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When do skew-products exist?*

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Abstract

The classical skew-product decomposition of planar Brownian motion represents the process in polar coordinates as an autonomously Markovian radial part and an angular part that is an independent Brownian motion on the unit circle time-changed according to the radial part. Theorem 4 of [10] gives a broad generalization of this fact to a setting where there is a diffusion on a manifold X with a distribution that is equivariant under the smooth action of a Lie group K. Under appropriate conditions, there is a decomposition into an autonomously Markovian "radial" part that lives on the space of orbits of K and an "angular" part that is an independent Brownian motion on the homogeneous space K/M, where M is the isotropy subgroup of a point of x, that is time-changed with a time-change that is adapted to the filtration of the radial part. We present two apparent counterexamples to [10, Theorem 4]. In the first counterexample the angular part is not a time-change of any Brownian motion on K/M, whereas in the second counterexample the angular part is the time-change of a Brownian motion on K/M but this Brownian motion is not independent of the radial part. In both of these examples K/M has dimension 1. The statement and proof of [10, Theorem 4] remain valid when K/M has dimension greater than 1. Our examples raise the question of what conditions lead to the usual sort of skew-product decomposition when K/M has dimension 1 and what conditions lead to there being no decomposition at all or one in which the angular part is a time-changed Brownian motion but this Brownian motion is not independent of the radial part.

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1 Introduction

The archetypal skew-product decomposition of a Markov process is the decomposition of a Brownian motion in the plane $(B_t)_{t\geq 0}$ into its radial and angular part

$$B_t = |B_t| \exp(i\theta_t). \tag{1.1}$$

Here the radial part $(|B_t|)_{t\geq 0}$ is a two-dimensional Bessel process and $\theta_t = y_{\tau_t}$, where $(y_t)_{t\geq 0}$ is a one-dimensional Brownian motion that is independent of the radial part $(|B_t|)_{t\geq 0}$ and τ is a time-change that is adapted to the filtration generated by the process $|B|$. Specifically, $\tau_t = \int_0^t \frac{1}{|B_s|^2} ds$. See Corollary 18.7 from [8] for more details.

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The most obvious generalization of this result is obtained in [5]. The process considered is any time-homogeneous diffusion $(x_t)_{t\geq 0}$ with state space $\mathbb{R}^3$ that satisfies the additional assumptions that almost surely every path does not pass through the origin at positive times and that $(x_t)_{t\geq 0}$ is isotropic in the sense that the law of $(x_t)_{t\geq 0}$ is equivariant under the group of orthogonal transformations; that is, if we consider a point $(r,\theta) \in \mathbb{R}^3$ in spherical coordinates, where $r \in \mathbb{R}_+$ is the radial coordinate and θ is a point on the unit sphere S^2, and if we take $k \in O(3)$, the orthogonal group on $\mathbb{R}^3$, then

$$P_{(r,k\theta)}\left(kA\right) = P_{(r,\theta)}\left(A\right)$$

for any Borel set A in path space $C(\mathbb{R}_+, \mathbb{R}^3)$. Here $P_x(A)$ is the probability a path started at x belongs to the Borel set A [5, (2.2)]. Theorem 1.2 of [5] states that we can decompose $(x_t)_{t\geq 0}$ as $x_t = r_t\theta_t$ where the radial motion $(r_t)_{t\geq 0}$ is a time-homogeneous Markov process on $\mathbb{R}_+$ and the angular process $(\theta_t)_{t\geq 0}$ can be written as $\theta_t = B_{\tau_t}$, with $(B_t)_{t\geq 0}$ a spherical Brownian motion independent of the radial part and with the time-change $(\tau_t)_{t\geq 0}$ adapted to the filtration generated by the radial part.

More generally, one can consider a group G acting on $\mathbb{R}^n$ and $(x_t)_{t\geq 0}$ a Markov process on $\mathbb{R}^n$ such that the distribution of $(x_t)_{t\geq 0}$ satisfies the equivariance condition

$$P_{gx}(gA) = P_x(A)$$

for any Borel set A in path space. The existence of a skew-product decomposition for this setting is explored in [2] when $(x_t)_{t\geq 0}$ is a Dunkl process and G is the group of distance preserving transformations of $\mathbb{R}^n$.

The paper [11] investigates the skew-product decomposition of a Brownian motion on a C^∞ Riemannian manifold (M, g) which can be written as a product of a radial manifold R and an angular manifold Θ, both of which are assumed to be smooth and connected. Provided the Riemannian metric respects the product structure of the manifold in a suitable manner, [11, Theorem 4] establishes the existence of a skew-product decomposition such that the radial motion is a Brownian motion with drift on R and the angular motion is a time-change of a Brownian motion on Θ that is independent of the radial motion.

A broadly applicable skew-product decomposition result is obtained in [10] for a general continuous Markov process $(x_t)_{t\geq 0}$ with state space a smooth manifold X and distribution that is equivariant under the smooth action of a Lie group K on X. Here the decomposition of $(x_t)_{t\geq 0}$ is into a radial part $(y_t)_{t\geq 0}$ that is a Markov process on the submanifold Y which is transversal to the orbits of K and an angular part $(z_t)_{t\geq 0}$ that is a process on a general K-orbit which can be identified with the homogeneous space K/M, where M is the isotropy subgroup of K that is assumed to be the same for all elements $x \in X$. Theorem 4 of [10] asserts that under suitable conditions the process $(x_t)_{t\geq 0}$ has the same distribution as $(B(a_t)y_t)_{t\geq 0}$, where the radial part $(y_t)_{t\geq 0}$ is a diffusion on Y, $(B_t)_{t\geq 0}$ is a Brownian motion on K/M that is independent of $(x_t)_{t\geq 0}$, and $(a_t)_{t\geq 0}$ a time-change that is adapted to the filtration generated by $(y_t)_{t\geq 0}$.

The present paper was motivated by our desire to understand better the structural features that give rise to skew-product decompositions of diffusions that are equivariant under the action of a group and what it is about the absence of these features which cause such a decomposition not to hold. In attempting to do so, we read the paper [10]. We found an apparent counterexample to the main result, Theorem 4 of that paper in which there is a decomposition of the process into an autonomously Markov radial process on Y and an angular part that is a Brownian motion on K/M time-changed according to the radial process, but this Brownian motion is **not**, contrary to the claim of [10], independent of the radial process, see Section 4 for an exposition of the counterexample.

This seeming contradiction appears because the assumption from [10] that K/M is irreducible is not strong enough to ensure the nonexistence of a nonzero M-invariant tangent vector in the special case when, as in our construction, K/M has dimension 1. It is the nonexistence of such a tangent vector that is used in the proof in [10] to deduce the independence of the radial process and the Brownian motion. Professor Liao pointed out to us that [10, Theorem 4] holds under the conditions in [10] for $\dim(K/M) > 1$ and that result also holds when K/M has dimension 1 if we further assume that there is no M-invariant tangent vector.

An anonymous referee pointed out an even simpler counterexample to [10, Theorem 4] which we present in Section 3. Namely, one takes

$$x_t = \Theta_t \begin{pmatrix} U_t \\ V_t \end{pmatrix}$$

where $\begin{pmatrix} U_t \\ V_t \end{pmatrix}$ is a planar Brownian motion and $\Theta_t \in SO(2)$ is the matrix that represents rotation about the origin through an angle t. We show that in this case that there is no skew-product decomposition for a somewhat different (and perhaps less interesting) reason: the angular part of $(x_t)_{t\geq 0}$ cannot be written as a time-changed Brownian motion on the unit circle in the plane. The apparent contradiction to [10, Theorem 4]is again due to the irreducibility of K/M being inadequate to ensure the non-existence of an M invariant tangent vector when K/M has dimension 1.

We present both of these counterexamples here because they illustrate two rather different ways in which things can go wrong. The latter counterexample shows that under what look like reasonable conditions one might fail to have a skew-product decomposition because the angular part can't be time-changed to be Brownian, whereas the former counterexample does involve an angular part that is a time-changed Brownian motion, but it is just that this Brownian motion isn't independent of the radial process. We hope that by presenting these two examples we will prompt further investigation into what general conditions lead to the subtle failure of the usual skew-product decomposition in the first counterexample and what ones lead to the grosser failure in the second counterexample.

The outline of the remainder of the paper is the following.

In Section 2 we check that the classical skew-product decomposition of planar Brownian motion fits in the setting from [10], even though the proof of [10, Theorem 4] does not, as we have noted, apply to ensure the existence of the skew-product decomposition when, as here, the dimension of K/M is 1.

In Section 3 we describe the counterexample mentioned above of a planar Brownian motion that is rotated at a constant rate for which the angular part is not a time-changed Brownian motion on the unit circle in the plane.

In Section 4 we construct the counterexample of a diffusion for which the angular part is a time-changed Brownian motion on the appropriate homogeneous space, but this Brownian motion is not independent of the radial part. Here the diffusion $(x_t)_{t\geq 0}$ has state space the manifold of 2×2 matrices that have a positive determinant. This diffusion can be represented via the well-known QR decomposition as the product of an autonomously Markov "radial" process $(T_t)_{t\geq 0}$ on the manifold of 2×2 upper-triangular matrices with positive diagonal entries and a time-changed "angular" process $(U_{R_t})_{t\geq 0}$, where $(U_t)_{t\geq 0}$ is a Brownian motion on the group $SO(2)$ of 2×2 orthogonal matrices with determinant one and the time-change $(R_t)_{t\geq 0}$ is adapted to the filtration of the radial process. However, the processes $(U_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are **not** independent.

We end this introduction by noting that analogous skew-product decompositions of superprocesses have been studied in [12, 4, 6]. The continuous Dawson-Watanabe (DW) superprocess is a rescaling limit of a system of branching Markov processes while the Fleming-Viot (FV) superprocess is a rescaling limit of the empirical distribution of a system of particles undergoing Markovian motion and multinomial resampling. It is shown in [4] that a FV process is a DW process conditioned to have total mass one. More generally, it is demonstrated in [12] that the distribution of the DW process conditioned on the path of its total mass process is equal to the distribution of a time-change of a FV process that has a suitable underlying time-inhomogeneous Markov motion. The latter result is extended to measure-valued processes that may have jumps in [6].

A sampling of other results involving skew-products can be found in [14, 9, 3, 1].

2 Example 1: Planar Brownian motion

Let $(x_t)_{t\geq 0}$ be a planar Brownian motion.

Following the notation of [10], we consider the following set-up.

1. Let $X = \mathbb{R}^2 \setminus \{(0,0)^T\}$.
2. Let K be the Lie group $SO(2)$ of 2×2 orthogonal matrices with determinant 1. This group acts on X by $A \mapsto Q^{-1}A$ for $Q \in K$ and $A \in X$.
3. The quotient of X with respect to the action of K can be identified with the positive x axis. Note that the orbits of K are just circles centered at the origin.
4. The isotropy subgroup of K for an element $x \in X$ is, as usual, the subgroup $\{k \in K : kx = x\}$. Since every element of X is an invertible matrix, this subgroup is always the trivial group consisting of just the identity. In particular, this subgroup is the same for every y in the interior of Y, as required in [10, pg 168]. We denote this subgroup by M.

It is straightforward to check that $(x_t)_{t\geq 0}$ satisfies all the assumptions of [10, Theorem 4]. We refer the reader to Sections 3 and 4 for details of how to verify these assumptions in more complicated examples.

Remark 2.1. In this example, $\dim(K/M) = 1$ and there is the skew-product decomposition (1.1).

3 Example 2: Rotated planar Brownian motion

Write $((U_t, V_t)^T)_{t\geq 0}$ for a planar Brownian started from $(x,y)^T$ (where T denotes transpose, so we are thinking of column vectors). The process $(x_t)_{t\geq 0} := \left((x_t^1, x_t^2)^T\right)_{t\geq 0}$ started from $(x,y)^T$ is defined by

$$\begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix} = \Theta_t \begin{pmatrix} U_t \\ V_t \end{pmatrix}, \tag{3.1}$$

where Θ_t is the matrix that represents rotating though an angle t. Thus,

$$\begin{aligned} x_t^1 &= \cos(t)U_t - \sin(t)V_t \\ x_t^2 &= \sin(t)U_t + \cos(t)V_t. \end{aligned} \tag{3.2}$$

Then,

$$\begin{aligned} dx_t^1 &= \cos(t)dU_t - U_t\sin(t)dt - \sin(t)dV_t - V_t\cos(t)dt \\ dx_t^2 &= \sin(t)dU_t + U_t\cos(t)dt + \cos(t)dV_t - V_t\sin(t)dt, \end{aligned}$$

which becomes

$$\begin{aligned} dx_t^1 &= \cos(t)dU_t - \sin(t)dV_t - Y_t dt \\ dx_t^2 &= \sin(t)dU_t + \cos(t)dV_t + X_t dt. \end{aligned}$$

If we define martingales $(B_t)_{t\geq 0}$ and $(C_t)_{t\geq 0}$ by

$$dB_t = \cos(t)dU_t - \sin(t)dV_t$$

and

$$dC_t = \sin(t)dU_t + \cos(t)dV_t,$$

then $[B]_t = t$, $[C]_t = t$ and $[B,C]_t = 0$, so the process $((B_t, C_t)^T)_{t\geq 0}$ is a planar Brownian motion and the process $\left((x_t^1, x_t^2)^T\right)_{t\geq 0}$ satisfies the SDE

$$\begin{aligned} dx_t^1 &= dB_t - Y_t dt \\ dx_t^2 &= dC_t + X_t dt. \end{aligned} \tag{3.3}$$

Following the notation of [10], we consider the following set-up.

1. Let $X = \mathbb{R}^2 \setminus \{(0,0)^T\}$.
2. Let K be the Lie group $SO(2)$ of 2×2 orthogonal matrices with determinant 1. This group acts on X by $A \mapsto Q^{-1}A$ for $Q \in K$ and $A \in X$.
3. The quotient of X with respect to the action of K can be identified with the positive x axis. Note that the orbits of K are just circles centered at the origin.
4. The isotropy subgroup of K for an element $x \in X$ is, as usual, the subgroup $\{k \in K : kx = x\}$. Since every element of X is an invertible matrix, this subgroup is always the trivial group consisting of just the identity. In particular, this subgroup is the same for every y in the interior of Y, as required in [10, pg 168]. We denote this subgroup by M.
5. Let $(x_t)_{t\geq 0}$ be the X-valued process that is defined in (3.1).

We now check that $(x_t)_{t\geq 0}$ satisfies all the assumptions of [10, Theorem 4]. These are as follows:

1. The process $(x_t)_{t\geq 0}$ is a Feller process with continuous sample paths.
2. The distribution of $(x_t)_{t\geq 0}$ is equivariant under the action of K. That is, for $k \in K$ the distribution of $(kx_t)_{t\geq 0}$ when $x_0 = x_*$ is the same as the distribution of $(x_t)_{t\geq 0}$ when $x_0 = kx_*$ [10, (2)].
3. The set Y is a submanifold of X that is transversal to the action of K [10, (3)].
4. For any $y \in Y^0$ (that is, the relative interior of Y – which in this case is just Y itself) T_yX, the tangent space of X at y, is the direct sum of tangent spaces $T_y(Ky) \bigoplus T_yY$ [10, (5)].
5. The homogeneous space K/M is irreducible; that is, the action of M on $T_o(K/M)$ (the tangent space at the coset o containing the identity) has no nontrivial invariant subspace [10, pg 177].

These assumptions are verified as follows:

1. This follows from the representation (3.3).
2. Since $\Theta_t \in SO(2)$ we have by (3.1) that for any $Q \in SO(2)$

$$Qx_t = Q\Theta_t \begin{pmatrix} U_t \\ V_t \end{pmatrix}.$$

   Since $Q\Theta_t \in SO(2)$ the condition holds because planar Brownian motion is equivariant under the action of $SO(2)$.
3. This is immediate.

4. $T_y(Ky) = \text{Span}\left\{(0,1)^T\right\}$ and $T_y(Y) = \text{Span}\left\{(1,0)^T\right\}$ so that

$$\mathbb{R}^2 = T_y X = T_y(Ky) \oplus T_y(Y)$$

5. The tangent space of Ky is one-dimensional so K/M is irreducible.

Consequently, $(x_t)_{t\geq 0}$ satisfies all the hypotheses of [10, Theorem 4].

Write $(R_t)_{t\geq 0}$ for the radial process

$$R_t := |(x_t^1, x_t^2)^T| = |(U_t, V_t)^T|,$$

and let $(L_t)_{t\geq 0}$ be the angular part of $((U_t, V_t)^T)_{t\geq 0}$. We can think of $(L_t)_{t\geq 0}$ as living on the unit circle in the complex plane. In polar coordinates, we have

$$x_t = (R_t, L_t \exp(it)).$$

By the usual skew-product for planar Brownian motion recalled in (1.1) we have that $L_t = \exp(iW_{T_t})$, where W is a standard Brownian motion on the line independent of R and T is a time-change defined from R. Therefore

$$x_t = (R_t, \exp(i(W_{T_t} + t))).$$

Proposition 3.1. *The process $(x_t)_{t\geq 0}$ cannot be written as*

$$x_t = (R_t, \exp(iZ_{S_t})),$$

where Z is a Brownian motion (possibly with drift) on the line independent of R and S is an increasing process adapted to the filtration generated by R.

Proof. If such a representation was possible, then we would have $Z_t = \tilde{Z}_t + at$ for some constant $a \in \mathbb{R}$, where $\tilde{Z}_t$ is a standard Brownian motion. This would imply that

$$\begin{aligned} \tilde{Z} &= W \\ S &= T \\ \exp(iaS_t) &= \exp(it). \end{aligned}$$

However, this is not possible: it would mean that

$$\exp(it) = \exp(iaT_t),$$

but T_t is certainly not a constant multiple of t for all $t \geq 0$. □

Remark 3.2. In this example K/M is the unit circle, which has dimension 1, and there is no skew-product decomposition. The angular part cannot be written as the time-change of any Brownian motion on the unit circle.

4 Example 3: A matrix valued process

Recall the well-known QR decomposition which says that any square matrix can be written as the product of an orthogonal matrix and an upper triangular matrix, and that this decomposition is unique for invertible matrices if we require the diagonal entries in the upper triangular matrix to be positive (see, for example, [7]). This decomposition is essentially a special case of the Iwasawa decomposition for semisimple Lie groups.

In the 2×2 case, uniqueness also holds for QR decomposition of invertible matrices if we require the orthogonal matrix to have determinant one and there are simple explicit formulae for the factors. Indeed, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.1}$$

and $\det A = ad - bc \neq 0$, then $A = \tilde{Q}\tilde{R}$, where

$$\tilde{Q} = \frac{1}{\sqrt{a^2 + c^2}} \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \in SO(2) \tag{4.2}$$

and

$$\tilde{R} = \begin{pmatrix} \sqrt{a^2 + c^2} & \frac{ab+cd}{\sqrt{a^2+c^2}} \\ 0 & \frac{ad-bc}{\sqrt{a^2+c^2}} \end{pmatrix}. \tag{4.3}$$

In this setting, we consider a 2×2 matrix of independent Brownian motions and time-change it to produce a Markov process with the property that if the determinant is positive at time 0, then it stays positive at all times. This ensures that uniqueness of the QR-factorization holds at all times and also that the time-changed process falls into the setting of [10].

Following the notation of [10], we consider the following set-up.

1. Let X be the manifold of 2×2 matrices over $\mathbb{R}$ with strictly positive determinant equipped with the topology it inherits as an open subset of $\mathbb{R}^{2\times 2} \cong \mathbb{R}^4$.
2. Let K be the Lie group $SO(2)$ of 2×2 orthogonal matrices with determinant 1. This group acts on X by $A \mapsto Q^{-1}A$ for $Q \in K$ and $A \in X$.
3. The quotient of X with respect to the action of K can, via the QR decomposition, be identified with the set Y of upper triangular 2×2 matrices with strictly positive diagonal entries.
4. The isotropy subgroup of K for an element $x \in X$ is, as usual, the subgroup $\{k \in K : kx = x\}$. Since every element of X is an invertible matrix, this subgroup is always the trivial group consisting of just the identity. In particular, this subgroup is the same for every y in the interior of Y, as required in [10, pg 168]. We denote this subgroup by M.
5. Let $(x_t)_{t\geq 0}$ be the X-valued process that satisfies the stochastic differential equation (SDE)

$$dx_t = \begin{pmatrix} dx_t^{1,1} & dx_t^{1,2} \\ dx_t^{2,1} & dx_t^{2,2} \end{pmatrix} = \begin{pmatrix} f(x_t)\, dA_t^{1,1} & f(x_t)\, dA_t^{1,2} \\ f(x_t)\, dA_t^{2,1} & f(x_t)\, dA_t^{2,2} \end{pmatrix}, \quad x_0 \in X, \tag{4.4}$$

   where $A_t^{1,1}$, $A_t^{1,2}$, $A_t^{2,1}$, and $A_t^{2,2}$ are independent standard one-dimensional Brownian motions, and $f(x) := \frac{\det(x)}{\mathrm{tr}(x'x)+1}$ with $\det$ and tr denoting the determinant and the trace. We establish below that (4.4) has a unique strong solution and that this solution does indeed take values in X.

It follows from the QR decomposition that $x_t = Q_t T_t$, where, in the terminology of [10], the "angular part" Q_t belongs to K and the "radial part" T_t belongs to Y. We will show that $(T_t)_{t\geq 0}$ is an autonomous diffusion on Y and that $Q_t = U_{R_t}$, where $(U_t)_{t\geq 0}$ is a Brownian motion on K and $(R_t)_{t\geq 0}$ is an increasing process adapted to the filtration generated by $(T_t)_{t\geq 0}$. However, we will establish that **it is not possible** to take the Brownian motion $(U_t)_{t\geq 0}$ to be independent of the process $(T_t)_{t\geq 0}$. This will contradict the claim of [10, Theorem 4] once we have also checked that the conditions of that result hold.

Note that if we consider f as a function on the space $\mathbb{R}^{2\times 2} \cong \mathbb{R}^4$ of all 2×2 matrices, then it has bounded partial derivatives, and hence it is globally Lipschitz continuous. Consequently, if we allow the initial condition in (4.4) to be an arbitrary element of $\mathbb{R}^{2\times 2}$, then the resulting SDE has a unique strong solution (see, for example, [13, Ch 5, Thm 11.2]). Moreover, the resulting process is a Feller process on $\mathbb{R}^{2\times 2}$ (see, for example, [13, Ch 5, Thm 22.5]).

We now check that $(x_t)_{t\geq 0}$ actually takes values in X. That is, we show that if x_0 has positive determinant, then x_t also has positive determinant for all $t \geq 0$. It follows from Itô's Lemma that

$$[\det(x.)]_t = \int_0^t \operatorname{tr}(x_s' x_s) f^2(x_s)\, ds,$$

$$[\operatorname{tr}(x'_. x.)]_t = \int_0^t 4\operatorname{tr}(x_s' x_s) f^2(x_s)\, ds,$$

and

$$[\det(x.), \operatorname{tr}(x'_. x.)] = \int_0^t 4\det(x_s) f^2(x_s)\, ds.$$

Thus, $((\det(x_t), \operatorname{tr}(x_t' x_t)))_{t\geq 0}$ is a Markov process and there exist independent standard one-dimensional Brownian motions $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$ such that

$$d\ \det(x_t) = \sqrt{\operatorname{tr}(x_t' x_t)} f(x_t)\, dB_t^1$$

and

$$\begin{aligned} d\operatorname{tr}(x_t' x_t) = {} & \frac{4\det(x_t) f(x_t)}{\sqrt{\operatorname{tr}(x_t' x_t)}}\, dB_t^1 + \sqrt{\frac{4\operatorname{tr}^2(x_t' x_t) - 16\det(x_t)^2}{\operatorname{tr}(x_t' x_t)}} f(x_t)\, dB_t^2 \\ & + 4 f^2(x_t)\, dt. \end{aligned}$$

When we substitute for f, the above equations transform into

$$d\ \det(x_t) = \frac{\det(x_t)\sqrt{\operatorname{tr}(x_t' x_t)}}{\operatorname{tr}(x_t' x_t) + 1}\, dB_t^1$$

and

$$\begin{aligned} d\operatorname{tr}(x_t' x_t) = {} & \frac{4(\det(x_t))^2}{\sqrt{\operatorname{tr}(x_t' x_t)}(\operatorname{tr}(x' x) + 1)}\, dB_t^1 + \sqrt{\frac{4\operatorname{tr}^2(x_t' x_t) - 16\det(x_t)^2}{\operatorname{tr}(x_t' x_t)}} \frac{\det(x_t)}{\operatorname{tr}(x_t' x_t) + 1}\, dB_t^2 \\ & + 4\left(\frac{\det(x_t)}{\operatorname{tr}(x_t' x_t) + 1}\right)^2 dt. \end{aligned}$$

In particular, the process $(\det(x_t))_{t\geq 0}$ is the stochastic exponential of the local martingale $(M_t)_{t\geq 0}$, where

$$M_t = \int_0^t \frac{\sqrt{\operatorname{tr}(x_s' x_s)}}{\operatorname{tr}(x_s' x_s) + 1}\, dB_s^1.$$

Since $x_0 \in X$, we have $\det(x_0) > 0$, and hence

$$\det(x_t) = \det(x_0) \exp\left(M_t - M_0 - \frac{1}{2}[M]_t\right)$$

is strictly positive for all $t \geq 0$. This shows that $(x_t)_{t\geq 0}$ takes values in X.

We now check that $(x_t)_{t\geq 0}$ satisfies all the assumptions of [10, Theorem 4]. These are as follows:

1. The process $(x_t)_{t\geq 0}$ is a Feller process with continuous sample paths.
2. The distribution of $(x_t)_{t\geq 0}$ is equivariant under the action of K. That is, for $k \in K$ the distribution of $(kx_t)_{t\geq 0}$ when $x_0 = x_*$ is the same as the distribution of $(x_t)_{t\geq 0}$ when $x_0 = kx_*$ [10, (2)].
3. The set Y is a submanifold of X that is transversal to the action of K [10, (3)].
4. For any $y \in Y^0$ (that is, the relative interior of Y – which in this case is just Y itself) T_yX, the tangent space of X at y, is the direct sum of tangent spaces $T_y(Ky) \bigoplus T_yY$ [10, (5)].
5. The homogeneous space K/M is irreducible; that is, the action of M on $T_o(K/M)$ (the tangent space at the coset o containing the identity) has no nontrivial invariant subspace [10, pg 177].

The verifications of (1)–(5) proceed as follows:

1. We have already observed that solutions of (4.4) with initial conditions in $\mathbb{R}^{2\times 2}$ form a Feller process and that this process stays in the open set X if it starts in X, and so $(x_t)_{t\geq 0}$ is a Feller process on X.
2. Suppose that $(x_t)_{t\geq 0}$ is a solution of (4.4) with $x_0 = x_*$ and $(\hat{x}_t)_{t\geq 0}$ is a solution of (4.4) with $\hat{x}_0 = kx_*$ for some $k \in K$. We have to show that if we set $\tilde{x}_t = k^{-1}\hat{x}_t$, then $(\tilde{x}_t)_{t\geq 0}$ has the same distribution as $(x_t)_{t\geq 0}$. Note that $\det \tilde{x}_t = \det \hat{x}_t$ and $\tilde{x}_t'\tilde{x}_t = \hat{x}_t'\hat{x}_t$, so that $f(\tilde{x}_t) = f(\hat{x}_t)$. Thus,
$$d\tilde{x}_t = f(\tilde{x}_t)k^{-1}\begin{pmatrix} dA_t^{1,1} & dA_t^{1,2} \\ dA_t^{2,1} & dA_t^{2,2} \end{pmatrix}, \quad \tilde{x}_0 = x_*.$$
Now the columns of the matrix
$$\begin{pmatrix} A_t^{1,1} & A_t^{1,2} \\ A_t^{2,1} & A_t^{2,2} \end{pmatrix}$$
are independent standard two-dimensional Brownian motions, and so the same is true of the columns of the matrix
$$k^{-1}\begin{pmatrix} A_t^{1,1} & A_t^{1,2} \\ A_t^{2,1} & A_t^{2,2} \end{pmatrix}$$
by the equivariance of standard two-dimensional Brownian motion under the action of $SO(2)$. Hence,
$$k^{-1}\begin{pmatrix} A_t^{1,1} & A_t^{1,2} \\ A_t^{2,1} & A_t^{2,2} \end{pmatrix} = \begin{pmatrix} \alpha_t^{1,1} & \alpha_t^{1,2} \\ \alpha_t^{2,1} & \alpha_t^{2,2} \end{pmatrix},$$
where $(\alpha_t^{1,1})_{t\geq 0}$, $(\alpha_t^{1,2})_{t\geq 0}$, $(\alpha_t^{2,1})_{t\geq 0}$, and $(\alpha_t^{2,2})_{t\geq 0}$ are independent standard Brownian motions. Since,
$$d\tilde{x}_t = f(\tilde{x}_t)\begin{pmatrix} d\alpha_t^{1,1} & d\alpha_t^{1,2} \\ d\alpha_t^{2,1} & d\alpha_t^{2,2} \end{pmatrix}, \quad \tilde{x_0} = x_0,$$
the existence and uniqueness of strong solutions to (4.4) establishes that the distributions of $(x_t)_{t\geq 0}$ and $(\tilde{x}_t)_{t\geq 0}$ are equal.
3. It follows from the existence of the QR decomposition for invertible matrices that X is the union of the orbits Ky for $y \in Y$, and it follows from the uniqueness of the decomposition for such matrices that the orbit Ky intersects Y only at y.

4. Since the tangent space of $K = SO(2)$ at the identity is the vector space of 2×2 skew-symmetric matrices and the tangent space of Y at the identity is the vector space of 2×2 upper-triangular matrices, we have to show that if W is a fixed invertible upper-triangular 2×2 matrix and M is a fixed 2×2 matrix, then

$$M = SW + V$$

for a unique skew-symmetric 2×2 matrix S and unique upper-triangular 2×2 matrix V. Let

$$M := \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \quad \text{and} \quad W := \begin{pmatrix} w_{11} & w_{12} \\ 0 & w_{22} \end{pmatrix}.$$

It is immediate that

$$S = \begin{pmatrix} 0 & -\frac{m_{21}}{w_{11}} \\ \frac{m_{21}}{w_{11}} & 0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} m_{11} & \frac{m_{12}w_{11}+m_{21}w_{22}}{w_{11}} \\ 0 & \frac{m_{22}w_{11}-m_{21}w_{12}}{w_{11}} \end{pmatrix}.$$

5. We have already noted that the tangent space of K at the identity is the vector space of skew-symmetric 2×2 matrices. This vector space is one-dimensional and so this condition holds trivially.

We have now shown that $(x_t)_{t \geq 0}$ satisfies all the hypotheses of [10, Theorem 4]. However, we have the following result.

Proposition 4.1. *In the decomposition $x_t = Q_t T_t$ the Y-valued process $(T_t)_{t\geq 0}$ is Markov and the K-valued process $(Q_t)_{t\geq 0}$ may be written as $Q_t = U_{R_t}$, where $(U_t)_{t\geq 0}$ is a K-valued Brownian motion and $(R_t)_{t\geq 0}$ is an increasing continuous process such that $R_0 = 0$ and $R_t - R_s$ is $\sigma\{T_u : s \leq u \leq t\}$-measurable for $0 \leq s < t < \infty$. However, there is no such representation in which $(T_t)_{t\geq 0}$ and $(U_t)_{t\geq 0}$ are independent.*

Proof. For all $t \geq 0$ we have $x_t = Q_t T_t$, where

$$Q_t = \frac{1}{\sqrt{(x_t^{11})^2 + (x_t^{21})^2}} \begin{pmatrix} x_t^{11} & -x_t^{21} \\ x_t^{21} & x_t^{11} \end{pmatrix} \in K$$

and

$$T_t = \begin{pmatrix} \sqrt{(x_t^{11})^2 + (x_t^{21})^2} & \frac{x_t^{11}x_t^{12}+x_t^{21}x_t^{22}}{\sqrt{(x_t^{11})^2+(x_t^{21})^2}} \\ 0 & \frac{\det(x_t)}{\sqrt{(x_t^{11})^2+(x_t^{21})^2}} \end{pmatrix} \in Y.$$

Note that $\det(x_t) = \det(T_t)$ and $\operatorname{tr}(x_t' x_t) = \operatorname{tr}(T_t' T_t)$, and so $f(x_t) = f(T_t)$. Note also that the complex-valued process $(x_t^{11} + i x_t^{21})_{t\geq 0}$ is an isotropic complex local martingale in the sense of [8, Ch 18], that is

$$[x^{11}] = [x^{21}]$$

and

$$[x^{22}, x^{21}] = 0.$$

In our case

$$d[x^{11}]_t = d[x^{21}]_t = f^2(T_t)\, dt.$$

By [8, Thm 18.5], $(\log(x_t^{11} + i x_t^{21})_{t\geq 0}$ is a well-defined isotropic complex local martingale that can be written as

$$\log(x_t^{11} + i x_t^{21}) = \log\left(T_t^{11}\right) + i\theta_t,$$

where

$$d[\theta]_t = d[\log(T^{11})]_t = \frac{1}{(T_t^{11})^2} d[x^{11}]_t = \left(\frac{f(T_t)}{T_t^{11}}\right)^2 dt.$$

By the classical result of Dambis, Dubins and Schwarz (see, for example, [8, Thm 18.4]), there exists a standard complex Brownian motion $(\tilde{B}_t + iB_t)_{t\geq 0}$ such that $\log(x_t^{11} + ix_t^{21}) = \tilde{B}_{R_t} + iB_{R_t}$, where

$$R_t = \int_0^t \left(\frac{f(T_s)}{T_s^{11}}\right)^2 ds, \quad t \geq 0.$$

So, $\theta_t = B_{R_t}$ and $\log(T_t^{11}) = \tilde{B}_{R_t}$. Hence,

$$\frac{x_t^{11} + ix_t^{21}}{\sqrt{(x_t^{11})^2 + (x_t^{21})^2}} = (\cos(\theta_t) + i\sin(\theta_t))$$

and

$$Q_t = \begin{pmatrix} \cos(B_{R_t}) & -\sin(B_{R_t}) \\ \sin(B_{R_t}) & \cos(B_{R_t}) \end{pmatrix}.$$

Consequently, $Q_t = U_{R_t}$, where

$$U_t = \begin{pmatrix} \cos(B_t) & -\sin(B_t) \\ \sin(B_t) & \cos(B_t) \end{pmatrix},$$

and $(B_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion.

Note that $(U_t)_{t\geq 0}$ is certainly a Brownian motion on $K = SO(2)$, and so we have uniquely identified the K-valued Brownian motion $(U_t)_{t\geq 0}$ and the increasing process $(R_t)_{t\geq 0}$ that appear in the claimed decomposition of $(x_t)_{t\geq 0}$.

To complete the proof, it suffices to suppose that $(U_t)_{t\geq 0}$ is independent of $(T_t)_{t\geq 0}$ and obtain a contradiction. An application of Itô's Lemma shows that the entries of $(U_t)_{t\geq 0}$ satisfy the system of SDEs

$$\begin{aligned}
dU_t^{1,1} &= -U_t^{2,1}\, dB_t - \frac{1}{2} U_t^{1,1}\, dt \\
dU_t^{2,1} &= U_t^{1,1}\, dB_t - \frac{1}{2} U_t^{2,1}\, dt \\
dU_t^{1,2} &= -U_t^{1,1}\, dB_t + \frac{1}{2} U_t^{2,1}\, dt = -dU_t^{2,1} \\
dU_t^{2,2} &= -U_t^{2,1}\, dB_t - \frac{1}{2} U_t^{1,1}\, dt = dU_t^{1,1}.
\end{aligned}$$

We apply Proposition 4.2 below to each of the four SDEs in the system describing $(U_t)_{t\geq 0}$, with, in the notation of that result, (ζ_t, H_t, K_t) being the respective triples $(U_t^{1,1}, U_t^{2,1}, U_t^{1,1})$, $(U_t^{2,1}, U_t^{1,1}, U_t^{2,1})$, $(U_t^{1,2}, U_t^{1,1}, U_t^{2,1})$, and $(U_t^{2,2}, U_t^{2,1}, U_t^{1,1})$. In each of the four applications, we let

- $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by $(U_t)_{t\geq 0}$,
- $(\mathcal{G}_t)_{t\geq 0}$ be the filtration generated by $(T_t)_{t\geq 0}$,
- $\beta_t = B_t$,
- $\rho_t = R_t$,
- $J_t = \left(\frac{f(T_t)}{T_t^{11}}\right)^2$,
- $\gamma_t = W_t = \int_0^t \sqrt{\frac{1}{R'_s}} dB_{R_s}$.

Let $\mathcal{H}_t = \mathcal{F}_{\rho_t} \vee \mathcal{G}_t$, $t \geq 0$, as in the Proposition 4.2. It follows by the assumed independence of $(U_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$, part (iii) of Proposition 4.2, and equation (4.5) that the entries of the time-changed process $Q_t = U_{R_t}$ satisfy the system of SDEs

$$\begin{aligned}
dQ_t^{1,1} &= -Q_t^{2,1}\sqrt{R_t'}\,dW_t - \frac{1}{2}Q_t^{1,1}R_t'\,dt = -Q_t^{2,1}\frac{f(T_t)}{T_t^{11}}\,dW_t - \frac{1}{2}Q_t^{1,1}\left(\frac{f(T_t)}{T_t^{11}}\right)^2 dt \\
dQ_t^{2,1} &= Q_t^{1,1}\sqrt{R_t'}\,dW_t - \frac{1}{2}Q_t^{2,1}R_t'\,dt = Q_t^{1,1}\frac{f(T_t)}{T_t^{11}}\,dW_t - \frac{1}{2}Q_t^{2,1}\left(\frac{f(T_t)}{T_t^{11}}\right)^2 dt \\
dQ_t^{1,2} &= -dQ_t^{2,1} = Q_t^{1,1}\sqrt{R_t'}\,dW_t - \frac{1}{2}Q_t^{2,1}R_t'\,dt = Q_t^{1,1}\frac{f(T_t)}{T_t^{11}}\,dW_t - \frac{1}{2}Q_t^{2,1}\left(\frac{f(T_t)}{T_t^{11}}\right)^2 dt \\
dQ_t^{2,2} &= dQ_t^{1,1} = -Q_t^{2,1}\sqrt{R_t'}\,dW_t - \frac{1}{2}Q_t^{1,1}R_t' = -Q_t^{2,1}\frac{f(T_t)}{T_t^{11}}\,dW_t - \frac{1}{2}Q_t^{1,1}\left(\frac{f(T_t)}{T_t^{11}}\right)^2 dt.
\end{aligned}$$

Set

$$\begin{aligned}
dw_t^1 &= \frac{x_t^{11}}{\sqrt{(x_t^{11})^2+(x_t^{21})^2}}\,dA_t^{11} + \frac{x_t^{21}}{\sqrt{(x_t^{11})^2+(x_t^{21})^2}}\,dA_t^{21} \\
dw_t^2 &= \frac{-x_t^{21}}{\sqrt{(x_t^{11})^2+(x_t^{21})^2}}\,dA_t^{11} + \frac{x_t^{11}}{\sqrt{(x_t^{11})^2+(x_t^{21})^2}}\,dA_t^{21} \\
dw_t^3 &= \frac{x_t^{11}}{\sqrt{(x_t^{11})^2+(x_t^{21})^2}}\,dA_t^{12} + \frac{x_t^{21}}{\sqrt{(x_t^{11})^2+(x_t^{21})^2}}\,dA_t^{22} \\
dw_t^4 &= \frac{-x_t^{21}}{\sqrt{(x_t^{11})^2+(x_t^{21})^2}}\,dA_t^{12} + \frac{x_t^{11}}{\sqrt{(x_t^{11})^2+(x_t^{21})^2}}\,dA_t^{22}.
\end{aligned}$$

The processes $(w_t^i)_{t\geq 0}$ are local martingales with $[w_t^i, w_t^j]_t = \delta_{ij}t$, and thus they are independent standard Brownian motions. An application of Itô's Lemma shows that $(T_t)_{t\geq 0}$ is a diffusion satisfying the following system of SDEs.

$$\begin{aligned}
dT_t^{11} &= f(T_t)\,dw_t^1 + \frac{f^2(T_t)}{T_t^{11}}\,dt \\
dT_t^{12} &= \frac{T_t^{22}f(T_t)}{T_t^{11}}\,dw_t^2 + f(T_t)dw_t^3 - \frac{T_t^{12}f^2(T_t)}{2(T_t^{11})^2}\,dt \\
dT_t^{22} &= \frac{T_t^{12}f(T_t)}{T_t^{11}}\,dw_t^2 + f(T_t)dw_t^4 - \frac{T_t^{22}f^2(T_t)}{2(T_t^{11})^2}\,dt.
\end{aligned}$$

The assumed independence of the processes $(U_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ and part (iv) of Proposition 4.2 give that $[Q^{i,j}, T^{k,l}] \equiv 0$ for all i, j, k and l. It follows from Itô's Lemma that

$$d(Q_tT_t)^{1,1} = dN_t + \frac{Q_t^{1,1}f^2(T_t)}{T_t^{1,1}}\left(1 - \frac{1}{2T_t^{1,1}}\right)dt,$$

where $(N_t)_{t\geq 0}$ is a continuous local martingale for the filtration $(\mathcal{H}_t)_{t\geq 0}$. This, however, is not possible because $(Q_tT_t)^{1,1} = x_t^{1,1}$ and the process $(x_t^{1,1})_{t\geq 0}$ is a continuous local martingale for the filtration $(\mathcal{H}_t)_{t\geq 0}$.

□

We required the following proposition that collects together some simple facts about time-changes.

Proposition 4.2. *Consider two filtrations $(\mathcal{F}_t)_{t\geq 0}$ and $(\mathcal{G}_t)_{t\geq 0}$ on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Set $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$ and $\mathcal{G}_\infty = \bigvee_{t\geq 0} \mathcal{G}_t$. Assume that the sub-$\sigma$-fields $\mathcal{F}_\infty$ and $\mathcal{G}_\infty$ are independent. Suppose that*

$$\zeta_t = \zeta_0 + \int_0^t H_s \, d\beta_s + \int_0^t K_s \, ds,$$

where ζ_0 is $\mathcal{F}_0$-measurable, the integrands $(H_t)_{t\geq 0}$ and $(K_t)_{t\geq 0}$ are $(\mathcal{F}_t)_{t\geq 0}$-adapted, and $(\beta_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion. Suppose further that $\rho_t = \int_0^t J_s \, ds$, where $(J_t)_{t\geq 0}$ is a nonnegative, $(\mathcal{G}_t)_{t\geq 0}$-adapted process such that ρ_t is finite for all $t \geq 0$ almost surely. For $t \geq 0$ put

$$\mathcal{F}_{\rho_t} = \sigma\{L_{s\wedge\rho_t} : s \geq 0 \text{ and } L \text{ is } (\mathcal{F}_t)_{t\geq 0}\text{-optional}\}.$$

Set $\mathcal{H}_t = \mathcal{F}_{\rho_t} \vee \mathcal{G}_t$, $t \geq 0$. Then the following hold.

(i) The process $(\beta_{\rho_t})_{t\geq 0}$ is a continuous local martingale for the filtration $(\mathcal{H}_t)_{t\geq 0}$ with quadratic variation $[\beta_{\rho_\cdot}]_t = \rho_t$.

(ii) The process $(\gamma_t)_{t\geq 0}$, where

$$\gamma_t = \int_0^t \sqrt{\frac{1}{J_s}} \, d\beta_{\rho_s},$$

is a Brownian motion for the filtration $(\mathcal{H}_t)_{t\geq 0}$.

(iii) If $\xi_t = \zeta_{\rho_t}$, $t \geq 0$, then

$$\xi_t = \xi_0 + \int_0^t H_{\rho_s} \sqrt{J_s} \, d\gamma_s + \int_0^t K_{\rho_s} J_s \, ds.$$

(iv) If $(\eta_t)_{t\geq 0}$ is a continuous local martingale for the filtration $(\mathcal{G}_t)_{t\geq 0}$, then $(\eta_t)_{t\geq 0}$ is also a continuous local martingale for the filtration $(\mathcal{H}_t)_{t\geq 0}$ and $[\eta, \gamma] \equiv 0$.

Remark 4.3. In this example $K/M = SO(2)$ has dimension 1 and there is a type of skew-product decomposition. The angular part can indeed be written as a time-change depending on the radial part of a Brownian motion on $SO(2)$. However, we cannot take this Brownian motion to be independent of the radial part.

5 Open problem

The apparent counterexamples to [10, Theorem 4] arise in Sections 3 and 4 because K/M is one-dimensional and hence trivially irreducible. When K/M has dimension greater than 1, irreducibility implies the nonexistence of a nonzero M-invariant tangent vector and it is this latter property that is actually used in the proof of [10, Theorem 4]. In the examples in Sections 2, 3 and 4 the group M is the trivial group consisting of just the identity and there certainly are nonzero M-invariant tangent vector.

Therefore, in view of the three examples we presented and Remarks 2.1, 3.2, 4.3 we propose the following open problem.

Question 5.1. Suppose that $(x_t)_{t\geq 0}$ is a continuous Markov process with state space a smooth manifold X and distribution that is equivariant under the smooth action of a Lie group K on X so that we can decompose $(x_t)_{t\geq 0}$ into a radial part $(y_t)_{t\geq 0}$ that is a Markov process on the submanifold Y which is transversal to the orbits of K and an angular part $(z_t)_{t\geq 0}$ that is a process on the homogeneous space K/M. Suppose further that $\dim(K/M) = 1$.

1. When can we write $z_t = B_{a_t}$ where $(B_t)_{t\geq 0}$ is a Brownian motion on K/M and $(a_t)_{t\geq 0}$ is a time-change that is adapted to the filtration generated by $(y_t)_{t\geq 0}$.
2. Under which conditions can we take the Brownian motion $(B_t)_{t\geq 0}$ to be independent of $(x_t)_{t\geq 0}$?

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