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Transience and recurrence of rotor-router walks on directed covers of graphs*

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Abstract

The aim of this note is to extend the result of Angel and Holroyd [1] concerning the transience and the recurrence of transfinite rotor-router walks, for random initial configuration of rotors on homogeneous trees. We address the same question on directed covers of finite graphs, which are also called trees with finitely many cone types or periodic trees. Furthermore, we provide an example of a directed cover such that the rotor-router walk can be either recurrent or transient, depending only on the planar embedding of the periodic tree.

Keywords: graphs; directed covers; rotor-router walks; multitype branching process; recurrence; transience.
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1 Introduction

Suppose we are given a finite connected and directed graph G with adjacency matrix $D = (d_{ij})_{i,j\in\mathsf{G}}$. Using G, one can construct a labelled rooted tree $\mathcal{T}$ in the following way. Start with a root vertex which is labelled with some $i \in \mathsf{G}$. Then define the tree recursively such that, if x is a vertex in $\mathcal{T}$ with label $i \in \mathsf{G}$, then x has d_{ij} successors with label j. The tree $\mathcal{T}$ is called the *directed cover* of G. Random walks on directed covers of graphs have been studied by Takacs [17], Nagnibeda and Woess [15]. On infinite graphs, their methods have been extended by Gilch and Müller [7].

Rotor-router walks have been first introduced into the physics literature under the name *Eulerian walks* by Priezzhev, D.Dhar et al [16] as a model of *self organized criticality*, a concept established by Bak, Tang and Wiesenfeld [2]. To define a rotor-router walk on a graph consider on each vertex of the graph an arrow (the rotor) pointing to one of the neighbours of the vertex. A particle performing a rotor-router walk first changes the rotor at its current position to point to the next neighbour, in a fixed order chosen at the beginning, and then moves to the neighbour the rotor is now pointing at. These walks have received increased attention in the last years, and in many settings there is remarkable agreement between the behaviour of rotor-router walks and the expected behaviour of random walks. For example, see Holroyd and Propp [9], who

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showed that many quantities associated to rotor-router walks such as normalized hitting frequencies, hitting times and occupation frequencies, are concentrated around their expected values for random walks.

For a bibliographical picture in this context, see also Cooper and Spencer [4], Doerr and Friedrich [5], Angel and Holroyd [1], and also Cooper, Doerr et al. [3]. On the other hand, rotor-router walks and random walks can also have striking differences. For example, in questions concerning recurrence and transience of rotor-router walks on homogeneous trees, this has been proven by Landau and Levine [12]. For random initial configurations on homogeneous trees, see Angel and Holroyd [1]. Furthermore, one can use rotor-router walks in order to solve questions regarding the behaviour of random walks: for example, in [10] we have used a special rotor-router process in order to determine the harmonic measure, that is, the exit distribution of a random walk from a finite subset of a graph.

In this note, we extend the result of Angel and Holroyd [1, Theorem 6] for rotor-router walks with random initial configuration of rotors on directed covers of graphs. The proofs are a generalization of [1] and are based on the extinction/survival of an appropriate multitype branching process (MBP). Such a MBP encodes the subtree on which rotor-router particles can reach infinity. We also give several examples where different phase transitions may appear. We give a graph G with two types of vertices and consider its directed cover $\mathcal{T}$ with all its possible planar embeddings in the plane. For the same random initial configuration of rotors on these trees, we show that the behaviour of the rotor-router walk depends dramatically on the planar embedding. This corresponds to the fact that different rotor sequence gives rise to different behaviour of the rotor-router walk. In [11] the authors study the rotor-router group on directed covers.

2 Preliminaries

Graphs and Trees. Let $\mathsf{G} = (V, E)$ be a locally finite and connected directed multigraph, with vertex set V and edge set E. For ease of presentation, we shall identify the graph G with its vertex set V, i.e., $i \in \mathsf{G}$ means $i \in V$. If (i, j) is an edge of G, we write $i \sim_{\mathsf{G}} j$, and write $\mathrm{d}(i, j)$ for the *graph distance*. Let $D = (d_{ij})_{i,j\in\mathsf{G}}$ be the *adjacency matrix* of G, where d_{ij} is the number of directed edges connecting i to j. We write d_i for the sum of the entries in the i-th row of D, that is $d_i = \sum_{j\in\mathsf{G}} d_{ij}$ is the *degree* of the vertex i.

A *tree* $\mathcal{T}$ is a connected, cycle-free graph. A *rooted tree* is a tree with a distinguished vertex r, called *the root*. For a vertex $x \in \mathcal{T}$, denote by $|x|$ the *height* of x, that is the graph distance from the root to x. For $h \in \mathbb{N}$, define the *truncated tree* $\mathcal{T}^h = \{x \in \mathcal{T} : |x| \leq h\}$ to be the subgraph of $\mathcal{T}$ induced by the vertices at height smaller or equal to h. For a vertex $x \in \mathcal{T} \setminus \{r\}$, denote by $x^{(0)}$ its *ancestor*, that is the unique neighbour of x closer to the root r. It will be convenient to attach an additional vertex $r^{(0)}$ to the root r, which will be considered in the following as a sink vertex. Additionally we fix a planar embedding of $\mathcal{T}$ and enumerate the neighbours of a vertex $x \in \mathcal{T}$ in counter-clockwise order $\left(x^{(0)}, x^{(1)}, \dots, x^{(d_x-1)}\right)$ beginning with the ancestor. We will call a vertex y a *descendant* of x, if x lies on the unique shortest path from y to the root r. A descendant of x, which is also a neighbour of x, will be called a *child*. The *principal branches* of $\mathcal{T}$ are the subtrees rooted at the children of the root r. The *wired tree* $\widetilde{\mathcal{T}}^h$ of height h is the multigraph obtained from $\mathcal{T}^h$ by collapsing all vertices $y \in \mathcal{T}$ with $\mathrm{d}(r, y) = h$, together with the ancestor $r^{(0)}$ of the root to a single vertex s, the sink. We do not collapse multiple edges. Let us introduce *the down and up sinks*

$$s_\downarrow = \{r^{(0)}\} \text{ and } s^\uparrow = \{x \in \mathcal{T}_i^h : |x| = h\}, \tag{2.1}$$

$\chi_i(k)$ ($i \downarrow$, $k \rightarrow$)	1	2
1	2	
2	2	1

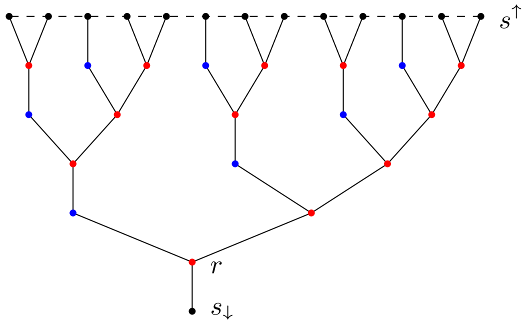


Figure 1: The wired Fibonacci tree $\widetilde{\mathcal{T}}_2^5$ of height 5 and root of type 2.

and $s = s_{\downarrow} \cup s^{\uparrow}$.

Directed Covers of Graphs. Suppose now that G is a finite, directed and strongly connected multigraph with adjacency matrix $D = (d_{ij})$. Let m be the cardinality of the vertices of G, and label the vertices of G by $\{1, 2, \dots, m\}$. The *directed cover* $\mathcal{T}$ of G is defined recursively as a rooted tree $\mathcal{T}$ whose vertices are labelled by the vertex set $\{1, 2, \dots, m\}$ of G. The root r of $\mathcal{T}$ is labelled with some $i \in$ G. Recursively, if x is a vertex in $\mathcal{T}$ with label $i \in$ G, then x has d_{ij} descendants with label j. We define the *label function* $\tau : \mathcal{T} \to$ G as the map that associates to each vertex in $\mathcal{T}$ its label in G. The label $\tau(x)$ of a vertex x will be also called the *type* of x. For a vertex $x \in \mathcal{T}$, we will not only need its type, but also the types of its children. In order to keep track of the type of a vertex and the types of its children we introduce the *generation function* $\chi = (\chi_i)_{i \in \mathsf{G}}$ with $\chi_i : \{1, \dots, d_i\} \to$ G. For a vertex x of type i, $\chi_i(k)$ represents the type of the k-th child $x^{(k)}$ of x, i.e.,

$$\text{if } \tau(x) = i \text{ then } \chi_i(k) = \tau(x^{(k)}), \text{ for } k = 1, \dots, d_i.$$

As the neighbours $\left(x^{(0)}, \dots, x^{(d_{\tau(x)})}\right)$ of any vertex x are drawn in clockwise order, the generation function χ also fixes the planar embedding of the tree and thus defines $\mathcal{T}$ uniquely as a planted plane tree. The tree $\mathcal{T}$ constructed in this way is called the *directed cover* of G. Such trees are also known as *periodic trees*, see Lyons [13], or *trees with finitely many cone types* in Nagnibeda and Woess [15]. The graph G is called the *base graph* or the *generating graph* for the tree $\mathcal{T}$. We write $\mathcal{T}_i$ for a tree with root r of type i, that is $\tau(r) = i$.

Example 2.1. *The* Fibonacci tree *is the directed cover of the graph* G *on two vertices with adjacency matrix* $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. *The* (α, β) bi-regular tree *with parameters* $\alpha, \beta \in \mathbb{N}$ *is the directed cover of the graph* G *on two vertices with adjacency matrix* $\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$.

Rotor-Router Walks. On a locally finite and connected graph G, a *rotor-router walk* is defined as follows. For each vertex $x \in$ G fix a cyclic ordering $c(x)$ of its neighbours: $c(x) = \left(x^{(0)}, x^{(1)}, \dots, x^{(d_x - 1)}\right)$, where $x \sim_{\mathsf{G}} x^{(i)}$ for all $i = 0, 1, \dots, d_x - 1$ and d_x is the degree of x. The ordering $c(x)$ is called the *rotor sequence* of x. A *rotor configuration* is a function $\rho : \mathsf{G} \to \mathsf{G}$, with $\rho(x) \sim_{\mathsf{G}} x$, for all $x \in$ G. By abuse of notation, we write $\rho(x) = i$ if the rotor at x points to the neighbour $x^{(i)}$. A rotor-router walk is defined by the following rule. Let x be the current position of the particle, and $\rho(x) = i$ the state of the rotor at x. In one step of the walk the following happens. First the position of the rotor at x is incremented to point to the next neighbour $x^{(i+1)}$ in the ordering $c(x)$, that is, $\rho(x)$ is set to $i+1$ (with addition performed modulo d_x). Then the particle moves to position $x^{(i+1)}$. The rotor-router walk is obtained by repeatedly applying this rule.

Recurrence and Transience Suppose now that the graph G is infinite and connected, and let o be a fixed vertex in G, the root. Start with a particle at o and let it perform a rotor-router walk stopped at the first return to o. Either the particle returns to o after a finite number of steps (*recurrence*), or it escapes to infinity without returning to o, and visiting each vertex only finitely many times (*transience*). In both cases, the positions of the rotors after the walk is complete are well defined. Before starting a new particle at the root, we do not reset the configuration of rotors. We then start a new particle at the root o, and repeat the above procedure, and so on. This type of rotor-router walk is called *transfinite rotor-router walk*, see Holroyd and Propp [9]. Let $E_n = E_n(\mathsf{G})$ be number of particles that escape to infinity after n rotor-router walks are run from o in this way. The following result, due to Schramm states that a rotor-router walk is no more transient than a random walk. See [9, Theorem 10] for a proof.

Theorem 2.2. *[Density bound–Schramm] For any locally finite graph, any starting vertex o, any cyclic order of neighbours and any initial rotor positions,*

$$\limsup_{n\to\infty} \frac{E_n}{n} \le \mathcal{E},$$

where $\mathcal{E}$ denotes the probability that a simple random walk *on G started at the origin o never returns to o.*

3 Recurrence and Transience of Rotor-Router Walks

We study now the behaviour of transfinite rotor-router walks on directed covers of graphs for random initial rotor configurations. In particular, we generalize a theorem of Holroyd and Angel [1, Theorem 6] which proves a transition between recurrent and transient phases for transfinite rotor-router walks on homogeneous trees $\mathbb{T}_b$ of degree b, with random initial rotor configuration $(\rho(v))_{v\in\mathbb{T}_b}$. The random variables $\rho(v)$ are i.i.d on $\{0,1,\dots,b\}$.

For the tree $\mathcal{T}$ with root r, the quantity $E_n(\mathcal{T},\rho)$ represents the number of particles that escape to infinity after n rotor-router walks are run from r with initial rotor configuration ρ, and $\mathcal{E}(\mathcal{T})$ represents the probability that a simple random walk on $\mathcal{T}$ started at r, never returns to r. The theorem then states the following.

Theorem 3.1 (Angel and Holroyd)**.** *For a random i.i.d. initial rotor configuration ρ on the homogeneous tree $\mathbb{T}_b$, writing v for an arbitrary vertex, we have almost surely*

(i) $\displaystyle\lim_{n\to\infty} \frac{E_n(\mathbb{T}_b,\rho)}{n} = \mathcal{E}(\mathbb{T}_b)$, *if* $\mathbb{E}\big[\rho(v)\big] < b-1$; *(Transience)*

(ii) $E_n(\mathbb{T}_b,\rho) = 0$ *for all* $n \ge 0$, *if* $\mathbb{E}\big[\rho(v)\big] \ge b-1$. *(Recurrence)*

The discontinuous phase transition above is related to a branching process. The main idea of the proof is to model the connected subtree of vertices, along which particles may move to infinity, as a Galton-Watson tree. In the case of directed covers, since we have vertices of different types, we need to model a multitype branching process (MBP).

Multitype Branching Processes. A multitype branching process (MBP) is a generalization of a Galton-Watson process, where one allows a finite number of distinguishable types of particles with different probabilistic behaviour. The particle types will coincide with the different types of vertices in the direct covers under consideration, and will be denoted by $\{1,\dots,m\}$.

A *multitype branching process* is a Markov process $(\boldsymbol{Z}_n)_{n\in\mathbb{N}_0}$ such that the states $\boldsymbol{Z}_n=(Z_n^1,\dots,Z_n^m)$ are m-dimensional vectors with non-negative components. The initial state $\boldsymbol{Z}_0$ is nonrandom. The i-th entry Z_n^i of $\boldsymbol{Z}_n$ represents the number of particles of type i in the n-th generation. The transition law of the process is as follows. If $\boldsymbol{Z}_0=\boldsymbol{e}_i$, where $\boldsymbol{e}_i$ is the m-dimensional vector whose $i-th$ component is 1 and all the others are 0, then $\boldsymbol{Z}_1$ will have the generating function $\boldsymbol{f}(\boldsymbol{z})=\big(f^1(\boldsymbol{z}),\dots,f^m(\boldsymbol{z})\big)$ with

$$f^i(\boldsymbol{z})=f^i(z_1,\dots,z_m)=\sum_{s_1,\dots,s_m\geq 0}p^i(s_1,\dots,s_m)z_1^{s_1}\cdots z_m^{s_m}, \tag{3.1}$$

and $0\leq z_1,\dots,z_m\leq 1$, where $p^i(s_1,\dots,s_m)$ is the probability that a particle of type i has s_j children of type j, for $j=1,\dots,m$. For $\boldsymbol{i}=(i_1,\dots,i_m)$ and $\boldsymbol{j}=(j_1,\dots,j_m)$, the one-step transition probabilities are given by

$$\boldsymbol{p}(\boldsymbol{i},\boldsymbol{j})=\mathbb{P}[\boldsymbol{Z}_{n+1}=\boldsymbol{j}|\boldsymbol{Z}_n=\boldsymbol{i}]=\text{coefficient of } \boldsymbol{z}^{\boldsymbol{j}} \text{ in } \big(\boldsymbol{f}(\boldsymbol{z})\big)^{\boldsymbol{i}}:=\prod_{k\in\mathsf{G}}f^k(\boldsymbol{z})^{i_k}.$$

For vectors $\boldsymbol{z},\boldsymbol{s}$, we write $\boldsymbol{z}^{\boldsymbol{s}}=(z_1^{s_1},\dots,z_m^{s_m})$. Let $M=(m_{ij})$ be the matrix of the first moments:

$$m_{ij}=\mathbb{E}[Z_1^j|\boldsymbol{Z}_0=\boldsymbol{e}_i]=\left.\frac{\partial f^i(z_1,\dots,z_m)}{\partial z_j}\right|_{\boldsymbol{z}=\boldsymbol{1}} \tag{3.2}$$

represents the expected number of offsprings of type j of a particle of type i in one generation. If there exists an n such that $m_{ij}^{(n)}>0$ for all i,j, then M is called *strictly positive* and the process $\boldsymbol{Z}_n$ is called *positive regular*. If each particle has exactly one child, then $\boldsymbol{Z}_n$ is called *singular*. The following is well known; see Harris [8].

Theorem 3.2. *Assume $\boldsymbol{Z}_n$ is positive regular and nonsingular, and let $\mathsf{r}(M)$ be the spectral radius of M. It $\mathsf{r}(M)\leq 1$, then the process $\boldsymbol{Z}_n$ dies with probability one. If $\mathsf{r}(M)>1$, then $\boldsymbol{Z}_n$ survives with positive probability.*

3.1 Nondeterministic Rotor Configurations on Directed Covers

Recall the setting we are working on: G is a finite graph with vertices labelled by $\{1,\dots,m\}$; for $i\in\mathsf{G}$, $\mathcal{T}_i$ is the directed cover of G with root r of type i. For a vertex $x\in\mathcal{T}_i$ with $\tau(x)=j\in\mathsf{G}$, we have denoted by $x^{(0)}$ its parent, and by $x^{(1)},\dots,x^{(d_j)}$ its d_j children. We choose the cyclic ordering $c(x)$ of the neighbours of x to be $(x^{(0)},x^{(1)},\dots,x^{(d_j)})$, and we allow the initial rotor to point at an arbitrary neighbour in this order. We will embed the tree in the plane in such a way that the rotors turn in counter-clockwise order, when following this rotor sequence. Recall also that for some rotor configuration ρ on $\mathcal{T}_i$, we write $\rho(x)=k$ if the rotor at x points to the neighbour $x^{(k)}$.

Let now $\mathcal{D}=(\mathcal{D}_1,\dots,\mathcal{D}_m)$ be a vector of probability distributions: for each $i\in\mathsf{G}$, $\mathcal{D}_i$ is a probability distribution with values in $\{0,\dots,d_i\}$. Consider a *random initial configuration* ρ of rotors on $\mathcal{T}_i$, such that $\big(\rho(x)\big)_{x\in\mathcal{T}_i}$ are independent random variables, and $\rho(x)$ has distribution $\mathcal{D}_j$ if the vertex x is of type j. Shortly

$$\rho(x)\overset{d}{\sim}\mathcal{D}_j \quad\Longleftrightarrow\quad \tau(x)=j. \tag{3.3}$$

If (3.3) is satisfied, we shall say that the rotor configuration ρ is $\mathcal{D}=(\mathcal{D}_1,\dots,\mathcal{D}_m)$-distributed, and we write $\rho\overset{d}{\sim}\mathcal{D}$. Performing transfinite rotor-router walks on $\mathcal{T}_i$ with

$\mathcal{D}$-distributed initial rotor configuration ρ, we observe a phase transition between the recurrent and transient regimes, similar to the case of homogeneous trees in Theorem 3.1. For defining the critical point of this phase transition, we need to introduce some additional definitions.

Consider a general tree $\mathcal{T}$ with rotor configuration ρ. For a vertex $x \in \mathcal{T}$ define the set of *good children* as $\{x^{(k)} : \rho(x) < k \leq d_{\tau(x)}\}$. This means that a rotor-router particle starting at a vertex will first visit all its good children before visiting its ancestor. An infinite sequence of vertices $(x_n)_{n\in\mathbb{N}}$ with each vertex being a child of the previous one, is called a *live path* if for every $n \geq 0$ the vertex x_{n+1} is a good child of x_n. An *end* of $\mathcal{T}$ is an infinite sequence of vertices $x_1, x_2, \dots$ each being the parent of the next. An end is called *live* if the subsequence $(x_i)_{i\geq j}$ starting at one of its vertices is a live path.

Denote by $E_\infty(\mathcal{T},\rho) = \lim_{n\to\infty} E_n(\mathcal{T},\rho)$ the total number of particle escaping to infinity, when one launches an infinite number of particles. Recall now an useful result for general trees $\mathcal{T}$, whose proof can be found in [1, Proposition 8].

Proposition 3.3. *The total number of escapes $E_\infty(\mathcal{T},\rho)$ equals the number of live ends in the initial rotor configuration ρ.*

Definition 3.4. *For $i \in \mathsf{G}$ and $k \in \{0,\dots,d_i\}$ denote by $\mathfrak{C}_i^j(k)$ the number of good children with type j of a vertex x with type i, if the rotor $\rho(x)$ at x is in position k, i.e.,*

$$\mathfrak{C}_i^j(k) = \#\{l \in \{k+1,\dots,d_i\} : \chi_i(l) = j\}.$$

We have that $\sum_{j\in\mathsf{G}} \mathfrak{C}_i^j(k) = d_i - k$. Using this definition we can now define a MBP which models connected subtrees consisting of only good children. In this MBP, $p^i(s_1,\dots,s_m)$ represents the probability that a vertex of type i has s_j good children of type j, with $j = 1,\dots,m$. Define the generating function of the MBP as in (3.1) and the probabilities p^i by

$$p^i(s_1,\dots,s_m) = \begin{cases} \mathcal{D}_i(k) & \text{if for all } j = 1,\dots,m:\ s_j = \mathfrak{C}_i^j(k), \text{ and } k \in \{0,\dots,d_i\}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.4}$$

with $\mathcal{D}_i(k) = \mathbb{P}[\rho(x) = k]$, for $k \in \{0,\dots,d_i\}$ and $i \in \mathsf{G}$. Let $M(\mathcal{D})$ be the first moment matrix — as defined in (3.2) — of the MBP with offspring probabilities given in (3.4).We are now ready to state our theorem, as an extension of [1, Theorem 6].

Theorem 3.5. *Let ρ be a random rotor configuration with distribution $\mathcal{D} = (\mathcal{D}_1,\dots,\mathcal{D}_m)$ on the directed cover $\mathcal{T}_i$ with root r of type i, of a finite graph G with m vertices. Let n particles perform transfinite rotor-router walks on $\mathcal{T}_i$. Then we have almost surely:*

1. *Recurrence: $E_n(\mathcal{T}_i,\rho) = 0$, for all $n \geq 0$ and $i \in \mathsf{G}$ if $\mathsf{r}\big(M(\mathcal{D})\big) \leq 1$;*
2. *Transience: $\lim_{n\to\infty} \dfrac{E_n(\mathcal{T}_i,\rho)}{n} = \mathcal{E}_i$ for all $i \in \mathsf{G}$ if $\mathsf{r}\big(M(\mathcal{D})\big) > 1$.*

The quantity $\mathcal{E}_i$ represents the probability that a simple random walk starting at the root r of $\mathcal{T}_i$ never returns to r, and $\mathsf{r}\big(M(\mathcal{D})\big)$ is the spectral radius of $M(\mathcal{D})$.

Proof of Theorem 3.5(a). For any fixed $x \in \mathcal{T}_i$ with $\tau(x) = j \in \mathsf{G}$ the set of descendants of x that can be reached via a path of good vertices forms a multitype branching process with offspring distributions $p^j(s_1,\dots,s_m)$ defined as in (3.4). The survival/extinction of this MBP is controlled by the matrix of the first moments $M(\mathcal{D}) = (m_{ij})_{i,j\in\mathsf{G}}$.

Since $\mathsf{r}(M(\mathcal{D})) \leq 1$, the extinction probability is 1 by Theorem 3.2 and the MBP dies almost surely, hence there are no live paths. Therefore by Proposition 3.3 there are no escapes almost surely and $E_n(\mathcal{T}_i,\rho) = 0$. This gives the recurrence of the rotor-router walk with random initial configuration ρ which is $\mathcal{D} = (\mathcal{D}_1,\dots,\mathcal{D}_m)$-distributed. $\square$

Transience. The transience part in Theorem 3.5 requires some additional work. Consider now the wired tree $\widetilde{\mathcal{T}}_i^h$ of height h, with sink $s = s_\downarrow \cup s^\uparrow$ as defined in (2.1). Let ρ be some rotor configuration on $\widetilde{\mathcal{T}}_i^h$. The proof of the transience uses the abelian property of rotor-router walks.

Denote by $e_r(\rho)$ the rotor configuration of the rotor-router group of $\widetilde{\mathcal{T}}_i^h$ resulting from starting a particle at the root r and letting it perform a rotor-router walk until it reaches the sink s. Write $e_r^n(\rho)$ for the corresponding rotor configuration when we let n particles perform a rotor-router walk starting at the root until hitting the sink. In addition, let $\sigma_i^n(\rho)$ be the particle configuration on $\widetilde{\mathcal{T}}_i^h$ obtained by routing n particles from the root r to the sink. As we route all n particles to the sink, the support of $\sigma_i^n(\rho)$ is contained in s. Define

$$n_i(h) = \max_{\rho \text{ rotor configuration}} \min\{n \geq 0 : s^\uparrow \subset \operatorname{supp} \sigma_i^n(\rho)\},$$

Hence $n_i(h)$ is the minimal number of particles needed such that every vertex of $s^\uparrow$, that is, every vertex at level h, has been hit by one rotor-router particle, maximized over all possible rotor-configurations of $\widetilde{\mathcal{T}}_i^h$. The next result gives an upper bound for $n_i(h)$.

Lemma 3.6. *For all $h \geq 1$ and $i \in \mathsf{G}$, we have*

$$n_i(h) \leq (D_{\max} + 1)^h,$$

where $D_{\max} = \max_{i \in \mathsf{G}} d_i$.

Proof. Using the results in [6], in particular Lemma 13 and Lemma 19, it is enough to take the maximum over all recurrent rotor configurations. We prove the statement by induction over the height h.

For $h = 1$, after one full turn of the rotor in $\widetilde{\mathcal{T}}_i^1$, every vertex of s has been visited, hence $n_i(1) = d_i + 1$. For $h > 1$, start with n particles at the origin, where $n = (d_i + 1)n'$. By the abelian property, if we first take one step for each of the particles and then route each particle from its new location to the sink, we also reach the same final particle configuration σ_i^n. Hence, we have

$$n_i(h) \leq (d_i + 1) \max_{1 \leq k \leq d_i} n_{\chi_i(k)}(h-1)$$

which completes the proof. □

We shall see below that it suffices to have at least one particle which stopped at each sink vertex of $s^\uparrow$, instead of exactly one particle like in [1] for homogeneous trees. They can specify there the exact numbers of particles needed at the origin. For our case, it suffices to have an exponential upper bound. In our setting, we are not only interested in the number of particles stopped at some level h, but also on the type of particles. This will be done next.

Number of Vertices of a Given Type on a Level. Denote by $w_{i,j}(n)$ the number of vertices of type j at height n in $\mathcal{T}_i$ and let $w(n)$ be the matrix with entries $(w_{i,j}(n))_{i,j \in \mathsf{G}}$. In addition, let $W_{i,j}(z)$ be the generating function of $w_{i,j}(n)$:

$$W_{i,j}(z) = \sum_{n=0}^{\infty} w_{i,j}(n) z^n, \quad \text{for } z \in \mathbb{C}. \tag{3.5}$$

Factorizing $w_{i,j}(n)$ with respect to the first level, we have

$$w_{i,j}(n) = \sum_{k \in \mathsf{G}} w_{i,k}(1) w_{k,j}(n-1) = \sum_{k \in \mathsf{G}} d_{ik} w_{k,j}(n-1).$$

Then $W_{i,j}(z)$ can be written as

$$\begin{aligned}W_{i,j}(z) &= \delta_i(j) + \sum_{n=1}^{\infty} w_{i,j}(n) z^n = \delta_i(j) + z\sum_{n=0}^{\infty} w_{i,j}(n+1) z^n \\ &= \delta_i(j) + z\sum_{n=0}^{\infty}\sum_{k\in\mathsf{G}} d_{ik} w_{k,j}(n) z^n = \delta_i(j) + z\sum_{k\in\mathsf{G}} d_{i,k} W_{k,j}(z), \quad \text{for } i,j\in\mathsf{G},\end{aligned}$$

where $\delta_i(j)$ equals 1 if $i=j$ and 0 otherwise. If we write $W(z) = (W_{i,j}(z))_{i,j\in\mathsf{G}}$, then the above implicit equation can be written in matrix form as $W(z) = I + zDW(z)$, where D is the adjacency matrix of the generating graph G and I is the identity matrix. Hence

$$W(z) = (I - zD)^{-1}.$$

Having explicitly the matrix $W(z)$ of generating function, we can compute the coefficients $w(n)$

$$w_{i,j}(n) = \frac{W_{i,j}^{(n)}(0)}{n!}.$$

In a tree $\mathcal{T}_i$ with root of type i, the total number of vertices at level n is $w_i(n) = \sum_{j\in\mathsf{G}} w_{i,j}(n)$.

Example 3.7. *In the case of Fibonacci tree, we have*

$$W(z) = \frac{1}{1-z-z^2}\begin{pmatrix}1-z & z\\ z & 1\end{pmatrix}$$

and

$$w(n) = \begin{pmatrix}F_{n-2} & F_{n-1}\\ F_{n-1} & F_n\end{pmatrix},$$

where F_n represents the n-th Fibonacci number. In a tree with root of type 1, the total number of vertices at level n is F_n and a tree with root of type 2 has F_{n+1} vertices at level n.

Knowing the number of vertices of a given type on the level h of a tree $\mathcal{T}_i$ with root of type i, we can now generalize [1, Corollary 23]. For sake of completeness, we state it here in the form needed for directed covers of graphs. Consider the MBP with offspring probabilities given as in (3.4), $M(\mathcal{D})$ its first moment matrix with spectral radius $\mathsf{r}(M(\mathcal{D}))$. The proof of the next corollary will follow mostly the line of the proof of [1, Corollary 23].

Corollary 3.8. *Let ρ be a random rotor configuration with distribution $\mathcal{D} = (\mathcal{D}_1,\dots,\mathcal{D}_m)$ on the directed cover $\mathcal{T}_i$ with root of type i, of a finite graph G with m vertices. Suppose $\mathsf{r}(M(\mathcal{D})) > 1$. Then there exists $\delta_i, c_i, C_i > 0$, such that for all n*

$$\mathbb{P}\big[E_n(\mathcal{T}_i,\rho) < \delta_i n\big] \le C_i e^{-c_i n}, \quad \textit{for all } i\in\mathsf{G}.$$

Proof. We shall prove the bound for $n\in\mathbb{N}$ of the form $(D_{\max}+1)^h$, where h is an integer and $D_{\max} = \max_{i\in\mathsf{G}} d_i$, since this implies the claimed result for all n, but with different constants.

The MBP with probabilities p^i as in (3.4) and $\mathsf{r}(M(\mathcal{D})) > 1$ survives with positive probability. Hence, for each $i\in\mathsf{G}$, with positive probability there exists a live path starting at the root of $\mathcal{T}_i$. Existence of a live path implies that the first particle escapes, hence

$$\mathsf{p}_i := \mathbb{P}[E_1(\mathcal{T}_i,\rho) = 1] > 0, \quad \text{for all } i\in\mathsf{G}.$$

For each $i,j \in \mathsf{G}$, denote by $X_{i,j}$ the set of vertices $v \in \mathcal{T}_i$ of type j at level $h+1$ such that there is a live path starting at v. Let $X_i = \bigcup_{j\in\mathsf{G}} X_{i,j}$ and $\#X_i$ be the cardinality of this set. For all j, the random variables $\#X_{i,j}$ are independent, with distribution Binomial$\big(w_{i,j}(h), \mathsf{p}_j\big)$. Let us first prove that

$$E_n(\mathcal{T}_i,\rho) \geq \#X_i, \quad \text{for all } i \in \mathsf{G} \tag{3.6}$$

and $n = (D_{\max}+1)^h$. From [9, Lemmas 18,19], it suffices to prove (3.6) for the truncated tree $\mathcal{T}_i^H$, with $H > h$, i.e.,

$$E_n(\mathcal{T}_i^H, s^{\uparrow}, \rho^H) \geq \#X_i, \quad \text{for all } i \in \mathsf{G}. \tag{3.7}$$

Here $E_n(\mathcal{T}_i^H, s^{\uparrow}, \rho^H)$ represents the number of particles that stop at $s^{\uparrow}$ (the vertices at level H of $\mathcal{T}_i^H$) when we start n rotor-router walks at the root of $\mathcal{T}_i$ and rotor configuration ρ^H (the restriction of ρ on $\mathcal{T}_i^H$). In the truncated tree $\mathcal{T}_i^H$, start $n = (D_{\max}+1)^h$ particles at the root, and stop them when they either enter the level $h+1$ or return to the root. By Lemma 3.6, there is at least one particle at each vertex on level $h+1$, and the rest of them are located at the root. Moreover, the vertices at distance greater than $h+1$ were not reached, and the rotors there are unchanged. Now for every vertex v in X_i restart one particle. Since there is a live path a v the particle will reach the level H without leaving the cone of v, at which point the particle is stopped again. Hence if we restart all particles which were stopped at level $h+1$ at least $\#X_i$ of them will reach level H before returning to the root. Because of the abelian property of rotor-router walks, (3.7) follows, therefore also (3.6).

Now fix $i \in \mathsf{G}$ and choose $k \in \mathsf{G}$ such that there are vertices of type k on the h-level of $\mathcal{T}_i$, that is $w_{i,k}(h) > 0$. We can then bound $\#X_i$ from below by $\#X_{i,k}$ which has distribution Binomial$\big(w_{i,k}(h), \mathsf{p}_k\big)$. Hence for all $\delta_i > 0$, using the Markov inequality, we have

$$\begin{aligned}
\mathbb{P}\big[E_n(\mathcal{T}_i,\rho) < \delta_i n\big] &\leq \mathbb{P}\big[\#X_i < \delta_i n\big] \leq \mathbb{P}\big[\#X_{i,k} < \delta_i n\big] \\
&= \mathbb{P}\Big[e^{-\#X_{i,k}} > e^{-\delta_i n}\Big] \leq \mathbb{E}\Big[e^{-\#X_{i,k}}\Big] e^{\delta_i n} \\
&\leq \big(1 - p_k + p_k e^{-1}\big)^{w_{i,k}(h)} e^{\delta_i n} \leq C_k^n e^{\delta_i n}.
\end{aligned}$$

In the last inequality we define $C_k = \big(1 - p_k + p_k e^{-1}\big) < 1$ and use that $n = (D+1)^h$ and $w_{i,k}(h) \leq n$. Hence for δ_i small enough we can choose $c_i > 0$ such that

$$\mathbb{P}\big[E_n(\mathcal{T}_i,\rho) < \delta_i n\big] \leq C_i e^{-c_i n},$$

which proves the statement. □

We shall also need [1, Lemma 25] which holds for general trees.

Lemma 3.9. *For a graph G with m vertices and $\mathcal{T}_i$ its directed cover with root r of type $i \in \mathsf{G}$, let $\mathcal{T}_{\chi_i(1)}, \dots, \mathcal{T}_{\chi_i(d_i)}$ be its principal branches rooted at the children $r^{(k)}$ of the root, with $k = 1, \dots, d_i$. Let ρ be some rotor configuration on $\mathcal{T}_i$ and ρ_k be its restriction on the tree $\mathcal{T}_{\chi_i(k)}$. For each $i \in \mathsf{G}$, let*

$$l_i = \liminf_{n\to\infty} \frac{E_n(\mathcal{T}_i,\rho)}{n} \quad \textit{and} \quad l_i^k = \liminf_{n\to\infty} \frac{E_n(\mathcal{T}_{\chi_i(k)},\rho_k)}{n}, \quad k = 1,\dots,d_i.$$

Then

$$l_i \geq 1 - \frac{1}{1 + \sum_{k=1}^{d_i} l_i^k} = 1 - \frac{1}{1 + \sum_{j\in\mathsf{G}} d_{ij} l_j}. \tag{3.8}$$

The probability $\mathcal{E}_i$ that a simple random walk (X_t) on $\mathcal{T}_i$ never returns to the root satisfies an relation similar to (3.8). If r is the root of $\mathcal{T}_i$, then $\mathcal{E}_i = \mathbb{P}_r[X_t \neq s_\downarrow, \forall t \geq 0]$. Factorizing the random walk on $\mathcal{T}_i$ with respect to the first step, we get

$$\big(1-\mathcal{E}_i\big)\Big(d_i + 1 - \sum_{j=1}^{m} d_{ij}\big(1-\mathcal{E}_j\big)\Big) = 1,$$

which gives

$$\mathcal{E}_i = 1 - \frac{1}{1+\sum_{j\in \mathsf{G}} d_{ij}\mathcal{E}_j} \tag{3.9}$$

We are now able to prove the transience part in Theorem 3.5.

Proof of Theorem 3.5(b). For each $i \in \mathsf{G}$, let

$$l_i = \liminf_{n\to\infty} \frac{E_n(\mathcal{T}_i,\rho)}{n}.$$

Using Borel-Cantelli Lemma for the events in Corollary 3.8, it follows that for each i, there exists δ_i such that

$$\mathbb{P}\Big[\limsup_{n\to\infty} \frac{E_n(\mathcal{T}_i,\rho)}{n} < \delta_i\Big] = 0.$$

Since

$$\mathbb{P}\Big[\limsup_{n\to\infty} \frac{E_n(\mathcal{T}_i,\rho)}{n} < \delta_i\Big] = 1 - \mathbb{P}\Big[\liminf_{n\to\infty} \frac{E_n(\mathcal{T}_i,\rho)}{n} \geq \delta_i\Big],$$

we have $\mathbb{P}[l_i \geq \delta_i] = 1$, with $\delta_i > 0$. Let a_i be some positive constants such that $l_i \geq a_i$ for all i. Then

$$l_i \geq 1 - \frac{1}{1+\sum_{j\in \mathsf{G}} d_{ij} a_j} \text{ a.s.}$$

Applying this repeatedly gives that for all i, l_i is greater or equal to the fixed point of the iteration

$$a_i \mapsto 1 - \frac{1}{1+\sum_{j\in \mathsf{G}} d_{ij} a_j}$$

from $\mathbb{R}^m \mapsto \mathbb{R}^m$. The return probabilities $\mathcal{E}_i$ are also solutions of the same fixed point equation (3.9). Hence $l_i \geq \mathcal{E}_i$. On the other hand, by Theorem 2.2 we have $l_i \leq \mathcal{E}_i$, which implies

$$\lim_{n\to\infty} \frac{E_n(\mathcal{T}_i,\rho)}{n} = \mathcal{E}_i, \text{ for all } i \in \mathsf{G},$$

if $\mathsf{r}(M(\mathcal{D})) > 1$, which proves the desired. □

Remark 3.10. *For a supercritical positive regular multitype branching process, in the event of nonextinction, the genealogical tree has branching number* $\mathsf{r}(M)$*. Like above,* M *represents the matrix of the first moments of the MBP, and* $\mathsf{r}(M)$ *its spectral radius.*

This means that in the transient case we have $\mathsf{r}(M(\mathcal{D})) = \mathrm{br}(\mathcal{D})$, where $\mathrm{br}(\mathcal{D})$ is the branching number of the genealogical tree with offspring distributions as given in (3.4). For more information on the relation between the spectral radius of a branching process and the branching number, see Lyons [14].

3.1.1 Examples

Example 3.11. *Let us first consider* a generalized Fibonacci tree *depending on the parameter* $\alpha \in \mathbb{N}$. *Consider the graph* G *with adjacency matrix*

$$D = \begin{pmatrix} 0 & \alpha \\ 1 & 1 \end{pmatrix},$$

and $\mathcal{T}_i$ *its directed cover with root of type* $i = 1, 2$. *Depending on the values of* α, *the rotor-router walk on* $\mathcal{T}_i$ *can be either transient or recurrent. If* $\alpha = 1$ *we get the Fibonacci tree, and for* $\alpha = 2$ *we get the binary tree. On such trees we take a random initial configuration of rotors which is uniformly distributed on the neighbours. Since these trees have* 2 *types of vertices (first type* 1 *with* α *children of type* 2, *and second type* 2 *with* 1 *child of type* 1 *and one of type* 2*), we have* $\mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2)$, *with* $\mathcal{D}_1 =$ *Uniform*$(0, \dots, \alpha)$ *and* $\mathcal{D}_2 =$ *Uniform*$(0, 1, 2)$. *Consider the following generation function* χ_i*:*

$$\chi_1(1) = \ldots = \chi_1(\alpha) = 2 \quad \textit{and} \quad \chi_2(1) = 1 \textit{ and } \chi_2(2) = 2.$$

The transition probabilities p^i, $i = 1, 2$ *defined in* (3.4)*, which model a MBP consisting of only good children are then given by:*

$$p^1(0,0) = \ldots = p^1(0,\alpha) = \frac{1}{\alpha} \quad \textit{and} \quad p^2(0,0) = p^2(1,0) = p^2(1,1) = \frac{1}{3}.$$

For the generating functions $f^i(z)$, *with* $z = (z_1, z_2)$ *we then get:*

$$f^1(z) = \frac{1}{\alpha+1}\left(1 + z_2 + z_2^2 + \cdots + z_2^{\alpha}\right)$$
$$f^2(z) = \frac{1}{3}(1 + z_1 + z_1 z_2).$$

The behaviour of rotor-router walks on directed covers of graphs is controlled by the matrix $M(\mathcal{D})$ *of the first moments of the MBP defined above. The entries of* $M(\mathcal{D})$ *can be computed using* (3.2)*, and for this particular example we have*

$$M(\mathcal{D}) = \begin{pmatrix} 0 & \alpha/2 \\ 2/3 & 1/3 \end{pmatrix}.$$

The spectral radius is $\mathsf{r}(M(\mathcal{D})) = \frac{1}{6}\left(1 + \sqrt{12\alpha + 1}\right)$. *Therefore, the rotor-router walk is*

- *recurrent for* $\alpha \le 2$
- *transient for* $\alpha > 2$.

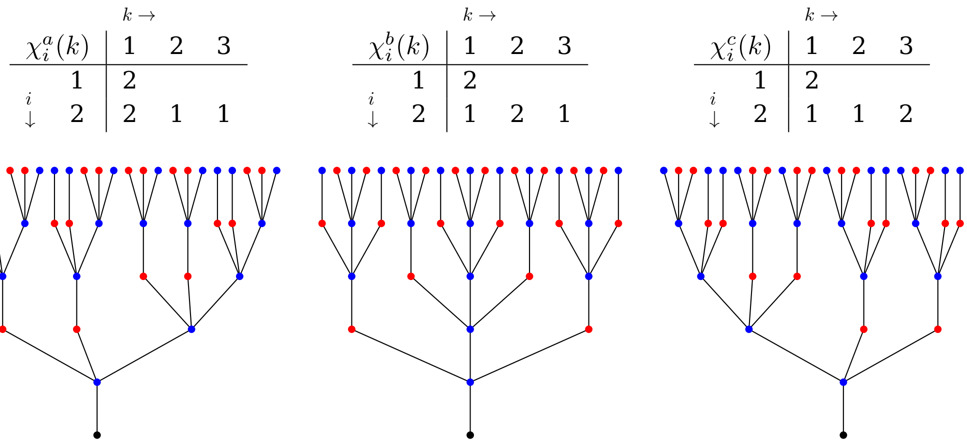
In particular, the rotor-router walk on the Fibonacci tree and on the binary tree is recurrent. Note here the contrast with the simple random walk which is transient.

The next example shows a case where different planar embeddings of the same tree, gives rise to changes in the recurrence/transience of the rotor-walk with random initial rotor configuration.

Example 3.12. *Consider a generating graph with* 2 *vertex types and adjacency matrix*

$$D = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}.$$

There are three possible planar embeddings χ^a, χ^b *and* χ^c, *which are shown in Figure 2 together with the directed covers they generate. On these trees we perform transfinite rotor-router walks where the rotors are initially distributed according to the uniform distribution on both types on vertices. The following table shows the generating functions, the first moment matrix and spectral radius of the associated MBP in all cases.*



(a) recurrent (b) critical (c) transient

Figure 2: Different planar embeddings of the same tree.

	generating function	1^{st} *moment matrix*	*spectral radius*
χ^a *Figure 2a*	$f^1(\boldsymbol{z}) = \frac{1}{2}(z_2+1)$ $f^2(\boldsymbol{z}) = \frac{1}{4}(z_1^2 z_2 + z_1^2 + z_1 + 1)$	$M = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{5}{4} & \frac{1}{4} \end{pmatrix}$	$\mathsf{r}(M) = \frac{\sqrt{41}+1}{8} < 1$
χ^b *Figure 2b*	$f^1(\boldsymbol{z}) = \frac{1}{2}(z_2+1)$ $f^2(\boldsymbol{z}) = \frac{1}{4}(z_1^2 z_2 + z_1 z_2 + z_1 + 1)$	$M = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}$	$\mathsf{r}(M) = 1$
χ^c *Figure 2c*	$f^1(\boldsymbol{z}) = \frac{1}{2}(z_2+1)$ $f^2(\boldsymbol{z}) = \frac{1}{4}(z_1^2 z_2 + z_1 z_2 + z_2 + 1)$	$M = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{3}{4} & \frac{3}{4} \end{pmatrix}$	$\mathsf{r}(M) = \frac{\sqrt{33}+3}{8} > 1$

Hence depending only on the planar embedding the rotor-router walk is either recurrent (for χ^a), recurrent in the critical case (for χ^b) or transient (for χ^c). Another interpretation of this example is: on the same tree (forgetting now about the planar embedding) different rotor sequence gives rise to different behaviour of the rotor-router walk, for random initial configuration of rotors.

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