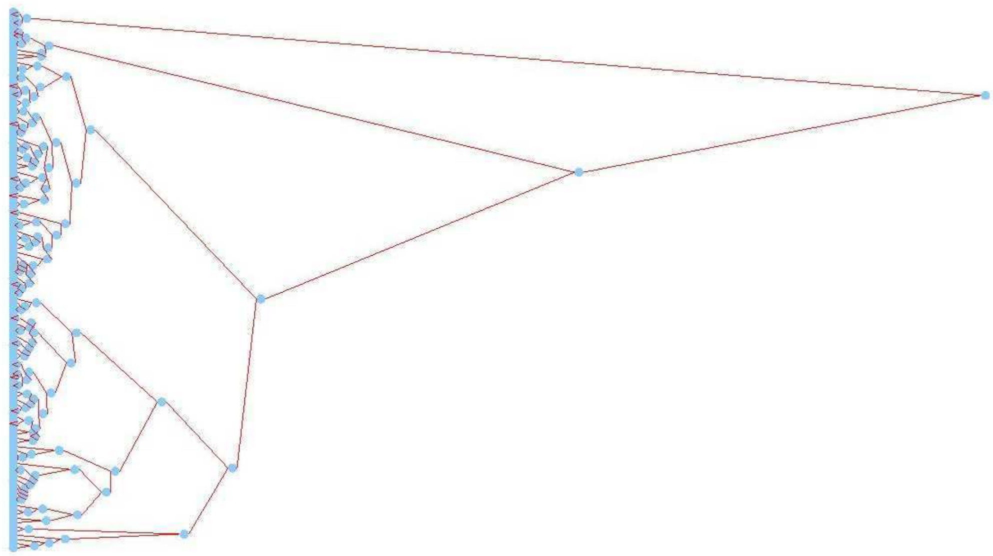

Figure 1: $N = 128$, $K(x, y) = 1$. We can see on this picture that there is no correlation between the masses of the particles, the times at which they coagulate, and which particles they coagulate with so the kernel must be independent on the masses of the particles, that is be constant.

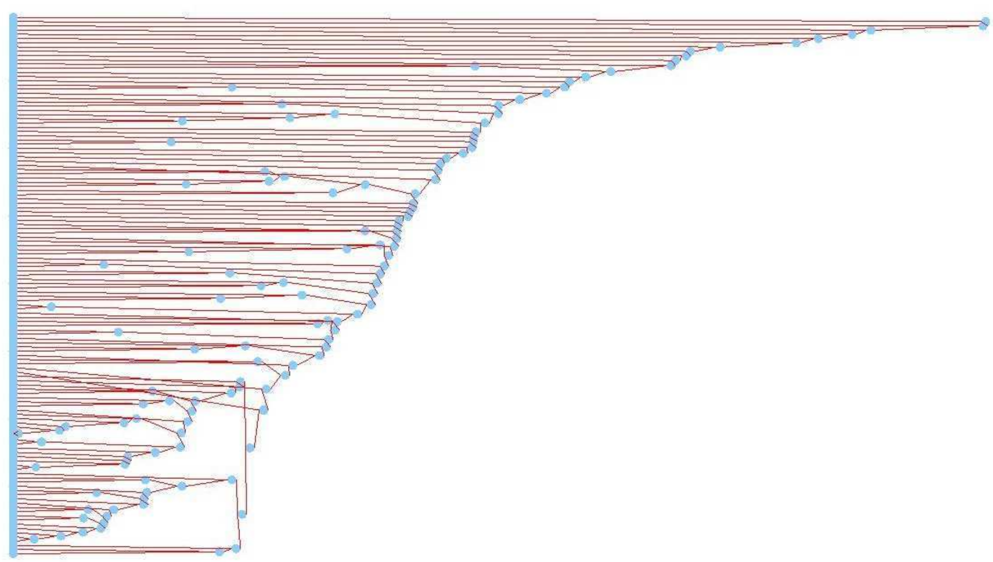

Figure 2: $N = 128$, $K(x, y) = xy$. The time spacing between coagulations decreases as the number of coagulations increases, meaning that coagulations happen quicker. This is because for $T \backsim Exp\left(K(x, y)\right)$, $\mathbb{E}(T) = 1/K(x, y) = 1/xy$ which is small when x, y are big. Also, the small particles tend to coagulate with each other because $K(x, y)$ is small when x, y are small.

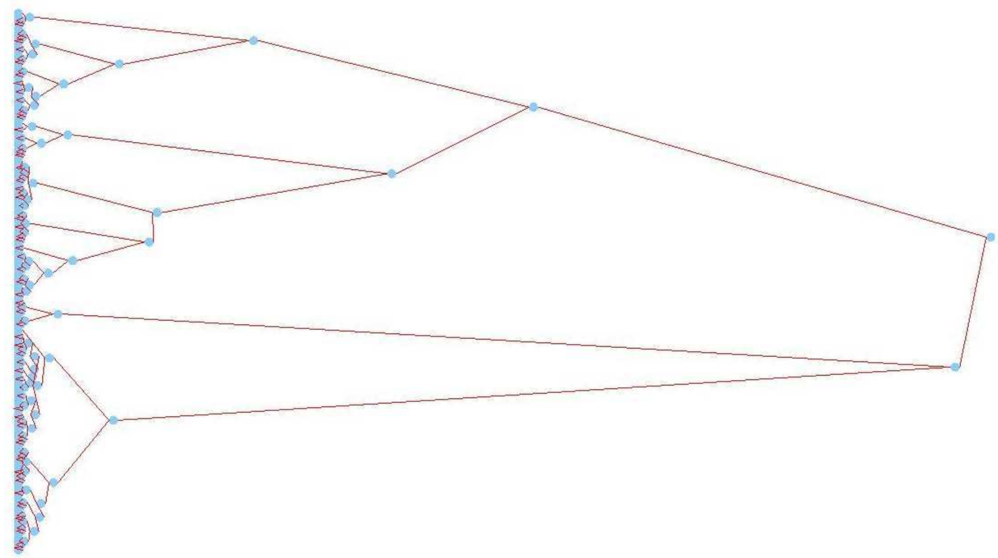

Figure 3: $N = 128$, $K(x, y) = 1/(x + y + 1)$. The time spacing between coagulations increases as the number of coagulations increases, meaning that coagulations happen slower. This is because for $T \backsim Exp\left(K(x, y)\right)$, $\mathbb{E}(T) = 1/K(x, y) = x + y + 1$ which is big when x, y are big. Also, the biggest particles tend to coagulate with each other because $K(x, y)$ is small when x, y are big. Thus after a certain number of coagulations we will end up with one big particle and some small particles which will coagulate with the big particle until no particles are left in the system.

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