Electron. Commun. Probab. **17** (2012), no. 59, 1–8.
DOI: 10.1214/ECP.v17-2318
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Large deviation exponential inequalities for supermartingales

Xiequan Fan* Ion Grama* Quansheng Liu*

Abstract

Let $(X_i, \mathcal{F}_i)_{i\geq 1}$ be a sequence of supermartingale differences and let $S_k = \sum_{i=1}^k X_i$. We give an exponential moment condition under which $\mathbb{P}(\max_{1\leq k\leq n} S_k \geq n) = O(\exp\{-C_1 n^{\alpha}\})$, $n \to \infty$, where $\alpha \in (0,1)$ is given and $C_1 > 0$ is a constant. We also show that the power α is optimal under the given moment condition.

Keywords: Large deviation; martingales; exponential inequality; Bernstein type inequality.
AMS MSC 2010: 60F10; 60G42; 60E15.
Submitted to ECP on September 17, 2012, final version accepted on December 9, 2012.

1 Introduction

Let $(X_i, \mathcal{F}_i)_{i\geq 1}$ be a sequence of martingale differences and let $S_k = \sum_{i=1}^k X_i, k \geq 1$. Under the Cramér condition $\sup_i \mathbb{E}e^{|X_i|} < \infty$, Lesigne and Volný [9] proved that

$$\mathbb{P}(S_n \geq n) \quad = \quad O(\exp\{-C_1 n^{\frac{1}{3}}\}), \qquad n \to \infty, \tag{1.1}$$

for some constant $C_1 > 0$. Here and throughout the paper, for two functions f and g, we write $f(n) = O(g(n))$ if there exists a constant $C > 0$ such that $|f(n)| \leq C|g(n)|$ for all $n \geq 1$. Lesigne and Volný [9] also showed that the power $\frac{1}{3}$ in (1.1) is optimal even for stationary and ergodic sequence of martingale differences, in the sense that there exists a stationary and ergodic sequence of martingale differences $(X_i, \mathcal{F}_i)_{i\geq 1}$ such that $\mathbb{E}e^{|X_1|} < \infty$ and $\mathbb{P}(S_n \geq n) \geq \exp\{-C_2 n^{\frac{1}{3}}\}$ for some constant $C_2 > 0$ and infinitely many n's. Liu and Watbled [10] proved that the power $\frac{1}{3}$ in (1.1) can be improved to 1 under the conditional Cramér condition $\sup_i \mathbb{E}(e^{|X_i|}|\mathcal{F}_{i-1}) \leq C_3$, for some constant C_3. It is natural to ask under what condition

$$\mathbb{P}(S_n \geq n) \quad = \quad O(\exp\{-C_1 n^{\alpha}\}), \qquad n \to \infty, \tag{1.2}$$

where $\alpha \in (0,1)$ is given and $C_1 > 0$ is a constant. In this paper, we give some sufficient conditions in order that (1.2) holds for supermartingales $(S_k, \mathcal{F}_k)_{k\geq 1}$.

The paper is organized as follows. In Section 2, we present the main results. In Sections 3-5, we give the proofs of the main results.

*Université de Bretagne-Sud, LMBA, UMR CNRS 6205, Vannes, France.
E-mail: fanxiequan@hotmail.com E-mail: ion.grama,quansheng.liu@univ-ubs.fr

2 Main Results

Our first result is an extension of the bound (1.1) of Lesigne and Volný [9].

Theorem 2.1. *Let $\alpha \in (0,1)$. Assume that $(X_i, \mathcal{F}_i)_{i\geq 1}$ is a sequence of supermartingale differences satisfying $\sup_i \mathbb{E} \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} \leq C_1$ for some constant $C_1 \in (0,\infty)$. Then, for all $x > 0$,*

$$\mathbb{P}\left(\max_{1\leq k\leq n} S_k \geq nx\right) \quad \leq \quad C(\alpha,x)\exp\left\{-\left(\frac{x}{4}\right)^{2\alpha} n^{\alpha}\right\}, \tag{2.1}$$

where

$$C(\alpha,x) = 2 + 35C_1\left(\frac{1}{x^{2\alpha}16^{1-\alpha}} + \frac{1}{x^2}\left(\frac{3(1-\alpha)}{2\alpha}\right)^{\frac{1-\alpha}{\alpha}}\right)$$

does not depend on n. In particular, with $x = 1$, it holds

$$\mathbb{P}\left(\max_{1\leq k\leq n} S_k \geq n\right) \quad = \quad O\left(\exp\{-\frac{1}{16}\,n^{\alpha}\}\right), \quad n\to\infty. \tag{2.2}$$

Moreover, the power α in (2.2) is optimal in the class of martingale differences: for each $\alpha \in (0,1)$, there exists a sequence of martingale differences $(X_i, \mathcal{F}_i)_{i\geq 1}$ satisfying $\sup_i \mathbb{E}\exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} < \infty$ and

$$\mathbb{P}\left(\max_{1\leq k\leq n} S_k \geq n\right) \quad \geq \quad \exp\{-3n^{\alpha}\}, \tag{2.3}$$

for all n large enough.

In fact, we shall prove that the power α in (2.2) is optimal even for stationary martingale difference sequences.

It is clear that when $\alpha = \frac{1}{3}$, the bound (2.2) implies the bound (1.1) of Lesigne and Volný.

Our second result shows that the moment condition $\sup_i \mathbb{E}\exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} < \infty$ in Theorem 2.1 can be relaxed to $\sup_i \mathbb{E}\exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\} < \infty$, where $X_i^+ = \max\{X_i, 0\}$, if we add a constraint on the sum of conditional variances

$$\langle S\rangle_k = \sum_{i=1}^{k} \mathbb{E}(X_i^2|\mathcal{F}_{i-1}).$$

Theorem 2.2. *Let $\alpha \in (0,1)$. Assume that $(X_i, \mathcal{F}_i)_{i\geq 1}$ is a sequence of supermartingale differences satisfying $\sup_i \mathbb{E}\exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\} \leq C_1$ for some constant $C_1 \in (0,\infty)$. Then, for all $x, v > 0$,*

$$\begin{aligned}\mathbb{P}\left(S_k \geq x \text{ and } \langle S\rangle_k \leq v^2 \text{ for some } k\in[1,n]\right)& \\ \leq \exp\left\{-\frac{x^2}{2(v^2+\frac{1}{3}x^{2-\alpha})}\right\} + nC_1\exp\{-x^{\alpha}\}.&\end{aligned} \tag{2.4}$$

For bounded random variables, some inequalities closely related to (2.4) can be found in Freedman [5], Dedecker [1], Dzhaparidze and van Zanten [3], Merlevède, Peligrad and Rio [11] and Delyon [2].

Adding a hypothesis on $\langle S\rangle_n$ to Theorem 2.2, we can easily obtain the following Bernstein type inequality which is similar to an inequality of Merlevède, Peligrad and Rio [12] for weakly dependent sequences.

Corollary 2.3. *Let $\alpha \in (0,1)$. Assume that $(X_i, \mathcal{F}_i)_{i\geq 1}$ is a sequence of supermartingale differences satisfying $\sup_i \mathbb{E}\exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\} \leq C_1$ and $\mathbb{E}\exp\{(\frac{\langle S\rangle_n}{n})^{\frac{\alpha}{1-\alpha}}\} \leq C_2$ for some constants $C_1, C_2 \in (0,\infty)$. Then, for all $x > 0$,*

$$\mathbb{P}\left(\max_{1\leq k\leq n} S_k \geq nx\right) \quad \leq \quad \exp\left\{-\frac{x^{1+\alpha}}{2\left(1+\frac{1}{3}x\right)}n^{\alpha}\right\} + (nC_1 + C_2)\exp\{-x^{\alpha}n^{\alpha}\}. \tag{2.5}$$

In particular, with $x = 1$, it holds

$$\mathbb{P}\left(\max_{1\leq k\leq n} S_k \geq n\right) \quad = \quad O\left(\exp\{-C\,n^{\alpha}\}\right), \quad n\to\infty, \tag{2.6}$$

where $C > 0$ is an absolute constant. Moreover, the power α in (2.6) is optimal for the class of martingale differences: for each $\alpha \in (0,1)$, there exists a sequence of martingale differences $(X_i, \mathcal{F}_i)_{i\geq 1}$ satisfying $\sup_i \mathbb{E}\exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\} < \infty$, $\sup_n \mathbb{E}\exp\{(\frac{\langle S\rangle_n}{n})^{\frac{\alpha}{1-\alpha}}\} < \infty$ and

$$\mathbb{P}\left(\max_{1\leq k\leq n} S_k \geq n\right) \quad \geq \quad \exp\{-3n^{\alpha}\} \tag{2.7}$$

for all n large enough.

Actually, just as (2.2), the power α in (2.6) is optimal even for stationary martingale difference sequences.

In the i.i.d. case, the conditions of Corollary 2.3 can be weakened considerably, see Lanzinger and Stadtmüller [8] where it is shown that if $\mathbb{E}\exp\{(X_1^+)^{\alpha}\} < \infty$ with $\alpha \in (0,1)$, then

$$\mathbb{P}\left(\max_{1\leq k\leq n} S_k \geq n\right) \quad = \quad O\left(\exp\{-C_{\alpha}\,n^{\alpha}\}\right), \quad n\to\infty. \tag{2.8}$$

3 Proof of Theorem 2.1

We shall need the following refined version of the Azuma-Hoeffding inequality.

Lemma 3.1. *Assume that $(X_i, \mathcal{F}_i)_{i\geq 1}$ is a sequence of martingale differences satisfying $|X_i| \leq 1$ for all $i \geq 1$. Then, for all $x \geq 0$,*

$$\mathbb{P}\left(\max_{1\leq k\leq n} S_k \geq x\right) \quad \leq \quad \exp\left\{-\frac{x^2}{2n}\right\}. \tag{3.1}$$

A proof can be found in Laib [7].

For the proof of Theorem 2.1, we use a truncating argument as in Lesigne and Volný [9]. Let $(X_i, \mathcal{F}_i)_{i\geq 1}$ be a sequence of supermartingale differences. Given $u > 0$, define

$$\begin{aligned}
X_i' &= X_i\mathbf{1}_{\{|X_i|\leq u\}} - \mathbb{E}(X_i\mathbf{1}_{\{|X_i|\leq u\}}|\mathcal{F}_{i-1}),\\
X_i'' &= X_i\mathbf{1}_{\{|X_i|> u\}} - \mathbb{E}(X_i\mathbf{1}_{\{|X_i|> u\}}|\mathcal{F}_{i-1}),\\
S_k' &= \sum_{i=1}^{k} X_i', \qquad S_k'' = \sum_{i=1}^{k} X_i'', \qquad S_k''' = \sum_{i=1}^{k} \mathbb{E}(X_i|\mathcal{F}_{i-1}).
\end{aligned}$$

Then $(X_i', \mathcal{F}_i)_{i\geq 1}$ and $(X_i'', \mathcal{F}_i)_{i\geq 1}$ are two martingale difference sequences and $S_k = S_k' + S_k'' + S_k'''$. Let $t \in (0,1)$. Since $S_k''' \leq 0$, for any $x > 0$,

$$\begin{aligned}
\mathbb{P}\left(\max_{1\leq k\leq n} S_k \geq x\right) &\leq \mathbb{P}\left(\max_{1\leq k\leq n} S_k' + S_k''' \geq xt\right) + \mathbb{P}\left(\max_{1\leq k\leq n} S_k'' \geq x(1-t)\right)\\
&\leq \mathbb{P}\left(\max_{1\leq k\leq n} S_k' \geq xt\right) + \mathbb{P}\left(\max_{1\leq k\leq n} S_k'' \geq x(1-t)\right).
\end{aligned} \tag{3.2}$$

Using Lemma 3.1 and the fact that $|X_i'| \leq 2u$, we have

$$\mathbb{P}\left(\max_{1\leq k\leq n} S_k' \geq xt\right) \quad \leq \quad \exp\left\{-\frac{x^2t^2}{8nu^2}\right\}. \tag{3.3}$$

Let $F_i(x) = \mathbb{P}(|X_i| \geq x), x \geq 0$. Since $\mathbb{E}\exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} \leq C_1$, we obtain, for all $x \geq 0$,

$$F_i(x) \leq \exp\{-x^{\frac{2\alpha}{1-\alpha}}\}\mathbb{E}\exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} \leq C_1\exp\{-x^{\frac{2\alpha}{1-\alpha}}\}.$$

Using the martingale maximal inequality (cf. e.g. p. 14 in [6]), we get

$$\mathbb{P}\left(\max_{1\leq k\leq n} S_k'' \geq x(1-t)\right) \quad \leq \quad \frac{1}{x^2(1-t)^2}\sum_{i=1}^n \mathbb{E}X_i''^2. \tag{3.4}$$

It is easy to see that

$$\begin{aligned}
\mathbb{E}X_i''^2 &= -\int_u^\infty t^2 dF_i(t)\\
&= u^2F_i(u) + \int_u^\infty 2tF_i(t)dt\\
&\leq C_1u^2\exp\{-u^{\frac{2\alpha}{1-\alpha}}\} + 2C_1\int_u^\infty t\exp\{-t^{\frac{2\alpha}{1-\alpha}}\}dt.
\end{aligned} \tag{3.5}$$

Notice that the function $g(t) = t^3\exp\{-t^{\frac{2\alpha}{1-\alpha}}\}$ is decreasing in $[\beta, +\infty)$ and is increasing in $[0, \beta]$, where $\beta = \left(\frac{3(1-\alpha)}{2\alpha}\right)^{\frac{1-\alpha}{2\alpha}}$. If $0 < u < \beta$, we have

$$\begin{aligned}
\int_u^\infty t\exp\{-t^{\frac{2\alpha}{1-\alpha}}\}dt &\leq \int_u^\beta t\exp\{-t^{\frac{2\alpha}{1-\alpha}}\}dt + \int_\beta^\infty t^{-2}t^3\exp\{-t^{\frac{2\alpha}{1-\alpha}}\}dt\\
&\leq \int_u^\beta t\exp\{-u^{\frac{2\alpha}{1-\alpha}}\}dt + \int_\beta^\infty t^{-2}\beta^3\exp\{-\beta^{\frac{2\alpha}{1-\alpha}}\}dt\\
&\leq \frac{3}{2}\beta^2\exp\{-u^{\frac{2\alpha}{1-\alpha}}\}.
\end{aligned} \tag{3.6}$$

If $\beta \leq u$, we have

$$\begin{aligned}
\int_u^\infty t\exp\{-t^{\frac{2\alpha}{1-\alpha}}\}dt &= \int_u^\infty t^{-2}t^3\exp\{-t^{\frac{2\alpha}{1-\alpha}}\}dt\\
&\leq \int_u^\infty t^{-2}u^3\exp\{-u^{\frac{2\alpha}{1-\alpha}}\}dt\\
&= u^2\exp\{-u^{\frac{2\alpha}{1-\alpha}}\}.
\end{aligned} \tag{3.7}$$

By (3.5), (3.6) and (3.7), we get

$$\mathbb{E}X_i''^2 \quad \leq \quad 3C_1(u^2+\beta^2)\exp\{-u^{\frac{2\alpha}{1-\alpha}}\}. \tag{3.8}$$

From (3.4), it follows that

$$\mathbb{P}\left(\max_{1\leq k\leq n} S_k'' \geq x(1-t)\right) \quad \leq \quad \frac{3nC_1}{x^2(1-t)^2}(u^2+\beta^2)\exp\{-u^{\frac{2\alpha}{1-\alpha}}\}. \tag{3.9}$$

Combining (3.2), (3.3) and (3.9), we obtain

$$\mathbb{P}\left(\max_{1\leq k\leq n} S_k \geq x\right) \quad \leq \quad 2\exp\left\{-\frac{x^2t^2}{8nu^2}\right\} + \frac{3nC_1}{(1-t)^2}\left(\frac{u^2}{x^2} + \frac{\beta^2}{x^2}\right)\exp\{-u^{\frac{2\alpha}{1-\alpha}}\}.$$

Taking $t = \frac{1}{\sqrt{2}}$ and $u = \left(\frac{x}{4\sqrt{n}}\right)^{1-\alpha}$, we get, for all $x > 0$,

$$\mathbb{P}\left(\max_{1\le k\le n} S_k \ge x\right) \quad \le \quad C_n(\alpha, x) \exp\left\{-\left(\frac{x^2}{16n}\right)^{\alpha}\right\},$$

where

$$C_n(\alpha, x) = 2 + 35nC_1\left(\frac{1}{x^{2\alpha}(16n)^{1-\alpha}} + \frac{\beta^2}{x^2}\right).$$

Hence, for all $x > 0$,

$$\mathbb{P}\left(\max_{1\le k\le n} S_k \ge nx\right) \quad \le \quad C(\alpha, x) \exp\left\{-\left(\frac{x}{4}\right)^{2\alpha} n^{\alpha}\right\},$$

where

$$C(\alpha, x) = 2 + 35C_1\left(\frac{1}{x^{2\alpha}16^{1-\alpha}} + \frac{1}{x^2}\left(\frac{3(1-\alpha)}{2\alpha}\right)^{\frac{1-\alpha}{\alpha}}\right).$$

This completes the proof of the first assertion of Theorem 2.1.

Next, we prove that the power α in (2.2) is optimal by giving a stationary sequence of martingale differences satisfying (2.3). We proceed as in Lesigne and Volný ([9], p. 150). Take a positive random variable X such that

$$\mathbb{P}(X > x) = \frac{2e}{1 + x^{\frac{1+\alpha}{1-\alpha}}} \exp\left\{-x^{\frac{2\alpha}{1-\alpha}}\right\} \tag{3.10}$$

for all $x > 1$. Using the formula $\mathbb{E}f(X) = f(1) + \int_1^{\infty} f'(t)\mathbb{P}(X > t)dt$ for $f(t) = \exp\{t^{\frac{2\alpha}{1-\alpha}}\}, t \ge 1$, we obtain

$$\mathbb{E}\exp\{X^{\frac{2\alpha}{1-\alpha}}\} = e + \frac{4e\,\alpha}{1-\alpha}\int_1^{\infty} \frac{t^{\frac{3\alpha-1}{1-\alpha}}}{1 + t^{\frac{1+\alpha}{1-\alpha}}}dt < \infty.$$

Assume that $(\xi_i)_{i\ge1}$ are Rademacher random variables independent of X, i.e. $\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = \frac{1}{2}$. Set $X_i = X\xi_i$, $\mathcal{F}_0 = \sigma(X)$ and $\mathcal{F}_i = \sigma(X, (\xi_k)_{k=1,\dots,i})$. Then, $(X_i, \mathcal{F}_i)_{i\ge1}$ is a stationary sequence of martingale differences satisfying

$$\sup_i \mathbb{E}\exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} = \mathbb{E}\exp\{X^{\frac{2\alpha}{1-\alpha}}\} < \infty.$$

For $\beta \in (0, 1)$, it is easy to see that

$$\mathbb{P}\left(\max_{1\le k\le n} S_i \ge n\right) \ge \mathbb{P}(S_n \ge n) \ge \mathbb{P}\left(\sum_{i=1}^{n} \xi_i \ge n^{\beta}\right)\mathbb{P}\left(X \ge n^{1-\beta}\right).$$

Since, for n large enough,

$$\mathbb{P}\left(\sum_{i=1}^{n} \xi_i \ge n^{\beta}\right) \ge \exp\left\{-n^{2\beta-1}\right\},$$

(cf. Corollary 3.5 in Lesigne and Volný [9]), we get, for n large enough,

$$\mathbb{P}\left(\max_{1\le k\le n} S_i \ge n\right) \quad \ge \quad \frac{2e}{1 + (n^{1-\beta})^{\frac{1+\alpha}{1-\alpha}}} \exp\left\{-n^{2\beta-1} - (n^{1-\beta})^{\frac{2\alpha}{1-\alpha}}\right\}. \tag{3.11}$$

Setting $2\beta - 1 = \alpha$, we obtain, for n large enough,

$$\mathbb{P}\left(\max_{1\le k\le n} S_i \ge n\right) \ge \frac{2e}{1 + n^{\frac{1+\alpha}{2}}} \exp\left\{-2n^{\alpha}\right\} \ge \exp\left\{-3n^{\alpha}\right\},$$

which proves (2.3). This ends the proof of Theorem 2.1.

4 Proof of Theorem 2.2

To prove Theorem 2.2, we need the following inequality.

Lemma 4.1 ([4], Remark 2.1)**.** *Assume that $(X_i, \mathcal{F}_i)_{i\geq 1}$ are supermartingale differences satisfying $X_i \leq 1$ for all $i \geq 1$. Then, for all $x, v > 0$,*

$$\mathbb{P}\left(S_k \geq x \text{ and } \langle S\rangle_k \leq v^2 \text{ for some } k \in [1,n]\right) \leq \exp\left\{-\frac{x^2}{2(v^2+\frac{1}{3}x)}\right\}. \tag{4.1}$$

Assume that $(X_i, \mathcal{F}_i)_{i\geq 1}$ are supermartingale differences. Given $u > 0$, set

$$X_i' = X_i \mathbf{1}_{\{X_i \leq u\}}, \quad X_i'' = X_i \mathbf{1}_{\{X_i > u\}}, \quad S_k' = \sum_{i=1}^k X_i' \text{ and } S_k'' = \sum_{i=1}^k X_i''.$$

Then, $(X_i', \mathcal{F}_i)_{i\geq 1}$ is also a sequence of supermartingale differences and $S_k = S_k' + S_k''$. Since $\langle S'\rangle_k \leq \langle S\rangle_k$, we deduce, for all $x, u, v > 0$,

$$\begin{aligned} & \mathbb{P}\left(S_k \geq x \text{ and } \langle S\rangle_k \leq v^2 \text{ for some } k \in [1,n]\right) \\ \leq \;& \mathbb{P}\left(S_k' \geq x \text{ and } \langle S\rangle_k \leq v^2 \text{ for some } k \in [1,n]\right) \\ & + \mathbb{P}\left(S_k'' \geq 0 \text{ and } \langle S\rangle_k \leq v^2 \text{ for some } k \in [1,n]\right) \\ \leq \;& \mathbb{P}\left(S_k' \geq x \text{ and } \langle S'\rangle_k \leq v^2 \text{ for some } k \in [1,n]\right) + \mathbb{P}\left(\max_{1\leq k\leq n} S_k'' \geq 0\right). \end{aligned} \tag{4.2}$$

Applying Lemma 4.1 to the supermartingale differences $(X_i'/u, \mathcal{F}_i)_{i\geq 1}$, we have, for all $x, u, v > 0$,

$$\mathbb{P}(S_k' \geq x \text{ and } \langle S'\rangle_k \leq v^2 \text{ for some } k \in [1,n]) \leq \exp\left\{-\frac{x^2}{2(v^2+\frac{1}{3}xu)}\right\}. \tag{4.3}$$

Using the exponential Markov's inequality and the condition $\mathbb{E}\exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\} \leq C_1$, we get

$$\begin{aligned} \mathbb{P}\left(\max_{1\leq k\leq n} S_k'' \geq 0\right) & \leq \sum_{i=1}^n \mathbb{P}(X_i > u) \\ & \leq \sum_{i=1}^n \mathbb{E}\exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}} - u^{\frac{\alpha}{1-\alpha}}\} \\ & \leq nC_1 \exp\{-u^{\frac{\alpha}{1-\alpha}}\}. \end{aligned} \tag{4.4}$$

Combining the inequalities (4.2), (4.3) and (4.4) together, we obtain, for all $x, u, v > 0$,

$$\begin{aligned} & \mathbb{P}(S_k \geq x \text{ and } \langle S\rangle_k \leq v^2 \text{ for some } k \in [1,n]) \\ & \qquad \leq \exp\left\{-\frac{x^2}{2(v^2+\frac{1}{3}xu)}\right\} + nC_1\exp\{-u^{\frac{\alpha}{1-\alpha}}\}. \end{aligned} \tag{4.5}$$

Taking $u = x^{1-\alpha}$, we get, for all $x, v > 0$,

$$\begin{aligned} & \mathbb{P}(S_k \geq x \text{ and } \langle S\rangle_k \leq v^2 \text{ for some } k \in [1,n]) \\ & \qquad \leq \exp\left\{-\frac{x^2}{2(v^2+\frac{1}{3}x^{2-\alpha})}\right\} + nC_1\exp\{-x^{\alpha}\}. \end{aligned} \tag{4.6}$$

This completes the proof of Theorem 2.2.

5 Proof of Corollary 2.3.

To prove Corollary 2.3 we make use of Theorem 2.2. It is easy to see that

$$\begin{aligned}\mathbb{P}\left(\max_{1\le k\le n} S_k \ge nx\right) &\le \mathbb{P}\left(\max_{1\le k\le n} S_k \ge nx, \langle S\rangle_n \le nv^2\right)\\ &\quad +\mathbb{P}\left(\max_{1\le k\le n} S_k \ge nx, \langle S\rangle_n > nv^2\right)\\ &\le \mathbb{P}(S_k \ge nx \text{ and } \langle S\rangle_k \le nv^2 \text{ for some } k\in[1,n])\\ &\quad +\mathbb{P}\left(\langle S\rangle_n > nv^2\right). \qquad (5.1)\end{aligned}$$

By Theorem 2.2, it follows that, for all $x, v > 0$,

$$\begin{aligned}\mathbb{P}\left(\max_{1\le k\le n} S_k \ge nx\right) &\le \exp\left\{-\frac{x^2}{2\left(n^{\alpha-1}v^2+\frac{1}{3}x^{2-\alpha}\right)}n^\alpha\right\}\\ &\quad +nC_1\exp\left\{-x^\alpha n^\alpha\right\}+\mathbb{P}(\langle S\rangle_n > nv^2),\end{aligned}$$

Using the exponential Markov's inequality and the condition $\mathbb{E}\exp\{(\frac{\langle S\rangle_n}{n})^{\frac{\alpha}{1-\alpha}}\}\le C_2$, we get, for all $v>0$,

$$\mathbb{P}\left(\langle S\rangle_n > nv^2\right) \le \mathbb{E}\exp\left\{\left((\frac{\langle S\rangle_n}{n})^{\frac{\alpha}{1-\alpha}}-v^{2\frac{\alpha}{1-\alpha}}\right)\right\}\le C_2\exp\{-v^{2\frac{\alpha}{1-\alpha}}\}.$$

Taking $v=(nx)^{\frac{1-\alpha}{2}}$, we obtain, for all $x>0$,

$$\mathbb{P}\left(\max_{1\le k\le n} X_k \ge nx\right) \le \exp\left\{-\frac{x^{1+\alpha}}{2\left(1+\frac{1}{3}x\right)}n^\alpha\right\}+(nC_1+C_2)\exp\{-x^\alpha n^\alpha\},$$

which gives inequality (2.5).

Next, we prove that the power α in (2.6) is optimal. Let $(X_i,\mathcal{F}_i)_{i\ge1}$ be the sequence of martingale differences constructed in the proof of the second assertion of Theorem 2.1. Then $\frac{\langle S\rangle_n}{n}=X^2$,

$$\sup_i \mathbb{E}\exp\left\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\right\}=\frac{1}{2}\mathbb{E}\exp\{X^{\frac{\alpha}{1-\alpha}}\}<\infty$$

and

$$\sup_n \mathbb{E}\exp\left\{(\frac{\langle S\rangle_n}{n})^{\frac{\alpha}{1-\alpha}}\right\}=\mathbb{E}\exp\{X^{\frac{2\alpha}{1-\alpha}}\}<\infty.$$

Using the same argument as in the proof of Theorem 2.1, we obtain, for n large enough,

$$\mathbb{P}\left(\max_{1\le k\le n} S_k \ge n\right) \ge \exp\left\{-3n^\alpha\right\}.$$

This ends the proof of Corollary 2.3.

References

[1] J., Dedecker. Exponential inequalities and functional central limit theorems for random fields. *ESAIM Probab. Statit.* **5** (2001), 77–104.

[2] B., Delyon. Exponential inequalities for sums of weakly dependent variables. *Electron. J. Probab.* **14** (2009), 752–779. MR-2495559

[3] K., Dzhaparidze and J. H., van Zanten. On Bernstein-type inequalities for martingales. *Stochastic Process. Appl.* **93** (2001), 109–117. MR-1819486

[4] X., Fan, I., Grama and Q., Liu. Hoeffding's inequality for supermartingales. *Stochastic Process. Appl.* **122** (2012), 3545–3559. MR-2956116

[5] D. A., Freedman. On tail probabilities for martingales. *Ann. Probab.* **3** (1975), 100–118. MR-0380971

[6] P., Hall and C. C., Heyde. *Martingale Limit Theory and Its Application*, Academic Press, 1980, 81–96.

[7] N., Laib. Exponential-type inequalities for martingale difference sequences. Application to nonparametric regression estimation. *Commun. Statist.-Theory. Methods*, **28** (1999), 1565–1576. MR-1707103

[8] H., Lanzinger and U., Stadtmüller. Maxima of increments of partial sums for certain subexponential distributions. *Stochastic Process. Appl.*, **86** (2000), 307–322. MR-1741810

[9] E., Lesigne and D., Volný. Large deviations for martingales. *Stochastic Process. Appl.* **96** (2001), 143–159. MR-1856684

[10] Q., Liu and F., Watbled. Exponential ineqalities for martingales and asymptotic properties of the free energy of directed polymers in a random environment. *Stochastic Process. Appl.* **119** (2009), 3101–3132. MR-2568267

[11] F., Merlevède, M., Peligrad and E., Rio. Bernstein inequality and moderate deviations under strong mixing conditions. *IMS Collections. High Dimensional Probability* **5** (2009), 273–292. MR-2797953

[12] F., Merlevède, M., Peligrad and E., Rio. A Bernstein type inequality and moderate deviations for weakly dependent sequences. *Probab. Theory Relat. Fields* **151** (2011), 435–474. MR-2851689

Acknowledgments. We would like to thank the two referees for their helpful remarks and suggestions.