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Continuity of the asymptotic shape of the supercritical contact process

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Abstract

We prove the continuity of the shape governing the asymptotic growth of the supercritical contact process in $\mathbb{Z}^d$, with respect to the infection parameter. The proof is valid in any dimension $d \geq 1$.

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1 Introduction

The contact process is a famous interacting particle system modelling the spread of an infection on the sites of $\mathbb{Z}^d$. The evolution in time depends on a fixed parameter $\lambda \in (0, +\infty)$ and is as follows: at each moment, an infected site becomes healthy at rate 1 while a healthy site becomes infected at a rate equal to λ times the number of its infected neighbors. There exists a critical value $\lambda_c(\mathbb{Z}^d) \in (0, +\infty)$ such that the infection, starting from the origin, infinitely expands with positive probability if and only if $\lambda > \lambda_c(\mathbb{Z}^d)$. See for instance Liggett's book [15] for a review on the contact process.

Durrett and Griffeath [5] proved that when the contact process on $\mathbb{Z}^d$ starting from the origin survives, the set of sites occupied before time t satisfies an asymptotic shape theorem, as in first-passage percolation. In [8], two of us extended this result to the case of the contact process in a random environment. The shape theorem can be stated as follows: provided that $\lambda > \lambda_c(\mathbb{Z}^d)$, there exists a norm μ_λ on $\mathbb{R}^d$ such that the set H_t of points already infected before time t satisfies:

$$\overline{\mathbb{P}}_\lambda \left(\exists T > 0 : \quad t \geq T \implies (1-\varepsilon)t\mathcal{S}(\lambda) \subset \tilde{H}_t \subset (1+\varepsilon)t\mathcal{S}(\lambda) \right) = 1,$$

where $\tilde{H}_t = \{z + u : \ (z, u) \in H_t \times [0,1]^d\}$, $\mathcal{S}(\lambda)$ is the unit ball for μ_λ and $\overline{\mathbb{P}}_\lambda$ is the law of the contact process with parameter λ, starting from the origin and conditioned to survive. The growth of the contact process is thus asymptotically linear in time, and governed by the shape $\mathcal{S}(\lambda)$.

The aim of this note is to prove the continuity of the map $\lambda \mapsto \mathcal{S}(\lambda)$. More precisely, we prove the following result: denote by $\mathbb{S}^{d-1}$ is the unit sphere for $\|.\|_1$ on $\mathbb{R}^d$, then

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Theorem 1.1. *For every $\lambda > \lambda_c(\mathbb{Z}^d)$,* $\lim_{\lambda' \to \lambda} \sup_{x \in \mathbb{S}^{d-1}} |\mu_{\lambda'}(x) - \mu_\lambda(x)| = 0.$

It is then easy to deduce the following continuity for the asymptotic shape. Denote by d_H the Hausdorff distance between non-empty compact sets in $\mathbb{R}^d$. For every $\lambda > \lambda_c(\mathbb{Z}^d)$,

$$\lim_{\lambda' \to \lambda} d_H(\mathcal{S}(\lambda'), \mathcal{S}(\lambda)) = 0.$$

Continuity properties for asymptotic shapes in random growth models have already been investigated. In first passage percolation, perhaps the most famous random growth model, Cox and Kesten [2, 3, 13] proved that the time constant is continuous with respect to the distribution of the passage-time of an edge. In a forthcoming paper, Garet, Marchand, Procaccia and Théret [10] extend their result to the case of possibly infinite passage times by renormalization techniques. In these two cases, thanks to a good subadditivity property, the quantity whose continuity is studied appears as an infimum of a decreasing sequence of continuous functions, which gives quite easily one half of the continuity.

Because of the possibility of extinction of the contact process, the subadditivity properties are not so obvious and we thus use the essential hitting time presented in Garet–Marchand [8]. Note that the one-dimensional case is simpler because the growth of the supercritical contact process in dimension 1 is characterized by the right-edge velocity: its continuity is proved in Liggett [14], Theorem 3.36. See also Durrett [4] for an analogous result about 2D oriented percolation.

In Section 2, we introduce the notation, build contact processes with distinct infection parameters on the same space thanks to the Harris construction and recall the definition and properties of the essential hitting time introduced in [8]. Section 3 is devoted to the proof of the left-continuity, while in Section 4 we prove the right-continuity.

2 Notation and known results

We work on the grid $\mathbb{Z}^d$, with $d \geq 1$, and we put an edge between any pair of sites at distance 1 for $\|\cdot\|_1$. We denote by $\mathbb{E}^d$ the set of these edges. To define the contact process, we use the Harris construction [12]. It allows to couple contact processes starting from distinct initial configurations and distinct parameters $\lambda \in (0, \lambda_{\max}]$, where $\lambda_{\max} > 0$ is fixed and finite, by building them from a single collection of Poisson measures on $\mathbb{R}_+$.

2.1 Construction of the Poisson measures

As the continuity is a local property, it will be sufficient in the sequel to build a coupling for contact processes with parameters in $(0, \lambda_{\max}]$, for a fixed and well chosen $\lambda_{\max} > 0$. Roughly speaking, to couple contact processes of parameter λ and λ' in $(0, \lambda_{\max}]$,

- we associate to each vertex of $\mathbb{Z}^d$ an independent Poisson point process of parameter 1, that corresponds to the recovery process for both contact processes;
- we associate to each edge of $\mathbb{E}^d$ an independent Poisson point process of parameter $\lambda_{\max}$;
- we associate to each atom of these Poisson point process of parameter $\lambda_{\max}$ a variable U that is uniform on $[0, \lambda_{\max}]$: this atom is part of the infection process for the contact process of parameter λ (respectively λ') if and only if $U \leq \lambda/\lambda_{\max}$ (respectively $U \leq \lambda'/\lambda_{\max}$).

Let us do this more formally. We endow $\mathbb{R}_+$ with the Borel σ-algebra $\mathcal{B}(\mathbb{R}_+)$, and we denote by M the set of locally finite counting measures $m = \sum_{i=0}^{+\infty} \delta_{t_i}$. We endow this set with the σ-algebra $\mathcal{M}$ generated by the maps $m \mapsto m(B)$, where B is a Borel set in $\mathbb{R}_+$.

We define the measurable space $(\Omega, \mathcal{F})$ by setting

$$\Omega = M^{\mathbb{E}^d} \times M^{\mathbb{Z}^d} \times ([0,\lambda_{\max}]^{\mathbb{N}})^{\mathbb{E}^d} \text{ and } \mathcal{F} = \mathcal{M}^{\otimes \mathbb{E}^d} \otimes \mathcal{M}^{\otimes \mathbb{Z}^d} \otimes ([0,\lambda_{\max}]^{\otimes \mathbb{N}})^{\otimes \mathbb{E}^d}.$$

On this space, we consider the probability measure defined by

$$\mathbb{P} = \mathcal{P}_{\lambda_{\max}}^{\otimes \mathbb{E}^d} \otimes \mathcal{P}_1^{\otimes \mathbb{Z}^d} \otimes (U([0,\lambda_{\max}])^{\otimes \mathbb{N}})^{\otimes \mathbb{E}^d},$$

where, for every $\lambda \in \mathbb{R}_+$, $\mathcal{P}_\lambda$ is the law of a Poisson point process on $\mathbb{R}_+$ with intensity λ and $U([a,b])$ is the uniform law on the compact set $[a,b]$.

Fix an edge e and consider $\omega_e \in M$. Denoting by $(S_i^e)_{i\geq 1}$ the atoms of ω_e, we build the classical coupling between the Poisson measures of the infection processes with different parameters $\lambda \in (0,\lambda_{\max}]$. Define

$$m_\lambda^e = m_\lambda(\omega_e, (U_i^e)_{i\geq 1}) = \sum_{i=1}^{+\infty} \mathbb{1}_{\{U_i^e \leq \frac{\lambda}{\lambda_{\max}}\}} \delta_{S_i^e}.$$

Under $\mathbb{P}$, the random variable m_λ is a Poisson point process with parameter λ. We then define, for $\lambda \leq \lambda_{\max}$, the application

$$\begin{array}{rcl} \Psi_\lambda : \Omega & \longrightarrow & M^{\mathbb{E}^d} \times M^{\mathbb{Z}^d} \\ ((\omega_e)_{e\in\mathbb{E}^d}, (\omega_z)_{z\in\mathbb{Z}^d}, (U_e^i)_{e\in\mathbb{E}^d, i\geq 1}) & \longmapsto & ((m_\lambda(\omega_e, (U_i^e)_{i\geq 1}))_{e\in\mathbb{E}^d}, (\omega_z)_{z\in\mathbb{Z}^d}). \end{array}$$

The law of Ψ_λ under $\mathbb{P}$ is then

$$\mathbb{P}_\lambda = \mathcal{P}_\lambda^{\otimes \mathbb{E}^d} \otimes \mathcal{P}_1^{\otimes \mathbb{Z}^d}.$$

We thus recover infection processes, indexed by $\mathbb{E}^d$, with parameter λ and recovering processes, indexed by $\mathbb{Z}^d$, with parameter 1. Note that the Poisson measures for recoverings, $(\omega_z)_{z\in\mathbb{Z}^d}$, do not depend on λ. The following lemma will be useful to compare the evolution of two contact processes with different parameters.

Lemma 2.1. *Let $t > 0$ and let S be a finite subset of $\mathbb{E}^d$. Assume $0 < \lambda' \leq \lambda \leq \lambda_{\max}$ and define*

$$\mathrm{Idem}(S,t,\lambda,\lambda') = \bigcap_{e\in S} \left\{ m_\lambda^e|_{[0,t]} = m_{\lambda'}^e|_{[0,t]} \right\}.$$

For each $\varepsilon > 0$, there exists $\delta = \delta(S,t,\varepsilon) > 0$ such that

$$\forall \lambda, \lambda' \in (0,\lambda_{\max}] \quad |\lambda' - \lambda| \leq \delta \ \Rightarrow\ \mathbb{P}(\mathrm{Idem}(S,t,\lambda,\lambda')) \geq 1 - \varepsilon.$$

Proof. Let $\lambda, \lambda' \in (0,\lambda_{\max}]$, and assume without loss of generality that $\lambda \leq \lambda'$.

For each $e \in \mathbb{E}^d$ and $t > 0$, set

$$D_t^e = \sum_{i=1}^{+\infty} \mathbb{1}_{\{\frac{\lambda}{\lambda_{\max}} < U_i^e \leq \frac{\lambda'}{\lambda_{\max}}\}} \mathbb{1}_{\{S_i^e \leq t\}},$$

$$\text{then } \mathbb{E}(D_t^e) = \frac{\lambda' - \lambda}{\lambda_{\max}} \mathbb{E}(\omega_e([0,t])) = \frac{\lambda' - \lambda}{\lambda_{\max}} \lambda_{\max} t = (\lambda' - \lambda)t.$$

Now,

$$\begin{aligned} \mathbb{P}(\mathrm{Idem}(S,t,\lambda,\lambda')^c) &\leq \sum_{e\in S} \mathbb{P}(m_\lambda^e|_{[0,t]} \neq m_{\lambda'}^e|_{[0,t]}) \\ &\leq \sum_{e\in S} \mathbb{P}(D_t^e \geq 1) \leq \sum_{e\in S} \mathbb{E}(D_t^e) \leq |S| t (\lambda' - \lambda), \end{aligned}$$

so we can take $\delta = 1/(t|S|\varepsilon)$. □

2.2 Graphical construction of the contact process

This construction is exposed in all details in Harris [12]; we just give here an informal description. Suppose that $\lambda \in (0, \lambda_{\max}]$ is fixed. Let $\omega = ((\omega_e)_{e\in\mathbb{E}^d}, (\omega_z)_{z\in\mathbb{Z}^d}, (U_e^i)_{e\in\mathbb{E}^d, i\geq 1}) \in \Omega$. Above each site $z \in \mathbb{Z}^d$, we draw a time line $\mathbb{R}_+$, and we put a cross at the times given by ω_z, corresponding to potential recoverings at site z. Above each edge $e \in \mathbb{E}^d$, we draw at the times given by $m_\lambda((\omega_e)_{e\in\mathbb{E}^d}, (U_e^i)_{e\in\mathbb{E}^d, i\geq 1})$ an horizontal segment between the extremities of the edge, corresponding to a potential infection through edge e (remember we fix the infection rate λ).

An open path is a connected oriented path which moves along the time line in the increasing time direction without passing a cross symbol, and along the horizontal segments corresponding to potential infections. In this description, the evolution of the contact process looks like a percolation process, oriented in time but not in space. For $x, y \in \mathbb{Z}^d$ and $t \geq 0$, we say that $y \in \xi_t^{\lambda,x}$ if and only if there exists an open path from $(x, 0)$ to (y, t), then we define:

$$\forall A \in \mathcal{P}(\mathbb{Z}^d) \quad \xi_t^{\lambda,A} \quad = \quad \bigcup_{x\in A} \xi_t^{\lambda,x}.$$

For instance, we obtain

$$(A \subset B, \quad \lambda' \leq \lambda) \quad \Rightarrow \quad (\forall t \geq 0 \quad \xi_t^{\lambda',A} \subset \xi_t^{\lambda,B}).$$

Harris proved that under $\mathbb{P}$, or under $\mathbb{P}_\lambda$, the process $(\xi_t^{\lambda,A})_{t\geq 0}$ is the contact process with infection rate λ, starting from initial configuration A.

2.3 Translations

For $t \geq 0$, we define the translation operator θ_t on a locally finite counting measure $m = \sum_{i=1}^{+\infty} \delta_{t_i}$ on $\mathbb{R}_+$ by setting

$$\theta_t m = \sum_{i=1}^{+\infty} \mathbb{1}_{\{t_i \geq t\}} \delta_{t_i - t}.$$

The translation θ_t induces an operator on Ω, still denoted by θ_t:
for every $\omega = ((\omega_e)_{e\in\mathbb{E}^d}, (\omega_z)_{z\in\mathbb{Z}^d}, (U_e^i)_{e\in\mathbb{E}^d, i\geq 1}) \in \Omega$, we set

$$\theta_t(\omega) = ((\theta_t\omega_e)_{e\in\mathbb{E}^d}, (\theta_t\omega_z)_{z\in\mathbb{Z}^d}, (U_e^{i+\omega_e([0,t])})_{e\in\mathbb{E}^d, i\geq 1}).$$

Since the Poisson point processes are translation invariant and $\omega_e([0, t])$ is independent from the (U_e^i)'s, $\mathbb{P}$ and $\mathbb{P}_\lambda$ are invariant under θ_t.

There is also a natural action of $\mathbb{Z}^d$ on Ω, which preserves $\mathbb{P}$ and $\mathbb{P}_\lambda$, and which consists in changing the observer's point of view: for $x \in \mathbb{Z}^d$, we define the translation operator T_x by setting:

$$\forall \omega \in \Omega \quad T_x(\omega) = ((\omega_{x+e})_{e\in\mathbb{E}^d}, (\omega_{x+z})_{z\in\mathbb{Z}^d}, (U_{x+e}^i)_{e\in\mathbb{E}^d; i\geq 1}),$$

where $x + e$ the edge e translated by vector x.

2.4 Notation and classical estimates for the contact process

For a set $A \subset \mathbb{Z}^d$, we define the life time τ_λ^A of the process starting from A by

$$\tau_\lambda^A = \inf\{t \geq 0 : \ \xi_t^{\lambda,A} = \varnothing\}.$$

If $y \in \mathbb{Z}^d$, we write τ_λ^y instead of $\tau_\lambda^{\{y\}}$ and we simply write τ_λ for τ_λ^0. With the graphical construction in mind, it is clear that $\{\tau_\lambda = +\infty\}$ if and only if there is an infinite path starting from $(0,0)$ in the graph that is built from potential infections that are present at rate λ. Then, it will be often more appealing to write $\{0 \overset{\lambda}{\leftrightarrow} \infty\}$ instead of $\{\tau_\lambda = +\infty\}$. The critical parameter for the contact process in $\mathbb{Z}^d$ is then

$$\begin{aligned}\lambda_c(\mathbb{Z}^d) &= \inf\{\lambda > 0 : \ \mathbb{P}_\lambda(\tau_\lambda = +\infty) > 0\} \\ &= \inf\{\lambda > 0 : \ \mathbb{P}_\lambda(0 \overset{\lambda}{\leftrightarrow} \infty) > 0\} \in (0, +\infty).\end{aligned}$$

The fact that $\lambda_c(\mathbb{Z}^d) < +\infty$ is due to Harris [11]. Define, for $\lambda > \lambda_c(\mathbb{Z}^d)$, the following conditional probability

$$\overline{\mathbb{P}}_\lambda(\cdot) = \mathbb{P}_\lambda(\cdot|\tau_\lambda = +\infty) = \frac{\mathbb{P}(\ \cdot \ \cap \{0 \overset{\lambda}{\leftrightarrow} \infty\})}{\mathbb{P}(0 \overset{\lambda}{\leftrightarrow} \infty)}.$$

For $A \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, we also define the first infection time $t^A_\lambda(x)$ of site x from set A by

$$t^A_\lambda(x) = \inf\{t \geq 0 : \ x \in \xi^{\lambda,A}_t\}.$$

It follows from Bezuidenhout–Grimmett [1] (see also Durrett [6]) that $\overline{\mathbb{P}}_\lambda(t^A_\lambda(x) < +\infty) = 1$ as soon as $A \neq \varnothing$. If $y \in \mathbb{Z}^d$, we write $t^y_\lambda(x)$ instead of $t^{\{y\}}_\lambda(x)$ and we simply write $t_\lambda(x)$ for $t^0_\lambda(x)$. The set of points infected before time t is then

$$H^\lambda_t = \{x \in \mathbb{Z}^d : \ t_\lambda(x) \leq t\} \quad \text{and } \tilde{H}^\lambda_t = \{x + u \ : \ (x,u) \in H^\lambda_t \times [0,1]^d\}.$$

The following estimates are classical for the supercritical contact process; they are mainly due to Bezuidenhout–Grimmett [1] and Durrett [6]. Here, we need an extra uniformity in the parameter λ (this uniformity is mainly obtained by stochastic comparison):

Proposition 2.2 (Proposition 5 in Garet–Marchand [8]).
Let $\lambda_{\min}, \lambda_{\max}$ with $\lambda_c(\mathbb{Z}^d) < \lambda_{\min} \leq \lambda_{\max}$. There exist $A, B, C, c, \rho > 0$ such that for every $\lambda \in [\lambda_{\min}, \lambda_{\max}]$, for every $x \in \mathbb{Z}^d$, for every $t \geq 0$,

$$\begin{aligned} \mathbb{P}(\tau_\lambda = +\infty) &\geq \rho, \\ \mathbb{P}(H^\lambda_t \not\subset [-Ct, Ct]^d) &\leq A\exp(-Bt), \\ \mathbb{P}(t < \tau_\lambda < +\infty) &\leq A\exp(-Bt), \\ \mathbb{P}\left(t_\lambda(x) \geq \frac{\|x\|}{c} + t, \ \tau_\lambda = +\infty\right) &\leq A\exp(-Bt). \end{aligned}$$

2.5 Essential hitting times and shape theorem

We now recall the definition of the essential hitting time $\sigma_\lambda(x)$. It was introduced in [8] to prove an asymptotic shape result for the supercritical contact process in random environment. See also Garet–Marchand [9] and Garet–Gouéré–Marchand [7] for further uses. The essential hitting time $\sigma_\lambda(x)$ is a time when the site x is infected from the origin 0 and also has an infinite life time.

We begin by an informal description: first wait until site x is occupied. If the progeny of the particle that occupies x at that time is infinite, then we have found $\sigma_\lambda(x)$. Otherwise, wait until this progeny dies, and then wait until site x is occupied again. If the progeny of the particle that occupies x at that new time is infinite, then we have found $\sigma_\lambda(x)$, otherwise we repeat the process until the whole population has disappeared or we have found through this process someone living at x and having an infinite progeny.

Formally, $\sigma_\lambda(x)$ is defined through a family of stopping times as follows: we set $u_0(x) = v_0(x) = 0$ and we define recursively two increasing sequences of stopping times $(u_n(x))_{n\geq 0}$ and $(v_n(x))_{n\geq 0}$ with $u_0(x) = v_0(x) \leq u_1(x) \leq v_1(x) \leq u_2(x) \ldots$:

- Assume that $v_k(x)$ is defined. We set $u_{k+1}(x) = \inf\{t \geq v_k(x) : \ x \in \xi_t^{\lambda,0}\}$.
- Assume that $u_k(x)$ is defined, with $k \geq 1$. We set $v_k(x) = u_k(x) + \tau_\lambda^x \circ \theta_{u_k(x)}$.

We then set

$$K_\lambda(x) = \min\{n \geq 0 : \ v_n(x) = +\infty \text{ or } u_{n+1}(x) = +\infty\}.$$

This quantity represents the number of steps before we stop: either we stop because we have just found an infinite $v_n(x)$, which corresponds to a time $u_n(x)$ when x is occupied and has infinite progeny, or we stop because we have just found an infinite $u_{n+1}(x)$, which says that after $v_n(x)$, site x is never infected anymore. Since $\overline{\mathbb{P}}_\lambda(t_\lambda^0(x) < +\infty) = 1$, it follows from the strong Markov property that u_{n+1} is never infinite when the contact process survives.

In [8], using (2.4) and (2.5), it is proved that $K_\lambda(x)$ is almost surely finite, which allows to define the essential hitting time $\sigma_\lambda(x)$ by setting

$$\sigma_\lambda(x) = u_{K_\lambda(x)}.$$

At the same time, we define the operator $\tilde{\Theta}_{x,\lambda}$ on Ω by:

$$\tilde{\Theta}_{x,\lambda} = \begin{cases} T_x \circ \theta_{\sigma_\lambda(x)} & \text{if } \sigma_\lambda(x) < +\infty, \\ T_x & \text{otherwise.} \end{cases}$$

The advantage of the essential hitting time $\sigma_\lambda(x)$, compared to $t_\lambda(x)$, is that $\theta_{\sigma_\lambda(x)}$ preserves $\overline{\mathbb{P}}_\lambda$ (see below). So, $\sigma_\lambda(x)$ can be seen as a regenerating time. It also enjoys good integrability properties. We now recall the main results of [8] we will need here. In the following, we fix $\lambda_{\min}, \lambda_{\max} > 0$ such that $\lambda_c(\mathbb{Z}^d) < \lambda_{\min} \leq \lambda_{\max}$.

Proposition 2.3 (Garet–Marchand [8], Theorems 1 and 3, Corollary 21, Theorem 22 and Lemma 29)**.**

- *For each $\lambda > \lambda_c(\mathbb{Z}^d)$, for every $x \in \mathbb{Z}^d$,*

$$\textit{the probability measure } \overline{\mathbb{P}}_\lambda \textit{ is invariant under the map } \tilde{\Theta}_{x,\lambda}. \tag{2.6}$$

- *There exist constants $(C_p)_{p\geq 1}$ such that for every $\lambda \in [\lambda_{\min}, \lambda_{\max}]$, for every $x \in \mathbb{Z}^d$, for every $p \geq 1$,*

$$\overline{\mathbb{E}}_\lambda[\sigma_\lambda(x)^p] \leq C_p(1 + \|x\|)^p. \tag{2.7}$$

- *For each $\lambda > \lambda_c(\mathbb{Z}^d)$, for every $x \in \mathbb{Z}^d$, there exists a deterministic $\mu_\lambda(x)$ such that*

$$\lim_{n\to+\infty} \frac{t_\lambda(nx)}{n} = \lim_{n\to+\infty} \frac{\sigma_\lambda(nx)}{n} = \mu_\lambda(x). \tag{2.8}$$

The convergence holds $\overline{\mathbb{P}}_\lambda$ almost surely, and also in $L^1(\overline{\mathbb{P}}_\lambda)$.

- *The function $x \mapsto \mu_\lambda(x)$ can be extended to a norm on $\mathbb{R}^d$. Let*

$$\mathcal{S}(\lambda) = \{x \in \mathbb{R}^d : \ \mu_\lambda(x) \leq 1\}.$$

- *For every $\varepsilon > 0$, $\overline{\mathbb{P}}_\lambda - a.s.$, for every t large enough,*

$$(1-\varepsilon)\mathcal{S}(\lambda) \subset \frac{\tilde{H}_t^\lambda}{t} \subset (1+\varepsilon)\mathcal{S}(\lambda). \tag{2.9}$$

The only drawback of this essential hitting time is that it is only almost subadditive, but we proved in Theorem 2 in [8] that the lack of subadditivity is well controlled: There exist $A, B > 0$ such that for any $\lambda \in [\lambda_{\min}, \lambda_{\max}]$, for any $x, y \in \mathbb{Z}^d$,

$$\forall t > 0 \quad \overline{\mathbb{P}}_\lambda(\sigma_\lambda(x+y) - (\sigma_\lambda(x) + \sigma_\lambda(y) \circ \tilde{\Theta}_{x,\lambda} \geq t) \leq A\exp(-B\sqrt{t}).$$

Thus there exists $M_1 > 0$ such that, for each $\lambda \in [\lambda_{\min}, \lambda_{\max}]$ and each $x \in \mathbb{Z}^d \backslash \{0\}$, the sequence $(\overline{\mathbb{E}}_\lambda \sigma_\lambda(nx) + M_1)_{n \geq 1}$ is subadditive, and with (2.8), we can represent $\mu_\lambda(x)$ as the following infimum:

$$\forall \lambda \in [\lambda_{\min}, \lambda_{\max}] \quad \forall x \in \mathbb{Z}^d \quad \mu_\lambda(x) = \inf_{n \geq 1} \frac{M_1 + \overline{\mathbb{E}}_\lambda(\sigma_\lambda(nx))}{n}. \tag{2.10}$$

As a corollary of (2.8), we obtain the following monotonicity property:

Corollary 2.4. *For each $x \in \mathbb{Z}^d$, $\lambda \mapsto \mu_\lambda(x)$ is non-increasing on $(\lambda_c(\mathbb{Z}^d), +\infty)$.*

Proof. Suppose $\lambda_c(\mathbb{Z}^d) < \lambda' < \lambda < +\infty$. Choose $\lambda_{\min}, \lambda_{\max}$ with $\lambda_c(\mathbb{Z}^d) < \lambda_{\min} < \lambda' < \lambda \leq \lambda_{\max}$. Use the construction of Subsection 2.2 to build the two contact processes with respective parameters λ and λ'. On the event $\{0 \overset{\lambda'}{\leftrightarrow} \infty\}$, which has positive probability, we have that for each $n \geq 1$, $\frac{t_\lambda(nx)}{n} \leq \frac{t_{\lambda'}(nx)}{n}$. Letting n go to infinity, we get $\mu_{\lambda'}(x) \leq \mu_\lambda(x)$. □

3 Left-Continuity

We prove here the left-continuity of μ_λ. More precisely, we prove that for each $\lambda_0 > \lambda_c(\mathbb{Z}^d)$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall \lambda \in [\lambda_0 - \delta, \lambda_0] \quad \forall x \in \mathbb{S}^{d-1} \quad |\mu_{\lambda_0}(x) - \mu_\lambda(x)| \leq \varepsilon.$$

When proving continuity theorems for the time constant in first passage percolation (see Cox and Kesten [2, 3, 13]), the left-continuity is usually considered as the easy part, due to the fact that the time constant is an infimum. In the case of the contact process, there are extra difficulties, because contact processes with different intensities can not be coupled in such a way that they die simultaneously.

Lemma 3.1. *Let $\lambda > \lambda_c(\mathbb{Z}^d)$. For each $x \in \mathbb{Z}^d$, $\overline{\lim}_{\lambda' \to \lambda^-} \overline{\mathbb{E}}_{\lambda'}(\sigma_{\lambda'}(x)) \leq \overline{\mathbb{E}}_\lambda(\sigma_\lambda(x))$.*

Proof. Fix $\lambda > \lambda_c(\mathbb{Z}^d)$. Choose $\lambda_{\min}$ such that $\lambda_c(\mathbb{Z}^d) < \lambda_{\min} < \lambda$ and set $\lambda_{\max} = \lambda$. Fix $x \in \mathbb{Z}^d$. Use the construction of Subsection 2.2. In this proof, for $\lambda' \in [\lambda_{\min}, \lambda]$, we write $\sigma_{\lambda'}$ instead of $\sigma_{\lambda'}(x)$ to simplify the notation. Fix $\lambda' \in [\lambda_{\min}, \lambda]$. Note that $\{0 \overset{\lambda'}{\leftrightarrow} +\infty\} \subset \{0 \overset{\lambda}{\leftrightarrow} +\infty\}$, and thus, for any non negative random variable X,

$$\overline{\mathbb{E}}_{\lambda'}(X) = \frac{\mathbb{E}\left(X,\ 0 \overset{\lambda'}{\leftrightarrow} \infty\right)}{\mathbb{P}(0 \overset{\lambda'}{\leftrightarrow} \infty)} \leq \frac{\mathbb{E}\left(X,\ 0 \overset{\lambda}{\leftrightarrow} \infty\right)}{\mathbb{P}(0 \overset{\lambda'}{\leftrightarrow} \infty)} = \frac{\mathbb{P}(0 \overset{\lambda}{\leftrightarrow} \infty)}{\mathbb{P}(0 \overset{\lambda'}{\leftrightarrow} \infty)} \overline{\mathbb{E}}_\lambda(X). \tag{3.1}$$

For any $\lambda' \in [\lambda_{\min}, \lambda]$, with (3.1) and the control (2.7) on the moments of $\sigma_{\lambda'}$, we get

$$\begin{aligned}
\overline{\mathbb{E}}_{\lambda'}(\sigma_{\lambda'}) &= \overline{\mathbb{E}}_{\lambda'}(\sigma_\lambda,\ \sigma_{\lambda'} = \sigma_\lambda) + \overline{\mathbb{E}}_{\lambda'}(\sigma_{\lambda'},\ \sigma_{\lambda'} \neq \sigma_\lambda) \\
&\le \frac{\mathbb{P}(0 \overset{\lambda}{\leftrightarrow} +\infty)}{\mathbb{P}(0 \overset{\lambda'}{\leftrightarrow} +\infty)} \overline{\mathbb{E}}_\lambda(\sigma_\lambda) + \sqrt{\overline{\mathbb{E}}_{\lambda'}(\sigma_{\lambda'}^2)\overline{\mathbb{P}}_{\lambda'}(\sigma_{\lambda'} \neq \sigma_\lambda)} \\
&\le \frac{\mathbb{P}(0 \overset{\lambda}{\leftrightarrow} +\infty)}{\mathbb{P}(0 \overset{\lambda'}{\leftrightarrow} +\infty)} \overline{\mathbb{E}}_\lambda(\sigma_\lambda) + \sqrt{C_2(1+\|x\|^2)} \sqrt{\frac{\mathbb{P}(0 \overset{\lambda}{\leftrightarrow} \infty)}{\mathbb{P}(0 \overset{\lambda'}{\leftrightarrow} \infty)} \overline{\mathbb{P}}_\lambda(\sigma_{\lambda'} \neq \sigma_\lambda)} \\
&\le \frac{\mathbb{P}(0 \overset{\lambda}{\leftrightarrow} +\infty)}{\mathbb{P}(0 \overset{\lambda'}{\leftrightarrow} +\infty)} \left(\overline{\mathbb{E}}_\lambda(\sigma_\lambda) + \sqrt{C_2(1+\|x\|^2)} \sqrt{\overline{\mathbb{P}}_\lambda(\sigma_{\lambda'} \neq \sigma_\lambda)} \right).
\end{aligned}$$

As $\lambda \mapsto \mathbb{P}(0 \overset{\lambda}{\leftrightarrow} +\infty)$ is continuous on $[\lambda_c(\mathbb{Z}^d), +\infty)$ (see Theorems 1.10.a and 1.6.d in [15]), if we prove that $\overline{\mathbb{P}}_\lambda(\sigma_{\lambda'} = \sigma_\lambda)$ tends to 1 when λ' goes to λ, we complete the proof.

We now build a "good" event $G(\lambda')$ such that $G(\lambda') \cap \{0 \overset{\lambda}{\leftrightarrow} \infty\} \subset \{\sigma_{\lambda'} = \sigma_\lambda\}$ and $\overline{\mathbb{P}}_\lambda(G(\lambda'))$ goes to 1 as λ' goes to λ. Since σ_λ is $\overline{\mathbb{P}}_\lambda$-a.s. finite and $H^\lambda_{\sigma_\lambda}$ is $\overline{\mathbb{P}}_\lambda$-a.s. a finite set, we can first choose $M > 0$ such that

$$\overline{\mathbb{P}}_\lambda(A_M) \ge 1 - \frac{\varepsilon}{3}, \text{ where } A_M = \{H^\lambda_{\sigma_\lambda} \subset [-M, M]^d,\ \sigma_\lambda \le M\}. \tag{3.2}$$

The event A_M says that, apart from the fact that $(x, \sigma_\lambda) \overset{\lambda}{\leftrightarrow} \infty$, the time σ_λ is determined by the configuration of the Poisson processes in the space-time box $[-M, M]^d \times [0, M]$. Then, with estimates (2.3) and (2.4) we choose $L > 0$ such that for each $\lambda' \in [\lambda_{\min}, \lambda]$

$$\overline{\mathbb{P}}_\lambda(B_L(\lambda')) \ge 1 - \frac{\varepsilon}{3}, \text{ with } B_L(\lambda') = \{H^{\lambda'}_L \subset [-CL, CL]^d\} \cap \{L < \tau_{\lambda'} < \infty\}^c. \tag{3.3}$$

Set $S = [-(M + CL), (M + CL)]^d \cap \mathbb{Z}^d$ and $t = M + L$. With Lemma 2.1, we can choose $\delta > 0$ such that

$$\forall \lambda' \in [\lambda - \delta, \lambda] \quad \overline{\mathbb{P}}_\lambda(\mathrm{Idem}(S, t, \lambda, \lambda')) \ge 1 - \varepsilon/3. \tag{3.4}$$

Finally, we consider, for every $\lambda' \in [\lambda - \delta, \lambda]$, the event

$$G(\lambda') = A_M \cap \tilde{\Theta}^{-1}_{x,\lambda}(B_L(\lambda')) \cap \mathrm{Idem}(S, t, \lambda, \lambda').$$

The choices (3.2), (3.3) and (3.4) we respectively made for M, L and δ, and the invariance property (2.6) ensure that

$$\forall \lambda' \in [\lambda - \delta, \lambda] \quad \overline{\mathbb{P}}_\lambda(G(\lambda')) \ge 1 - \varepsilon.$$

It now remains to see that $G(\lambda') \cap \{0 \overset{\lambda}{\leftrightarrow} \infty\} \subset \{\sigma_{\lambda'} = \sigma_\lambda\}$. On the event $G(\lambda') \cap \{0 \overset{\lambda}{\leftrightarrow} \infty\}$, the point (x, σ_λ) has a progeny for parameter λ that is still alive at time $\sigma_\lambda + L$. But the event $\mathrm{Idem}(S, t, \lambda, \lambda')$ ensures that the infection at rate λ' in the box $S \times [0, t]$ behaves exactly like the infection at rate λ in the same box, so the point (x, σ_λ) has a progeny for parameter λ' that is also still alive at times $\sigma_\lambda + L$. The event $\tilde{\Theta}^{-1}_{x,\lambda}(B_L(\lambda'))$ says then that $(x, \sigma_\lambda) \overset{\lambda'}{\leftrightarrow} \infty$, and, with $\mathrm{Idem}(S, t, \lambda, \lambda')$, this implies that $\sigma_{\lambda'} = \sigma_\lambda$. This completes the proof. □

Lemma 3.2. *For each $x \in \mathbb{Z}^d$, $\lambda \mapsto \mu_\lambda(x)$ is left-continuous on $(\lambda_c(\mathbb{Z}^d), +\infty)$.*

Proof. Fix $x \in \mathbb{Z}^d$. Since, from Corollary 2.4, the application $\lambda \mapsto \mu_\lambda(x)$ is non-increasing on $(\lambda_c(\mathbb{Z}^d), +\infty)$, we can define

$$L = \lim_{\lambda' \to \lambda^-} \mu_{\lambda'}(x).$$

Obviously $L \geq \mu_\lambda(x)$ and we must prove $L \leq \mu_\lambda(x)$. Put $\lambda_n = \lambda - 1/n$. Using the representation (2.10) of $\mu_\lambda(x)$ as an infimum, we have

$$\begin{aligned} L = \inf_{n\geq 1} \mu_{\lambda_n}(x) &= \inf_{n\geq 1} \inf_{k\geq 1} \frac{\overline{\mathbb{E}}_{\lambda_n}(\sigma_{\lambda_n}(kx)) + M_1}{k} \\ &= \inf_{k\geq 1} \inf_{n\geq 1} \frac{\overline{\mathbb{E}}_{\lambda_n}(\sigma_{\lambda_n}(kx)) + M_1}{k} = \inf_{k\geq 1} \left(\frac{M_1}{k} + \inf_{n\geq 1} \frac{\overline{\mathbb{E}}_{\lambda_n}(\sigma_{\lambda_n}(kx))}{k} \right). \end{aligned}$$

By Lemma 3.1, for each k, $\inf_{n\geq 1} \overline{\mathbb{E}}_{\lambda_n}(\sigma_{\lambda_n}(kx)) \leq \overline{\mathbb{E}}_\lambda(\sigma_\lambda(kx))$, so

$$L \leq \inf_{k\geq 1} \left(\frac{M_1}{k} + \frac{\overline{\mathbb{E}}_\lambda(\sigma_\lambda(kx))}{k} \right) = \mu_\lambda(x),$$

which completes the proof. □

By homogeneity of μ_λ, the result of Lemma 3.2 also holds for all $x \in \mathbb{R}^d$, thus the difference between (3) and Lemma 3.2 is the uniformity of the control. For all $\lambda > 0$, since μ_λ is a norm and by symmetry of the model, we have for all $x, y \in \mathbb{R}^d$,

$$|\mu_\lambda(x) - \mu_\lambda(y)| \leq \mu_\lambda(x-y) \leq \|x-y\|_1 \mu_\lambda(e_1)\,,$$

where $e_1 = (1, 0, \ldots, 0)$. Fix $\lambda_0 \in (\lambda_c, +\infty)$ and $\varepsilon > 0$. By Lemma 3.2 we know that $\lim_{\lambda\to\lambda_0^-} \mu_\lambda(e_1) = \mu_{\lambda_0}(e_1)$, thus there exists $\delta > 0$ such that for all $\lambda \in [\lambda_0 - \delta, \lambda_0]$, for all $x, y \in \mathbb{R}^d$, we have $|\mu_\lambda(x) - \mu_\lambda(y)| \leq 2\|x-y\|_1 \mu_{\lambda_0}(e_1)$. We obtain the existence of $\eta > 0$ such that for all $x, y \in \mathbb{R}^d$ satisfying $\|x - y\|_1 \leq \eta$, we have

$$\sup_{\lambda\in[\lambda_0-\delta,\lambda_0]} \{|\mu_\lambda(x) - \mu_\lambda(y)|\} \leq \varepsilon\,.$$

There exists a finite set of points $y_1, \ldots, y_m$ in $\mathbb{R}^d$ such that

$$\mathbb{S}^{d-1} \subset \bigcup_{i=1}^{m} \{x \in \mathbb{R}^d \,:\, \|x - y_i\|_1 \leq \eta\}\,,$$

thus for all $\lambda \in [\lambda_0 - \delta, \lambda_0]$ we obtain

$$\sup_{x\in\mathbb{S}^{d-1}} |\mu_\lambda(x) - \mu_{\lambda_0}(x)| \leq 2\varepsilon + \max_{i=1,\ldots,m} |\mu_\lambda(y_i) - \mu_{\lambda_0}(y_i)|\,.$$

By homogeneity of μ_λ, the result of Lemma 3.2 also holds for $y_i, i \in \{1, \ldots, m\}$. This concludes the proof of (3).

We can notice that the previous argument also applies to the study of the right-continuity of μ_λ. However, as we will see in the next section, we do not need it since we perform directly the study of the right-continuity of μ_λ uniformly in all directions.

4 Right-continuity

We prove here the right-continuity of μ_λ. More precisely, we prove that for each $\lambda_0 > \lambda_c(\mathbb{Z}^d)$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall \lambda \in [\lambda_0, \lambda_0 + \delta] \quad \forall x \in \mathbb{S}^{d-1} \quad |\mu_{\lambda_0}(x) - \mu_\lambda(x)| \leq \varepsilon. \tag{4.1}$$

As we will see, the right-continuity of the asymptotic shape of the contact process can be obtained by a slight modification of a part of the proof of the large deviations inequality for the contact process established by Garet and Marchand in [9].

Let $\lambda_0 > \lambda_c(\mathbb{Z}^d)$ be fixed. Fix $\lambda_{\min}, \lambda_{\max}$ with $\lambda_c(\mathbb{Z}^d) < \lambda_{\min} \leq \lambda_0 < \lambda_{\max}$.

Let $\alpha, \varepsilon > 0$ and L, N be positive integers. Consider $\lambda \geq \lambda_0$ and close to λ_0. We define the following event, relative to the space-time box $B_N = B_N(0,0) = ([-N,N]^d \cap \mathbb{Z}^d) \times [0, 2N]$:

$$A_{\lambda,\lambda_0}^{\alpha,L,N,\varepsilon} = \left\{ \forall (x_0,t_0) \in B_N \quad \xi_{\alpha LN - t_0}^{x_0,\lambda} \circ \theta_{t_0} \subset x_0 + (1+\varepsilon)(\alpha LN - t_0)\mathcal{S}(\lambda_0) \right\}$$
$$\cap \left\{ \forall (x_0,t_0) \in B_N \quad \bigcup_{0 \leq s \leq \alpha LN - t_0} \xi_s^{x_0,\lambda} \circ \theta_{t_0} \subset]-LN, LN[^d \right\}.$$

Consider first $A_{\lambda_0,\lambda_0}^{\alpha,L,N,\varepsilon}$. The first part of the event ensures that the descendants, at time αLN, of any point (x_0,t_0) in the box B_N are included in $x_0 + (1+\varepsilon)(\alpha LN)\mathcal{S}(\lambda)$: it is a sharp control, requiring the asymptotic shape Theorem for parameter λ_0. The second part ensures that the descendants, at all times in $[0, \alpha LN]$, of the whole box B_N are included in $]-LN, LN[^d$: the bound is rough, only based on the (at most) linear growth of the process with parameter λ_0. Thus, the "good growth" event $A_{\lambda_0,\lambda_0}^{\alpha,L,N,\varepsilon}$ is typical, and the following Lemma has been proved, using essentially (2.9) and (2.3) :

Lemma 4.1 ([9]). *Fix $\lambda_0 > \lambda_c(\mathbb{Z}^d)$. There exists $\alpha = \alpha(\lambda_0) \in (0,1)$ such that for every $\varepsilon \in (0,1)$, every $L_0 > 0$, there exists an integer $L > L_0$ such that*

$$\lim_{N \to +\infty} \mathbb{P}(A_{\lambda_0,\lambda_0}^{\alpha,L,N,\varepsilon}) = 1.$$

Garet and Marchand used Lemma 4.1 to prove the upper large deviations for the contact process: for every $\lambda_0 \in [\lambda_{\min}, \lambda_{\max}]$, provided that $\alpha = \alpha(\lambda_0)$ is fixed as in Lemma 4.1, then for L greater than some $L_0 = L_0(\varepsilon, \lambda_0)$, they prove that there exists $p_1 = p_1(\lambda_0, \varepsilon, L) > 0$ such that

$$\mathbb{P}(A_{\lambda_0,\lambda_0}^{\alpha,L,N,\varepsilon/3}) > p_1 \Longrightarrow \exists A, B \quad \forall t > 0 \quad \mathbb{P}(\xi_t^{0,\lambda_0} \not\subset (1+\varepsilon)t\mathcal{S}(\lambda_0)) \leq A\exp(-Bt).$$

The idea of the proof is classical and as follows: a too fast infection from $(0,0)$ to $\mathbb{Z}^d \times \{n\}$ uses a too fast path, along which we find a number of order θn of "bad growth" events, *i.e.* translated versions of $(A_{\lambda_0,\lambda_0}^{\alpha,L,N,\varepsilon/3})^c$. The proof ends with a Peierls argument: the event $(A_{\lambda_0,\lambda_0}^{\alpha,L,N,\varepsilon/3})^c$ is local, thus its translated events are only locally dependent. If their probability is small enough, the probability that there exists a path from $(0,0)$ to $\mathbb{Z}^d \times \{n\}$ with at least θn "bad growth" events decreases exponentially fast in n.

Let's come back to the right-continuity. Fix $\lambda_0 > \lambda_c(\mathbb{Z}^d)$ and $\varepsilon > 0$. Take α given by Lemma 4.1, $L \geq L_0(\varepsilon, \lambda_0)$ large enough, and $p_1(\lambda_0, \varepsilon, L) > 0$ as before. The very same Peierl argument ensures that for any $\lambda \geq \lambda_0$, we have

$$\mathbb{P}(A_{\lambda,\lambda_0}^{\alpha,L,N,\varepsilon/3}) > p_1 \Longrightarrow \exists A, B \quad \forall t > 0 \quad \mathbb{P}(\xi_t^{0,\lambda} \not\subset (1+\varepsilon)t\mathcal{S}(\lambda_0)) \leq A\exp(-Bt).$$

Remember that the event $A_{\lambda,\lambda_0}^{\alpha,L,N,\varepsilon/3}$ is local. Thus, applying Lemma 2.1 with the set $S = [-LN, LN]^d \cap \mathbb{Z}^d$ and $t = \alpha LN$, we obtain the existence of $\lambda_1 \in (\lambda_0, \lambda_{\max}]$ such that for every $\lambda \in [\lambda_0, \lambda_1]$, $\mathbb{P}(A_{\lambda,\lambda_0}^{\alpha,L,N,\varepsilon/3}) > p_1$. Thus,

$$\forall \lambda \in [\lambda_0, \lambda_1] \quad \exists A, B \quad \forall t > 0 \quad \mathbb{P}(\xi_t^{0,\lambda} \not\subset (1+\varepsilon)t\mathcal{S}(\lambda_0)) \leq A\exp(-Bt),$$

from which we deduce (for a detailed proof, see the passage from (62) to (63) in [9]):

$$\forall \lambda \in [\lambda_0, \lambda_1] \quad \exists A, B \quad \forall t > 0 \quad \overline{\mathbb{P}}_\lambda(H_t^{0,\lambda} \not\subset (1+\varepsilon)t\mathcal{S}(\lambda_0)) \leq A\exp(-Bt).$$

Fix $\lambda \in [\lambda_0, \lambda_1]$ and $\eta > 0$. With the asymptotic shape result (2.9), choose t large enough to have $A\exp(-Bt) < 1/2$ and $\overline{\mathbb{P}}_\lambda((1-\eta)t\mathcal{S}(\lambda) \not\subset H_t^{0,\lambda}) < 1/2$. Then the event

$\{(1-\eta)t\mathcal{S}(\lambda) \subset H_t^{0,\lambda} \subset (1+\varepsilon)t\mathcal{S}(\lambda_0)\}$ has positive probability; particularly, $(1-\eta)\mathcal{S}(\lambda) \subset (1+\varepsilon)\mathcal{S}(\lambda_0)$, and, letting η tend to 0, we have

$$\forall \lambda \in [\lambda_0, \lambda_1] \quad \mathcal{S}(\lambda) \subset (1+\varepsilon)\mathcal{S}(\lambda_0),$$

or equivalently $\forall \lambda \in [\lambda_0, \lambda_1],\ \forall x \in \mathbb{R}^d,\ \mu_{\lambda_0}(x) \le (1+\varepsilon)\mu_\lambda(x)$. This completes the proof of (4.1).

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