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# Sharp lower bounds on the least singular value of a random matrix without the fourth moment condition*

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**Abstract**

We obtain non-asymptotic lower bounds on the least singular value of $\mathbf{X}_{pn}^\top/\sqrt{n}$, where $\mathbf{X}_{pn}$ is a $p\times n$ random matrix whose columns are independent copies of an isotropic random vector $X_p$ in $\mathbb{R}^p$. We assume that there exist $M>0$ and $\alpha\in(0,2]$ such that $\mathbb{P}(|(X_p,v)|>t)\leqslant M/t^{2+\alpha}$ for all $t>0$ and any unit vector $v\in\mathbb{R}^p$. These bounds depend on $y=p/n$, $\alpha$, $M$ and are asymptotically optimal up to a constant factor.

**Keywords:** Random matrices; Singular values; Heavy-tailed distributions.
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## 1 Introduction

In this paper we obtain sharp lower bounds on the least singular value of a random matrix with independent heavy-tailed rows.

For precise statements, we need to introduce some notation. Let $X_p$ be an isotropic random vector in $\mathbb{R}^p$, i.e. $\mathbb{E}X_pX_p^\top=I_p$ for a $p\times p$ identity matrix $I_p$. Let also $\mathbf{X}_{pn}$ be a $p\times n$ random matrix whose columns $\{X_{pk}\}_{k=1}^n$ are independent copies of $X_p$. Denote by $s_p(n^{-1/2}\mathbf{X}_{pn}^\top)$ the least singular value of the matrix $n^{-1/2}\mathbf{X}_{pn}^\top$.

The celebrated Bai-Yin theorem states that, with probability one,

$$s_p(n^{-1/2}\mathbf{X}_{pn}^\top)=1-\sqrt{y}+o(1)$$

when $n\to\infty$, $p=p(n)$ satisfies $p/n\to y\in(0,1)$, and the entries of $X_p$ are independent copies of a random variable $\xi$ with $\mathbb{E}\xi=0$, $\mathbb{E}\xi^2=1$, and $\mathbb{E}\xi^4<\infty$. In [5], Tikhomirov extended this result to the case $\mathbb{E}\xi^4=\infty$. Several authors have studied non-asymptotic versions of this theorem, relaxing the independence assumption, and obtained bounds of the form

$$s_p(n^{-1/2}\mathbf{X}_{pn}^\top)\geqslant 1-Cy^a|\log y|^b$$

that hold with large probability for some $C,a,b>0$ and all small enough $y=p/n$. See papers [2], [3], [4], and [6]. For general isotropic random vectors $X_p$ with dependent entries not having finite fourth moments, the optimal values of $a$ and $b$ are unknown. Assuming that there exist $M>0$ and $\alpha\in(0,2]$ such that

$$\mathbb{P}(|(X_p,v)|>t)\leqslant\frac{M}{t^{2+\alpha}}\quad\text{for all } t>0 \text{ and any unit (in the } l^2\text{-norm) vector } v\in\mathbb{R}^p,\quad(1.1)$$

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we derive the optimal values of $a$ and $b$ in this paper.

The paper is organized as follows. Section 2 contains the main results of the paper. Section 3 deals with the proofs. An Appendix with proofs of auxiliary results is given in Section 4.

## 2 Main results

Our main lower bound is a corollary of Theorem 2.1 in [6]. It is given below.

**Theorem 2.1.** *Let $C \geqslant 1$ and $n > p \geqslant 1$. If (1.1) holds for $M = C^{\alpha/2}$ and some $\alpha \in (0, 2]$, then, with probability at least $1 - e^{-p}$,*

$$s_p(n^{-1/2}\mathbf{X}_{pn}^\top) \geqslant 1 - 14 \begin{cases} K_\alpha (Cy)^{\alpha/(2+\alpha)}, & \alpha \in (0,2) \\ \sqrt{Cy \log(C/y)} & \alpha = 2 \text{ and } C/y > e \\ \sqrt{Cy}, & \alpha = 2 \text{ and } C/y \leqslant e \end{cases}$$

*where $y = p/n$ and $K_\alpha = 1/(\alpha(1 - \alpha/2))^{2/(2+\alpha)}$.*

The next theorem contains our main upper bound for a class of random vectors

$$X_p = \eta Z_p \quad \text{for } Z_p = (z_1, \ldots, z_p) \text{ with i.i.d. entries } \{z_i\}_{i=1}^p \text{ independent of } \eta. \tag{2.1}$$

**Theorem 2.2.** *Let (2.1) hold for each $p \geqslant 1$, where $\{z_i\}_{i=1}^\infty$ are independent copies of a random variable $z$ with $\mathbb{E}z = 0$, $\mathbb{E}z^2 = 1$, and $\eta$ is a random variable with $\mathbb{E}\eta^2 = 1$. If there exist $\alpha \in (0, 2]$ and $C > 0$ such that*

$$\mathbb{P}(|\eta| > t) \geqslant \frac{C^{\alpha/2}}{t^{2+\alpha}} \quad \text{for all large enough } t > 0, \tag{2.2}$$

*then, for each small enough $y > 0$,*

$$s_p(n^{-1/2}\mathbf{X}_{pn}^\top) \leqslant 1 + o(1) - \frac{1}{2} \begin{cases} K_\alpha (Cy)^{\alpha/(2+\alpha)}, & \alpha \in (0,2) \\ \sqrt{Cy \log(C/y)}, & \alpha = 2 \end{cases}$$

*almost surely as $n \to \infty$, where $p = p(n) = yn + o(n)$ and $K_\alpha$ is given in Theorem 2.1.*

Theorem 2.2 and the next proposition show that, when $y$ is small enough, the lower bounds in Theorem 2.1 are asymptotically optimal up to a constant factor (equal to 14).

**Proposition 2.3.** *For any given $C > 1/4$ and $\alpha \in (0, 2]$, there exists a random variable $\eta$ such that $\mathbb{E}\eta^2 = 1$, (2.2) holds, and*

$$\mathbb{P}(|(X_p, v)| > t) \leqslant \frac{(\kappa C)^{\alpha/2}}{t^{2+\alpha}} \quad \text{for all } t > 0 \text{ and any unit vector } v \in \mathbb{R}^p,$$

*where $X_p = \eta Z_p$, $Z_p$ is a standard normal vector in $\mathbb{R}^p$ that is independent of $\eta$, and $\kappa > 0$ is a universal constant.*

The proof of Proposition 2.3 is given at the end of the paper, before the Appendix.

## 3 Proofs

We will use below the following fact. By definition, $s_p(n^{-1/2}\mathbf{X}_{pn}^\top)$ is the square root of $\lambda_p(n^{-1}\mathbf{X}_{pn}\mathbf{X}_{pn}^\top)$, where $\lambda_p(A)$ is the least eigenvalue of a $p \times p$ matrix $A$. In addition,

$$\text{if } a \geqslant 1 - b \text{ for some } a, b \geqslant 0, \text{ then } \sqrt{a} \geqslant 1 - b.$$

Moreover, if $a \leqslant 1 - b$ for some $a, b \geqslant 0$, then $\sqrt{a} \leqslant 1 - b/2$. Thus, to prove Theorems 2.1 and 2.2 we need to derive appropriate lower and upper bounds only for $\lambda_p(n^{-1}\mathbf{X}_{pn}\mathbf{X}_{pn}^\top)$.

*Proof of Theorem 2.1.* By Theorem 2.1 in [6], for all $a > 0$ and $y = p/n \in (0,1)$,

$$\lambda_p(n^{-1}\mathbf{X}_{pn}\mathbf{X}_{pn}^\top) \geqslant c_p(a) - \frac{C_p(a)}{a} - 5ay + \frac{\sqrt{C_p(2a)}Z}{\sqrt{n}},$$

where $Z = Z(p,n,a)$ is a random variable with $\mathbb{E}Z = 0$ and $\mathbb{P}(Z < -t) \leqslant e^{-t^2/2}$, $t > 0$,

$$c_p(a) = \inf \mathbb{E}\min\{(X_p,v)^2, a\} \quad \text{and} \quad C_p(a) = \sup \mathbb{E}(X_p,v)^2 \min\{(X_p,v)^2, a\}$$

with $\inf$ and $\sup$ taken over all unit vectors $v \in \mathbb{R}^p$.

Since $\mathbb{P}\big(Z < -\sqrt{2p}\big) \leqslant e^{-p}$ and $y = p/n$, we have, with probability at least $1 - e^{-p}$,

$$\lambda_p(n^{-1}\mathbf{X}_{pn}\mathbf{X}_{pn}^\top) \geqslant c_p(a) - \frac{C_p(a)}{a} - 5ay - \sqrt{2yC_p(2a)}. \tag{3.1}$$

To estimate $c_p(a)$ and $C_p(a)$, we will use the following lemma that is proved in the Appendix.

**Lemma 3.1.** *Let $a > 0$, $X_p$ be an isotropic random vector in $\mathbb{R}^p$, and (1.1) hold for some $M > 0$ and $\alpha \in (0,2]$. If $\alpha \in (0,2)$, then*

$$c_p(a) \geqslant 1 - \frac{2M}{\alpha}\, a^{-\alpha/2} \quad \textit{and} \quad C_p(a) \leqslant (2/\alpha + 4/(2-\alpha))Ma^{1-\alpha/2}.$$

*In addition, if $\alpha = 2$, then*

$$c_p(a) \geqslant 1 - \frac{M}{a} \quad \textit{and} \quad C_p(a) \leqslant 2M + M\log(a^2/M)\, I(a^2 > M).$$

First, assume that $\alpha \in (0,2)$. Using (1.1) and Lemma 3.1, we get

$$c_p(a) - \frac{C_p(a)}{a} \geqslant 1 - \left[\frac{4}{\alpha} + \frac{4}{2-\alpha}\right]\frac{M}{a^{\alpha/2}} = 1 - \frac{8Ma^{-\alpha/2}}{\alpha(2-\alpha)}.$$

Taking

$$a = \left[\frac{2My^{-1}}{\alpha(2-\alpha)}\right]^{2/(2+\alpha)} = K_\alpha(M/y)^{2/(2+\alpha)},$$

we have

$$ay = \frac{2Ma^{-\alpha/2}}{\alpha(2-\alpha)} \quad \text{and} \quad c_p(a) - \frac{C_p(a)}{a} \geqslant 1 - 4ay.$$

In addition,

$$\frac{C_p(2a)}{2a} \leqslant \left[\frac{2}{\alpha} + \frac{4}{2-\alpha}\right]M(2a)^{-\alpha/2} \leqslant \left[\frac{4}{\alpha} + \frac{4}{2-\alpha}\right]Ma^{-\alpha/2} = \frac{8Ma^{-\alpha/2}}{\alpha(2-\alpha)} = 4ay$$

and

$$\sqrt{2yC_p(2a)} \leqslant \sqrt{2y(8a^2y)} = 4ay = 4K_\alpha(M^{2/\alpha}y)^{\alpha/(2+\alpha)}.$$

Since $C = M^{2/\alpha}$, we infer from (3.1) that, with probability at least $1 - e^{-p}$,

$$\lambda_p(n^{-1}\mathbf{X}_{pn}\mathbf{X}_{pn}^\top) \geqslant 1 - 13ay = 1 - 13K_\alpha(Cy)^{\alpha/(2+\alpha)}.$$

Thus we get the desired lower bounds for $\alpha \in (0,2)$.

Suppose now $\alpha = 2$. Then $M = C^{\alpha/2} = C \geqslant 1$ and $\log(a^2/C) \leqslant \log(a^2)$ for any $a > 0$. Lemma 3.1 implies that

$$c_p(a) - \frac{C_p(a)}{a} \geqslant 1 - \frac{3C + C\log(a^2)I(a^2 > C)}{a}.$$

Consider two possibilities $\log(C/y) > 1$ and $\log(C/y) \leqslant 1$.

Assuming that $\log(C/y) \leqslant 1$ and taking $a = \sqrt{C/y}$, we have $a^2 > C$, $\log(a^2) \leqslant 1$, and

$$\frac{3C + C\log(a^2)}{a} \leqslant \frac{4C}{a} = 4\sqrt{Cy}.$$

Additionally, we get $5ay = 5\sqrt{Cy}$,

$$C_p(2a) \leqslant 2C + C\log(4a^2) \leqslant (3+\log 4)C \leqslant 9C/2 \quad \text{and} \quad \sqrt{2yC_p(2a)} \leqslant 3\sqrt{Cy}.$$

As a result, we conclude from (3.1) that, with probability at least $1 - e^{-p}$,

$$\lambda_p(n^{-1}\mathbf{X}_{pn}\mathbf{X}_{pn}^\top) \geqslant 1 - 12\sqrt{Cy}.$$

Suppose $\log(C/y) > 1$. Set $a = \sqrt{(C/y)\log(C/y)}$. Then $a^2 > C$, $\sqrt{C/y} \leqslant a \leqslant C/y$, and

$$\frac{3C + C\log(a^2)}{a} \leqslant \frac{3C}{\sqrt{C/y}} + \frac{C\log(C/y)^2}{a} \leqslant 5\sqrt{Cy\log(C/y)}.$$

Similarly, $C_p(2a) \leqslant 2C + C\log(4a^2) \leqslant 7C/2 + C\log(a^2) \leqslant (7/2+2)C\log(C/y)$ and

$$\sqrt{2yC_p(2a)} \leqslant 4\sqrt{Cy\log(C/y)}.$$

Noting that $5ay = 5\sqrt{Cy\log(C/y)}$, we infer that, with probability at least $1 - e^{-p}$,

$$\lambda_p(n^{-1}\mathbf{X}_{pn}\mathbf{X}_{pn}^\top) \geqslant 1 - 14\sqrt{Cy}.$$

Thus we have proved the theorem.

□

*Proof of Theorem 2.2.* We will use the following lemma (for the proof, see the Appendix).

**Lemma 3.2.** *Under the conditions of Theorem 2.2,*

$$\lambda_p(n^{-1}\mathbf{X}_{pn}\mathbf{X}_{pn}^\top) \leqslant \max\{0, \sup_{s>0}\lambda(s)\} + o(1) \quad \textit{a.s.}, \quad n\to\infty, \tag{3.2}$$

*where* $p = p(n)$, $p/n \to y \in (0,1)$, *and* $\lambda(s) = -y/s + \mathbb{E}\eta^2/(1+s\eta^2)$.

We estimate $\lambda = \lambda(s)$ given in Lemma 3.2 as follows. Set $\zeta = \eta^2$. Since $\mathbb{E}\zeta = 1$,

$$\lambda(s) + \frac{y}{s} = \mathbb{E}\frac{\zeta}{1+s\zeta} = 1 + \mathbb{E}\Big(\frac{\zeta}{1+s\zeta} - \zeta\Big) = 1 - \mathbb{E}\frac{s\zeta^2}{1+s\zeta}.$$

It follows from the inequality $x/(1+x) \geqslant \min\{x,1\}/2$, $x \geqslant 0$, and (4.1) that

$$\mathbb{E}\frac{s\zeta^2}{1+s\zeta} \geqslant \frac{1}{2}\,\mathbb{E}\zeta\min\{s\zeta,1\} = \frac{1}{2s}\,[\mathbb{E}(s\zeta-1)I(s\zeta>1) + \mathbb{E}\min\{(s\zeta)^2,1\}].$$

As a result, for all $s > 0$, we get the following upper bound

$$\lambda(s) \leqslant 1 - \frac{y}{s} - \frac{1}{2s}\,[\mathbb{E}(s\zeta-1)I(s\zeta>1) + \mathbb{E}\min\{(s\zeta)^2,1\}]. \tag{3.3}$$

Recall also that, by (2.2) and the definition of $\zeta$ $(=\eta^2)$, there exists $t_0 \geqslant 1$ such that

$$\mathbb{P}(\zeta > t) \geqslant \frac{C^{\alpha/2}}{t^{1+\alpha/2}} \quad \text{for all } t \geqslant t_0. \tag{3.4}$$

As in the proof of Lemma 3.2 (see the Appendix), we get that

$$\lambda'(s) = (y - h(s))/s^2, \quad s > 0,$$

where $h(s) = \mathbb{E}(s\zeta)^2/(1+s\zeta)^2$ is a continuous strictly increasing function on $\mathbb{R}_+$ with $h(0) = 0$ and $h(\infty) = \mathbb{P}(\zeta > 0) > 0$. Hence, if $y < \mathbb{P}(\zeta > 0)$, $\lambda(s)$ achieves its maximum in $s = b$ with $b = h^{-1}(y)$.

Let $\alpha \in (0,2)$ and take $y$ small enough to make $b = h^{-1}(y) \leqslant 1/(2^{1/(1-\alpha/2)}t_0)$. Then $1/b > t_0$ and, by (3.4),

$$\mathbb{E}(b\zeta - 1)I(b\zeta > 1) = \int_1^\infty \mathbb{P}(b\zeta > t)\,dt \geqslant \int_1^\infty \frac{C^{\alpha/2}}{(t/b)^{1+\alpha/2}}\,dt = \frac{2}{\alpha}\,(Cb)^{\alpha/2}b.$$

Moreover, $(1/b)^{1-\alpha/2}/2 > t_0^{1-\alpha/2}$ and, by (3.4),

$$\begin{aligned}\mathbb{E}\min\{(b\zeta)^2, 1\} &= \int_0^1 \mathbb{P}((b\zeta)^2 > t)\,dt = 2b^2\int_0^{1/b} z\mathbb{P}(\zeta > z)\,dz \\ &\geqslant 2b^2\int_{t_0}^{1/b}\frac{C^{\alpha/2}}{z^{\alpha/2}}\,dz = 2C^{\alpha/2}b^2\frac{(1/b)^{1-\alpha/2} - t_0^{1-\alpha/2}}{1-\alpha/2} \\ &\geqslant 2C^{\alpha/2}b^2\frac{(1/b)^{1-\alpha/2}/2}{1-\alpha/2} = \frac{(Cb)^{\alpha/2}b}{1-\alpha/2}.\end{aligned}$$

By (3.3), $\lambda(b) \leqslant g(b)$, where $g(b) = 1 - y/b - Kb^{\alpha/2}$ and

$$K = \frac{C^{\alpha/2}}{2}\left(\frac{1}{\alpha/2} + \frac{1}{1-\alpha/2}\right) = \frac{C^{\alpha/2}}{\alpha(1-\alpha/2)}.$$

By Young's inequality,

$$(K^{2/\alpha}y)^{\frac{\alpha}{2+\alpha}} = \left(\frac{y}{b}\right)^{\frac{\alpha}{2+\alpha}}(Kb^{\alpha/2})^{\frac{2}{2+\alpha}} \leqslant \frac{y/b}{(2+\alpha)/\alpha} + \frac{Kb^{\alpha/2}}{(2+\alpha)/2} \leqslant \frac{y}{b} + Kb^{\alpha/2}$$

and

$$\lambda(b) \leqslant g(b) \leqslant 1 - (K^{2/\alpha}y)^{\alpha/(2+\alpha)}.$$

The right-hand side of the last inequality can be made positive for small enough $y$. Hence, combining the above bounds with Lemma 3.2, we get the desired upper bound for $\lambda_p(n^{-1}\mathbf{X}_{pn}\mathbf{X}_{pn}^\top)$ when $\alpha \in (0,2)$ (see also the beginning of Section 3).

Let now $\alpha = 2$ and take $y$ small enough to make $b = h^{-1}(y) \leqslant 1/t_0^2$. Since $t_0 \geqslant 1$, we have $1/b \geqslant t_0^2 \geqslant t_0$ and, hence, the same arguments as above yield

$$\mathbb{E}(b\zeta - 1)I(b\zeta > 1) = \int_1^\infty \mathbb{P}(b\zeta > t)\,dt \geqslant \int_1^\infty \frac{C}{(t/b)^2}\,dt = Cb^2,$$

$$\mathbb{E}\min\{(b\zeta)^2, 1\} \geqslant 2b^2\int_{t_0}^{1/b}\frac{C}{z}\,dz = 2Cb^2\log\frac{1}{bt_0} \geqslant 2Cb^2\log\frac{1}{\sqrt{b}} = Cb^2\log(1/b).$$

Therefore, it follows from (3.3) that $\lambda(b) \leqslant g(b)$, where

$$g(s) = 1 - \frac{y}{s} - \frac{Cs}{2}(\log(1/s) + 1), \quad s > 0.$$

Differentiating $g$ yields

$$g'(s) = \frac{y}{s^2} - \frac{C}{2}(\log(1/s) + 1) + \frac{Cs}{2}\frac{1}{s} = \frac{2y - Cs^2\log(1/s)}{2s^2}.$$

If $2y/C$ is small enough, then $g = g(s)$ has a unique local maximum in $s_1$ and a unique local minimum in $s_2$, where $s_1 < s_2$, and $s_1, s_2$ are solutions to the equation $f(s) = 2y/C$ with $f(s) = s^2 \log(1/s)$.

The function $f = f(s)$ is increasing on $[0, 1/\sqrt{e}]$, decreasing on $[1/\sqrt{e}, \infty]$ and has $f(0) = f(1) = 0$. Hence, $s_2 > 1/2$ and $b = h^{-1}(y) < 1/2$ when $y$ is small enough. Thus,

$$\lambda(b) \leqslant g(b) \leqslant 1 - \frac{y}{s_1} - \frac{Cs_1}{2}(\log(1/s_1) + 1) \leqslant 1 - \frac{y}{s_1} - \frac{Cs_1^2 \log(1/s_1)}{2s_1} = 1 - \frac{2y}{s_1}.$$

Let us bound $s_1$ from above. Take $s_0 = \sqrt{(4y/C)/\log(C/y)}$. If $y$ is small enough, then $s_0 < 1/\sqrt{e}$ as well as

$$s_0^2 \log(1/s_0) = \frac{4y/C}{\log(C/y)}\Big[\frac{1}{2}\log(C/y) + \frac{1}{2}\log\Big(\frac{1}{4}\log(C/y)\Big)\Big] = \frac{2y}{C} + \frac{2y \log\log \sqrt[4]{C/y}}{C\log(C/y)} > \frac{2y}{C}.$$

Therefore, $s_1 < s_0$ and

$$\lambda(b) \leqslant 1 - \frac{2y}{s_1} \leqslant 1 - \frac{2y}{s_0} = 1 - \sqrt{Cy\log(C/y)}.$$

The right-hand side of the last inequality can be made positive for small enough $y$. Hence, combining the above bounds with Lemma 3.2, we get the desired upper bound for $\lambda_p(n^{-1}\mathbf{X}_{pn}\mathbf{X}_{pn}^\top)$ in the case with $\alpha = 2$ (see also the beginning of Section 3). $\square$

*Proof of Proposition 2.3.* Let $t_0 = (1 + 2/\alpha)^{-1}$ and $q = C/t_0^{1+2/\alpha}$. If $\alpha \in (0, 2]$, then

$$q \geqslant C \inf_{\alpha\in(0,2]} (1 + 2/\alpha)^{1+2/\alpha} = 4C > 1.$$

Let $\eta = \sqrt{\xi\zeta}$, where $\xi$ and $\zeta$ are independent random variables,

$$\mathbb{P}(\xi = q) = q^{-1} \quad \text{and} \quad \mathbb{P}(\xi = 0) = 1 - q^{-1},$$

$\zeta$ has the Pareto distribution

$$\mathbb{P}(\zeta > t) = \begin{cases} (t_0/t)^{1+\alpha/2}, & t \geqslant t_0, \\ 1, & t < t_0. \end{cases}$$

It is easy to see that $\mathbb{E}\xi = 1$. Moreover, $\mathbb{P}(\zeta > t) \leqslant (t_0/t)^{1+\alpha/2}$ for all $t > 0$ and

$$\mathbb{E}\zeta = \int_0^\infty \mathbb{P}(\zeta > t)\,dt = t_0 + \int_{t_0}^\infty (t_0/t)^{1+\alpha/2}\,dt = t_0 + \frac{2t_0}{\alpha} = 1.$$

Hence, $\mathbb{E}\eta^2 = \mathbb{E}\xi\,\mathbb{E}\zeta = 1$. In addition, (2.2) holds since, for all large enough $t > 0$,

$$\mathbb{P}(|\eta| > t) = q^{-1}\mathbb{P}(\zeta > t^2/q) = q^{-1}(qt_0/t^2)^{1+\alpha/2} = \frac{q^{\alpha/2}t_0^{1+\alpha/2}}{t^{2+\alpha}} = \frac{C^{\alpha/2}}{t^{2+\alpha}}.$$

We also have

$$|(X_p, v)| = \sqrt{\xi\zeta}|(Z_p, v)| \overset{d}{=} \sqrt{\xi\zeta}|Z| \quad \text{for all unit vectors } v \in \mathbb{R}^p,$$

where $Z \sim \mathcal{N}(0,1)$ is independent of $(\xi, \zeta)$, «$\overset{d}{=}$» means equality in law. Hence, if $t > 0$,

$$\mathbb{P}(\sqrt{\xi\zeta}|Z| > t) = \mathbb{E}\mathbb{P}(s\zeta > t^2)|_{s=\xi Z^2} \leqslant \mathbb{E}\big(st_0/t^2\big)^{1+\alpha/2} I(s > 0)|_{s=\xi Z^2} \leqslant$$
$$\leqslant \frac{\mathbb{E}(t_0\xi Z^2)^{1+\alpha/2}}{t^{2+\alpha}} = \frac{t_0^{1+\alpha/2}q^{\alpha/2}\mathbb{E}|Z|^{2+\alpha}}{t^{2+\alpha}} = \frac{C^{\alpha/2}\mathbb{E}|Z|^{2+\alpha}}{t^{2+\alpha}} \leqslant \frac{(\kappa C)^{\alpha/2}}{t^{2+\alpha}},$$

where
$$\kappa = \sup_{\alpha\in(0,2]} (\mathbb{E}|Z|^{2+\alpha})^{2/\alpha}.$$

Let us show that $\kappa < \infty$. If $Z \sim \mathcal{N}(0,1)$, then
$$f(\alpha) = \mathbb{E}|Z|^{2+\alpha} = \frac{2^{\frac{2+\alpha}{2}}\Gamma\Big(\dfrac{3+\alpha}{2}\Big)}{\sqrt{\pi}}$$
is a smooth function on $[0,2]$ with $f(0) = 1$ and, in particular, $f'(0)$ exists and is finite. The function $g(\alpha) = f(\alpha)^{2/\alpha}$ is continuous on $(0,2]$ and
$$g(\alpha) = (1 + f'(0)\alpha + o(\alpha))^{2/\alpha} \to \exp\{2f'(0)\}, \quad \alpha \to 0+.$$
As a result, $\kappa = \sup\{g(\alpha) : \alpha \in (0,2]\}$ is finite. This finishes the proof of the proposition.

□

# 4 Appendix

*Proof of Lemma 3.1.* If $U$ is a non-negative random variable with $\mathbb{E}U = 1$, then
$$\mathbb{E}\min\{U,a\} = \int_0^a \mathbb{P}(U>t)\,dt = \mathbb{E}U - \int_a^\infty \mathbb{P}(U>t)\,dt \geqslant 1 - \int_a^\infty \frac{M}{t^{1+\alpha/2}}\,dt = 1 - \frac{2M}{\alpha a^{\alpha/2}},$$
where $M = \sup\{t^{1+\alpha/2}\mathbb{P}(U>t) : t>0\}$. Putting $U = (X_p, v)^2$ for a given unit vector $v \in \mathbb{R}^p$ and taking the infimum over such $v$, we obtain the desired lower bound for $c_p(a)$.

Similarly, we have
$$\begin{aligned}\mathbb{E}U\min\{U,a\} =& a\mathbb{E}(U-a)I(U>a) + a^2\mathbb{P}(U>a) + \mathbb{E}U^2 I(U \leqslant a)\\ =& a\mathbb{E}(U-a)I(U>a) + \mathbb{E}\min\{U^2,a^2\}\\ =& I_1 + I_2,\end{aligned} \tag{4.1}$$
where
$$I_1 = a\int_a^\infty \mathbb{P}(U>t)\,dt \leqslant a\int_a^\infty \frac{M}{t^{1+\alpha/2}}\,dt = \frac{2M}{\alpha}\,a^{1-\alpha/2}, \quad I_2 = \int_0^{a^2} \mathbb{P}(U^2>t)\,dt.$$
If $\alpha \in (0,2)$, then $I_2$ can be bounded as follows
$$I_2 \leqslant \int_0^{a^2} \frac{M dt}{t^{1/2+\alpha/4}} = \frac{Ma^{1-\alpha/2}}{1/2-\alpha/4}.$$
Similarly, if $\alpha = 2$, then
$$I_2 \leqslant M + I(a^2 > M)\int_M^{a^2} \frac{M dt}{t} = M + M\log(a^2/M)I(a^2>M).$$
Thus, we have proved that
$$\mathbb{E}U\min\{U,a\} \leqslant M \cdot \begin{cases}(2/\alpha + 4/(2-\alpha))a^{1-\alpha/2}, & \alpha \in (0,2),\\ 2 + \log(a^2/M)I(a^2>M), & \alpha = 2.\end{cases}$$
Putting $U = (X_p, v)^2$ for a given unit vector $v \in \mathbb{R}^p$ and taking the supremum over such $v$, we get the desired upper bound for $C_p(a)$.

□

*Proof of Lemma 3.2.* We have $n^{-1}\mathbf{X}_{pn}\mathbf{X}_{pn}^\top = n^{-1}\mathbf{Z}_{pn}\mathbf{T}_n\mathbf{Z}_{pn}^\top$, where $\mathbf{Z}_{pn}$ is a $p\times n$ matrix with i.i.d. entries, $\mathbf{T}_n$ is a $n\times n$ diagonal matrix whose diagonal entries are independent copies of $\zeta=\eta^2$, and $\mathbf{Z}_{pn}$ is independent of $\mathbf{T}_n$.

By the Glivenko-Cantelli theorem, the empirical spectral distribution of $\mathbf{T}_n$ converges a.s. to the distribution of $\zeta$. By Theorem 4.3 in [1], there is a non-decreasing cádlág function $F=F(\lambda)$, $\lambda\in\mathbb{R}$, such that $F(\lambda)=0$ for $\lambda<0$, $F(\infty)\leqslant 1$, and

$$\mathbb{P}\Big(\lim_{n\to\infty}\frac{1}{p}\sum_{k=1}^{p} I(\lambda_{kn}\leqslant\lambda)=F(\lambda)\Big)=1 \quad \text{for all continuity points } \lambda \text{ of } F, \tag{4.2}$$

where $p=p(n)=yn+o(n)$ and $\{\lambda_{kn}\}_{k=1}^p$ is the set of eigenvalues of $p^{-1}\mathbf{X}_{pn}\mathbf{X}_{pn}^\top$. The Stieltjes transform

$$f(z)=\int_{\mathbb{R}}\frac{F(d\lambda)}{\lambda-z},\quad z\in\mathbb{C}^+=\{w\in\mathbb{C}:\Im z>0\}, \tag{4.3}$$

of $F$ can be defined explicitly as a unique solution in $\mathbb{C}^+$ to the equation

$$f(z)=-\Big(z-\frac{1}{y}\mathbb{E}\frac{\zeta}{1+f(z)\zeta}\Big)^{-1} \text{ or, equivalently, } z=-\frac{1}{f(z)}+\frac{1}{y}\mathbb{E}\frac{\zeta}{1+f(z)\zeta}. \tag{4.4}$$

Define

$$\mathcal{S}_G=\{\lambda\geqslant 0:\ G(\lambda+\varepsilon)>G(\lambda-\varepsilon)\quad \text{for any small enough } \varepsilon>0\}$$

for a non-decreasing cádlág function $G=G(\lambda)$, $\lambda\in\mathbb{R}$. In other words, $\mathcal{S}_G$ is the set of points of increase of $G$. Obviously, $\mathcal{S}_G$ is a closed set. Using (4.2) and setting $G=F$ as well as

$$a=\inf\{\lambda\geqslant 0:\ \lambda\in\mathcal{S}_F\},$$

we conclude that $a\in\mathcal{S}_F$ and

$$\lambda_p(n^{-1}\mathbf{X}_{pn}\mathbf{X}_{pn}^\top)=\frac{p}{n}\,\lambda_p(p^{-1}\mathbf{X}_{pn}\mathbf{X}_{pn}^\top)\leqslant ay+o(1)\quad\text{a.s.} \tag{4.5}$$

when $n\to\infty$.

Consider the function

$$z(s)=-\frac{1}{s}+\frac{1}{y}\mathbb{E}\frac{\zeta}{1+s\zeta}$$

defined for $s\in D$, where $D$ consists of all $s\in\mathbb{R}\setminus\{0\}$ with $-s^{-1}\notin\mathcal{S}_G$ for $G(\lambda)=\mathbb{P}(\zeta\leqslant\lambda)$, $\lambda\in\mathbb{R}$. This function differs from $\lambda=\lambda(s)$ given in Lemma 3.2 by the factor $y$, i.e. $\lambda(s)=yz(s)$ for all $s>0$. Therefore, to finish the proof, we only need to show that

$$a=\max\{0,\sup_{s>0}z(s)\}.$$

Let us show that $a=0$ when $z(s)\leqslant 0$ for all $s>0$. The latter can be reformulated as follows: if $a>0$, then there is $s>0$ satisfying $z(s)>0$. Suppose $a>0$. Then $a/2\in\mathbb{R}\setminus\mathcal{S}_F$ and $F(a/2)=0$. Hence,

$$f(a/2)=\int_{\mathbb{R}}\frac{F(d\lambda)}{\lambda-a/2}>0\quad\text{and}\quad\lim_{\varepsilon\to 0+}f(a/2+i\varepsilon)=f(a/2)>0.$$

Taking $z=a/2+i\varepsilon$ in (4.4) and tending $\varepsilon$ to zero, we get $a/2=z(s)>0$ for $s=f(a/2)$.

Assume further that there is $s>0$ satisfying $z(s)>0$ or, equivalently,

$$g(s)=\mathbb{E}\frac{s\zeta}{1+s\zeta}>y.$$

The function $g = g(s)$ is continuous and strictly increasing on $\mathbb{R}_+$. It changes from zero to $\mathbb{P}(\zeta > 0)$ when $s$ changes from zero to infinity. The same can be said about

$$h(s) = \mathbb{E}\frac{(s\zeta)^2}{(1+s\zeta)^2}.$$

Hence, $y < \mathbb{P}(\zeta > 0)$ and there is $b = b(y) > 0$ that solves $h(b) = y$. By the Lebesgue dominated convergence theorem,

$$z'(s) = \frac{1}{s^2} - \frac{1}{y}\mathbb{E}\frac{\zeta^2}{(1+s\zeta)^2} = \frac{y - h(s)}{ys^2} \quad \text{for any } s > 0.$$

Therefore, $b$ is a strict global maximum point of $z = z(s)$ on $\{s : s > 0\}$.

The rest of the proof is based on Lemma 6.1 in [1] which states that $z'(s) > 0$ and $s \in D$ if $s = f(\lambda)$ for some $\lambda \in \mathbb{R} \setminus \mathcal{S}_F$. Moreover, $\{z(s) : \, s \in D, \, z'(s) > 0\} \subseteq \mathbb{R} \setminus \mathcal{S}_F$.

We will now prove that $a \leqslant z(b)$. Suppose the contrary, i.e. $a > z(b)$. By definition, $F(\lambda) = 0$ for all $\lambda < a$. Set $z_0 = z(b)$. Then $z_0 \in \mathbb{R} \setminus \mathcal{S}_F$,

$$s_0 = f(z_0) = \int_{\mathbb{R}} \frac{F(d\lambda)}{\lambda - z_0} > 0,$$

and, by the above lemma, $z'(s_0) > 0$. Taking $z = z_0 + i\varepsilon$ in (4.4) and tending $\varepsilon$ to zero, we arrive at $z(b) = z_0 = z(f(z_0)) = z(s_0)$. Since $z'(s_0) > 0$ and $s_0 > 0$, we get the contradiction to the fact that $b$ is a strict global maximum point of $z = z(s)$ on $\{s : s > 0\}$.

Let us finally prove that $a \geqslant z(b)$. The function $z = z(s)$ is continuous and strictly increasing on the set $(0, b)$ with $z(0+) = -\infty$ and $z(b-) = z(b)$. By the above lemma,

$$z((0,b)) = (-\infty, z(b)) \subseteq \mathbb{R} \setminus \mathcal{S}_F.$$

Thus, $z(b) \leqslant a$. This finishes the proof. □

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