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Some infinite divisibility properties of the reciprocal of planar Brownian motion exit time from a conex

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Abstract

With the help of the Gauss-Laplace transform for the exit time from a cone of planar Brownian motion, we obtain some infinite divisibility properties for the reciprocal of this exit time.

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1 Introduction

Let $(Z_t = X_t + iY_t, t \geq 0)$ denote a standard planar Brownian motion§, starting from $x_0 + i0, x_0 > 0$, where $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ are two independent linear Brownian motions, starting respectively from x_0 and 0.
It is well known [7] that, since $x_0 \neq 0$, $(Z_t, t \geq 0)$ does not visit a.s. the point 0 but keeps winding around 0 infinitely often. Hence, the continuous winding process $\theta_t = \mathrm{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$ is well defined. Using a scaling argument, we may assume $x_0 = 1$, without loss of generality, since, with obvious notation:

$$\left(Z_t^{(x_0)}, t \geq 0\right) \stackrel{(law)}{=} \left(x_0 Z^{(1)}_{(t/x_0^2)}, t \geq 0\right). \tag{1.1}$$

From now on, we shall take $x_0 = 1$.
Furthermore, there is the skew product representation:

$$\log |Z_t| + i\theta_t \equiv \int_0^t \frac{dZ_s}{Z_s} = (\beta_u + i\gamma_u)\Big|_{u=H_t=\int_0^t \frac{ds}{|Z_s|^2}}, \tag{1.2}$$

where $(\beta_u + i\gamma_u, u \geq 0)$ is another planar Brownian motion starting from $\log 1 + i0 = 0$ (for further study of the Bessel clock H, see [15]).
We may rewrite (1.2) as:

$$\log |Z_t| = \beta_{H_t}; \quad \theta_t = \gamma_{H_t}. \tag{1.3}$$

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§When we write: Brownian motion, we always mean real-valued Brownian motion, starting from 0. For 2-dimensional Brownian motion, we indicate planar or complex BM.

One now easily obtains that the two σ-fields $\sigma\{|Z_t|, t \geq 0\}$ and $\sigma\{\beta_u, u \geq 0\}$ are identical, whereas $(\gamma_u, u \geq 0)$ is independent from $(|Z_t|, t \geq 0)$.
Bougerol's celebrated identity in law ([5, 1] and [16] (p. 200)), which says that:

$$\text{for fixed } t, \quad \sinh(\beta_t) \stackrel{(law)}{=} \delta_{A_t(\beta)} \tag{1.4}$$

where $(\beta_u, u \geq 0)$ is 1-dimensional BM, $A_u(\beta) = \int_0^u ds \exp(2\beta_s)$ and $(\delta_v, v \geq 0)$ is another BM, independent of $(\beta_u, u \geq 0)$, will also be used. We define the random times $T_c^{|\theta|} \equiv \inf\{t : |\theta_t| = c\}$, and $T_c^{|\gamma|} \equiv \inf\{t : |\gamma_t| = c\}$, $(c > 0)$. From the skew-product representation (1.3) of planar Brownian motion, we obtain [11]:

$$A_{T_c^{|\gamma|}}(\beta) \equiv \int_0^{T_c^{|\gamma|}} ds \exp(2\beta_s) = H_u^{-1} \Big|_{u=T_c^{|\gamma|}} = T_c^{|\theta|} \, . \tag{1.5}$$

Then, Bougerol's identity (1.4) for the random time $T_c^{|\theta|}$ yields the following [13, 14]:

Proposition 1.1. *The distribution of $T_c^{|\theta|}$ is characterized by:*

$$E\left[\sqrt{\frac{2c^2}{\pi T_c^{|\theta|}}} \exp\left(-\frac{x}{2T_c^{|\theta|}}\right)\right] = \frac{1}{\sqrt{1+x}} \varphi_m(x), \tag{1.6}$$

for every $x \geq 0$, with $m = \frac{\pi}{2c}$, and

$$\varphi_m(x) = \frac{2}{(G_+(x))^m + (G_-(x))^m}, \quad \text{with } \; G_\pm(x) = \sqrt{1+x} \pm \sqrt{x}. \tag{1.7}$$

Comment and Terminology:
If $S > 0$ a.s. is independent from a Brownian motion $(\delta_u, u \geq 0)$, we call the density of δ_S, which is:

$$E\left[\frac{1}{\sqrt{2\pi S}} \exp\left(-\frac{x^2}{2S}\right)\right] \tag{1.8}$$

the Gauss-Laplace transform of S (see e.g. [6] ex.4.18, or [3]). Thus, formula (1.6) expresses - up to simple changes -the Gauss-Laplace transform of $T_c^{|\theta|}$.

We also recall several notions which will be used throughout the following text:

a) A stochastic process $\zeta = (\zeta_t, t \geq 0)$ is called a *Lévy process* if $\zeta_0 = 0$ a.s., it has stationary and independent increments and it is almost surely right continuous with left limits. A Lévy process which is increasing is called a *subordinator*.

b) Following e.g. [11], a probability measure π on $\mathbb{R}$ (resp. a real-valued random variable with law π) is said to be *infinitely divisible* if, for any $n \geq 1$, there is a probability measure π_n such that $\pi = \pi_n^{*n}$ (resp. if $\zeta_1, \ldots, \zeta_n$ are n i.i.d. random variables, $\zeta \stackrel{(law)}{=} \zeta_1 + \ldots + \zeta_n$). For instance, Gaussian, Poisson and Cauchy variables are infinitely divisible.
It is well-known that (e.g. [2]), π is infinitely divisible if and only if, its Fourier transform $\hat{\pi}$ is equal to $\exp(\psi)$, with:

$$\psi(u) = ibu - \frac{\sigma^2 u^2}{2} + \int \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) \nu(dx),$$

where $b \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a Radon measure on $\mathbb{R} \setminus \{0\}$ such that:

$$\int \frac{x^2}{1+x^2} \nu(dx) < \infty.$$

This expression of $\hat{\pi}$ is known as the *Lévy-Khintchine formula* and the measure ν as the *Lévy measure.*

c) Following [4] (p.29) and [8], a positive random variable Γ is a *generalized Gamma convolution* (GGC) if there exists a positive Radon measure μ on $]0,\infty[$ such that:

$$E\left[e^{-\lambda\Gamma}\right] = \exp\left(-\int_0^\infty \left(1-e^{-\lambda x}\right)\frac{dx}{x}\int_0^\infty e^{-xz}\mu(dz)\right) \tag{1.9}$$

$$= \exp\left(-\int_0^\infty \log\left(1+\frac{\lambda}{z}\right)\mu(dz)\right), \tag{1.10}$$

with:

$$\int_{]0,1]} |\log x|\mu(dx) \quad \text{and} \quad \int_{[1,\infty[} \frac{\mu(dx)}{x} < \infty. \tag{1.11}$$

We remark that (1.10) follows immediately from (1.9) using the elementary Frullani formula (see e.g. [10], p.6). The measure μ is called *Thorin's measure* associated with Γ.

We return now to the case of planar Brownian motion and the exit times from a cone. Below, we state and prove the following:

Proposition 1.2. *For every integer m, the function $x \to \varphi_m(x)$, is the Laplace transform of an infinitely divisible random variable K; more specifically, the following decompositions hold:*

- *for $m = 2n+1$,*

$$K = \frac{\mathcal{N}^2}{2} + \sum_{k=1}^n a_k \mathbf{e}_k, \quad a_k = \frac{1}{\sin^2\left(\frac{\pi}{2}\frac{2k-1}{2n+1}\right)}; \; k = 1,2,\dots,n, \tag{1.12}$$

- *for $m = 2n$,*

$$K = \sum_{k=1}^n b_k \mathbf{e}_k, \quad b_k = \frac{1}{\sin^2\left(\frac{\pi}{2}\frac{2k-1}{2n}\right)}; \; k = 1,2,\dots,n, \tag{1.13}$$

where $\mathcal{N}$ is a centered, reduced Gaussian variable and $\mathbf{e}_k$, $k \le n$ are n independent exponential variables, with expectation 1.

Looking at formula (1.6), it is also natural to consider:

$$\tilde{\varphi}_m(x) \equiv \frac{1}{\sqrt{1+x}}\,\varphi_m(x). \tag{1.14}$$

We note that:

$$\tilde{K} \equiv \frac{\mathcal{N}^2}{2} + K, \tag{1.15}$$

admits the RHS of (1.6) as its Laplace transform. Hence,

- for $m = 2n+1$,

$$\tilde{K} \overset{(law)}{=} \mathbf{e}_0 + \sum_{k=1}^n a_k \mathbf{e}_k, \tag{1.16}$$

- for $m = 2n$,

$$\tilde{K} \overset{(law)}{=} \frac{\mathcal{N}^2}{2} + \sum_{k=1}^{n} b_k \mathbf{e}_k, \tag{1.17}$$

with obvious notation.
In Section 2 we first illustrate Proposition 1.2 for $m = 1$ and $m = 2$; we may also verify equation (1.6) by using the laws of $T_c^{|\theta|}$, for $c = \pi/2$ and $c = \pi/4$, which are well known [11].
In Section 3, we prove Proposition 1.2, where the Chebyshev polynomials play an essential role, we calculate the Lévy measure in the Lévy-Khintchine representation of φ_m and we obtain the following asymptotic result:

Proposition 1.3. *With c denoting a positive constant, the distribution of $T_{c\varepsilon}^{|\theta|}$, for every $x \geq 0$, follows the asymptotics:*

$$\left(E\left[\sqrt{\frac{2(c\varepsilon)^2}{\pi T_{c\varepsilon}^{|\theta|}}} \exp\left(-\frac{x}{2T_{c\varepsilon}^{|\theta|}} \right) \right] \right)^{1/\varepsilon} \overset{\varepsilon \to 0}{\longrightarrow} \frac{1}{\left(\sqrt{x} + \sqrt{1+x}\right)^{\pi/2c}}, \tag{1.18}$$

which, from [8], is the Laplace transform of a subordinator $\left(\Gamma_t\left(\mathbb{G}_{1/2}\right), t \geq 0\right)$ with Thorin measure that of the arc sine law, taken at $t = \pi/2c$.

Finally, we state a conjecture concerning the case where m is not necessarily an integer.

2 Examples

2.1 $\boxed{m = 1 \Rightarrow c = \frac{\pi}{2}}$

Then:

$$\tilde{\varphi}_1(x) = \frac{1}{1+x}, \tag{2.1}$$

is the Laplace transform of an exponential variable $\mathbf{e}_1$.
Indeed, with $(Z_t = X_t + iY_t = |Z_t| \exp(i\theta_t), t \geq 0)$ a planar BM starting from $(1, 0)$, $T_{\pi/2}^{|\theta|} = \inf\{t : X_t = 0\} = \inf\{t : X_t^0 = 1\}$,
with $(X_t^0, t \geq 0)$ denoting another one-dimensional BM starting from 0. Formula (1.6) states that:

$$E\left[\sqrt{\frac{2}{\pi T_{\pi/2}^{|\theta|}}} \exp\left(-\frac{x}{2T_{\pi/2}^{|\theta|}} \right) \right] = \frac{1}{1+x}. \tag{2.2}$$

However, we know that: $T_{\pi/2}^{|\theta|} \overset{(law)}{=} \frac{1}{N^2}$, $N \sim \mathcal{N}(0, 1)$.
The LHS of the previous equality (2.2) gives:

$$E\left[\sqrt{\frac{2}{\pi}} |N| \exp\left(-\frac{x}{2} N^2 \right) \right] = \int_0^\infty dy\; y\; e^{-\frac{x+1}{2} y^2} = \frac{1}{1+x}, \tag{2.3}$$

thus, we have verified directly that (2.2) holds.

2.2 $\boxed{m = 2 \Rightarrow c = \frac{\pi}{4}}$

Similarly,

$$\tilde{\varphi}_2(x) = \frac{1}{\sqrt{1+x}} \frac{1}{1+2x}, \tag{2.4}$$

is the Laplace transform of the variable $\frac{\mathcal{N}^2}{2} + 2\mathbf{e}_1$.
Again, this can be shown directly; indeed, with obvious notation:

$$\begin{aligned} T^{|\theta|}_{\pi/4} &= \inf\{t : X_t + Y_t = 0, \text{ or } X_t - Y_t = 0\} \\ &= \inf\{t : \frac{X^0_t + Y_t}{\sqrt{2}} = \frac{1}{\sqrt{2}}, \text{ or } \frac{X^0_t - Y_t}{\sqrt{2}} = \frac{1}{\sqrt{2}}\} \\ &= T_{1/\sqrt{2}} \wedge \tilde{T}_{1/\sqrt{2}} \overset{(law)}{=} \frac{1}{2}\left(T \wedge \tilde{T}\right). \end{aligned}$$

Hence, formula (1.6) now writes, in this particular case:

$$E\left[\sqrt{\frac{\pi}{4(T\wedge\tilde{T})}} \exp\left(-\frac{x}{T\wedge\tilde{T}}\right)\right] = \frac{1}{\sqrt{1+x}}\frac{1}{1+2x}. \tag{2.5}$$

This is easily proven, using: $T \overset{(law)}{=} \frac{1}{N^2}, \tilde{T} \overset{(law)}{=} \frac{1}{\tilde{N}^2}$, which yields:

$$\begin{aligned} &E\left[\left(|N| \vee |\tilde{N}|\right) \exp\left(-x\left(N^2 \vee \tilde{N}^2\right)\right)\right] = 2E\left[|N| \exp\left(-xN^2\right) 1_{(|N|\geq|\tilde{N}|)}\right] \\ &= C \int_0^\infty du\, u\, e^{-xu^2} e^{-\frac{u^2}{2}} \int_0^u dy\, e^{-\frac{y^2}{2}}. \end{aligned}$$

Fubini's theorem now implies that (2.5) holds.

Remark 2.1. *In a first draft, we continued looking at the cases:* $m = 3, 4, 5, 6, \ldots$, *in a direct manner. But, these studies are now superseded by the general discussion in Section 3.*

2.3 A "small" generalization

As we just wrote in Remark 2.1, before finding the proof of Proposition 1.2 (see below, Subsection 3.1), we kept developing examples for larger values of m, and in particular, we encountered quantities of the form:

$$\frac{1}{\mathcal{P}_{u,v}(x)}, \text{ with } \mathcal{P}_{u,v}(x) = 1 + ux + vx^2. \tag{2.6}$$

These quantities turn out to be the Laplace transforms of variables of the form $a\mathbf{e} + b\mathbf{e}'$, with $a, b > 0$ constants and $\mathbf{e}, \mathbf{e}'$ two independent exponential variables. In this Subsection, we characterize the polynomials $\mathcal{P}_{u,v}(x)$ such that this is so.

Lemma 2.2. a) *A necessary and sufficient condition for* $1/\mathcal{P}_{u,v}$ *to be the Laplace transform of the law of* $a\mathbf{e} + b\mathbf{e}'$, *is:*

$$u, v > 0 \quad \text{and} \quad \Delta \equiv u^2 - 4v \geq 0. \tag{2.7}$$

b) *Then, we obtain:*

$$a = \frac{u - \sqrt{\Delta}}{2} \quad ; \quad b = \frac{u + \sqrt{\Delta}}{2}. \tag{2.8}$$

Proof. i) $1/\mathcal{P}_{u,v}$ is the Laplace transform of $a\mathbf{e} + b\mathbf{e}'$, then:

$$\mathcal{P}_{u,v}(x) = (1 + ax)(1 + bx).$$

Both $u = a + b$ and $v = ab$ are positive.
Moreover, $\mathcal{P}_{u,v}$ admits two real roots, thus $\Delta \equiv u^2 - 4v \geq 0$; i.e.: (2.7) is satisfied.

ii) Conversely, if the two conditions (2.7) are satisfied, then the 2 roots of the polynomial are $-1/a$ and $-1/b$. Hence, $\mathcal{P}_{u,v}(x) = C(1 + ax)(1 + bx)$, where C is a constant. However, from the definition of $\mathcal{P}_{u,v}$ (2.6), we have: $\mathcal{P}_{u,v}(0) = 1$, hence $C = 1$. Thus, $1/\mathcal{P}_{u,v}$ is the Laplace transform of $a\mathbf{e} + b\mathbf{e}'$.

iii) To show b), we note that:

$$\left\{-\frac{1}{a}, -\frac{1}{b}\right\} = \left\{\frac{-u - \sqrt{\Delta}}{2v}, \frac{-u + \sqrt{\Delta}}{2v}\right\}.$$

as well as: $\left(u - \sqrt{\Delta}\right)\left(u + \sqrt{\Delta}\right) = 4v$, which finishes the proof of the second part of the Lemma. □

3 A discussion of Proposition 1.2 in terms of the Chebyshev polynomials

3.1 Proof of Proposition 1.2

$a)$ Assuming, to begin with, the validity of our Proposition 1.2, for any integer m, the function φ_m should admit the following representation:

$$\varphi_m(x) = \frac{1}{D_m(x)}, \tag{3.1}$$

where

- for $m = 2n + 1$, $D_m(x) = \sqrt{1 + x}P_n(x)$, with $P_n(x) = \prod_{k=1}^n (1 + a_k x)$,
- for $m = 2n$, $D_m(x) = Q_n(x)$, with $Q_n(x) = \prod_{k=1}^n (1 + b_k x)$.

In particular, P_n and Q_n are polynomials of degree n, each of which has its n zeros, that is $(-1/a_k; k = 1, 2, \ldots, n)$, resp. $(-1/b_k; k = 1, 2, \ldots, n)$, on the negative axis $\mathbb{R}_-$.
It is not difficult, from the explicit expression of $D_m(x) = \frac{1}{2}\left((G_+(x))^m + (G_-(x))^m\right)$, to find the polynomials P_n and Q_n. They are given by the formulas:

$$\begin{cases} P_n(x) = \sum_{k=0}^n C_{2n+1}^{2k+1}(1 + x)^k x^{n-k}, \\ Q_n(x) = \sum_{k=0}^n C_{2n}^{2k}(1 + x)^k x^{n-k}. \end{cases} \tag{3.2}$$

In order to prove Proposition 1.2, we shall make use of Chebyshev's polynomials of the first kind (see e.g. [12] ex.1.1.1 p.5 or [9] ex.25, p.195):

$$\begin{aligned} T_m(y) &\equiv \frac{\left(y + \sqrt{y^2 - 1}\right)^m + \left(y - \sqrt{y^2 - 1}\right)^m}{2} \\ &\equiv \begin{cases} \cos\left(m \arg\cos(y)\right), & y \in [-1, 1] \\ \cosh\left(m \arg\cosh(y)\right), & y \geq 1 \\ (-1)^m \cosh\left(m \arg\cosh(-y)\right), & y \leq 1. \end{cases} \end{aligned} \tag{3.3}$$

b) We now start the proof of Proposition 1.2 in earnest. First, we remark that:

$$\varphi_m(x) = \frac{1}{T_m\left(\sqrt{1+x}\right)}, \tag{3.4}$$

hence:

$$D_m(x) = T_m\left(\sqrt{1+x}\right), \tag{3.5}$$

with $x \geq -1$, thus we are interested only in the positive zeros of T_m, and we study separately the cases m odd and m even.

$\boxed{m = 2n+1}$

$$D_{2n+1}(y) \equiv \sqrt{1+y}P_n(y) = T_{2n+1}\left(\sqrt{1+y}\right)$$

and the zeros of T_{2n+1} are: $x_k = \cos\left(\frac{\pi}{2}\frac{2k-1}{2n+1}\right)$, $k = 1, 2, \ldots, (2n+1)$. However, x_k is positive if and only if $k = 1, 2, \ldots, n$, thus:

$$y_k = x_k^2 - 1 = \cos^2\left(\frac{\pi}{2}\frac{2k-1}{2n+1}\right) - 1 = -\sin^2\left(\frac{\pi}{2}\frac{2k-1}{2n+1}\right); k = 1, 2, \ldots, n.$$

Finally:

$$a_k = \frac{1}{\sin^2\left(\frac{\pi}{2}\frac{2k-1}{2n+1}\right)};\ k = 1, 2, \ldots, n, \tag{3.6}$$

and

$$P_n(x) = \prod_{k=1}^{n}\left(1 + \frac{x}{\sin^2\left(\frac{\pi}{2}\frac{2k-1}{2n+1}\right)}\right). \tag{3.7}$$

$\boxed{m = 2n}$ Similarly, we obtain:

$$b_k = \frac{1}{\sin^2\left(\frac{\pi}{2}\frac{2k-1}{2n}\right)};\ k = 1, 2, \ldots, n, \tag{3.8}$$

and

$$Q_n(x) = \prod_{k=1}^{n}\left(1 + \frac{x}{\sin^2\left(\frac{\pi}{2}\frac{2k-1}{2n}\right)}\right). \tag{3.9}$$

□

3.2 Search for the Lévy measure of φ_m and proof of Proposition 1.3

We have proved that φ_m is infinitely divisible. In this Subsection, we shall calculate its Lévy measure. For this purpose, we shall make use of the following (recall that $\mathbf{e}_k$, $k \leq n$ are n independent exponential variables, with expectation 1):

Lemma 3.1. *With* $(c_k,\ k = 1, 2, \ldots, n)$ *denoting a sequence of positive constants, the Laplace transform of* $\sum_{k=1}^{n} c_k \mathbf{e}_k$ *is* $\prod_{k=1}^{n} \frac{1}{(1+c_k x)}$*, which is an infinitely divisible random variable with Lévy measure:*

$$\frac{dz}{z}\sum_{k=1}^{n} e^{-z/c_k}\ .$$

Proof. Using the elementary Frullani formula (see e.g. [10], p.6), we have:

$$\begin{aligned}\prod_{k=1}^{n}\frac{1}{(1+c_kx)} &= \exp\left\{-\sum_{k=1}^{n}\log\left(1+c_kx\right)\right\} = \exp\left\{-\sum_{k=1}^{n}\int_0^\infty\frac{dy}{y}\,e^{-y}\left(1-e^{-c_kxy}\right)\right\}\\ &\overset{z=c_ky}{=} \exp\left\{-\sum_{k=1}^{n}\int_0^\infty\frac{dz}{z}\,e^{-z/c_k}\left(1-e^{-xz}\right)\right\},\end{aligned}$$

which finishes the proof. □

We return now to the proof of Proposition 1.3 and we study separately the cases m odd and m even and we apply Lemma 3.1 with $c_k = a_k$ and $c_k = b_k$ respectively.

$\boxed{m = 2n+1}$ Lemma 3.1 yields that, $\prod_{k=1}^{n}\frac{1}{(1+a_kx)}$ is the Laplace transform of an infinitely divisible random variable with Lévy measure:

$$\nu_+(dz) = \frac{dz}{z}\sum_{k=1}^{n}e^{-z/a_k}. \tag{3.10}$$

Moreover:

$$\frac{1}{\left(\prod_{k=1}^{n}\left(1+a_kx\right)\right)^{1/n}} = \exp\left\{-\int_0^\infty\frac{dz}{z}\left(1-e^{-xz}\right)\frac{1}{n}\sum_{k=1}^{n}\exp\left\{-\frac{z}{a_k}\right\}\right\}, \tag{3.11}$$

and $\frac{1}{\left(\prod_{k=1}^{n}(1+a_kx)\right)^{1/n}}$, for $n\to\infty$, converges to the Laplace transform of a variable which is a generalized Gamma convolution (GGC) with Thorin measure density:

$$\begin{aligned}\mu_+(z) &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\exp\left\{-\frac{z}{a_k}\right\} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\exp\left\{-z\ \sin^2\left(\frac{\pi}{2}\frac{2k-1}{2n+1}\right)\right\}\\ &= \int_0^1 du\ \exp\left\{-z\sin^2\left(\frac{\pi}{2}u\right)\right\} \overset{v=\frac{\pi}{2}u}{=} \frac{2}{\pi}\int_0^{\pi/2}dv\ \exp\left\{-z\sin^2\left(v\right)\right\}\\ &\overset{h=\sin^2 v}{=} \frac{1}{\pi}\int_0^1\frac{dh}{\sqrt{h(1-h)}}\,e^{-hz},\end{aligned} \tag{3.12}$$

which, following the notation in [8], is the Laplace transform of the variable $\mathbb{G}_{1/2}$ which is arc sine distributed on $[0,1]$.

$\boxed{m = 2n}$ Lemma 3.1 yields that, $\prod_{k=1}^{n}\frac{1}{(1+b_kx)}$ is the Laplace transform of an infinitely divisible random variable with Lévy measure:

$$\nu_-(dz) = \frac{dz}{z}\sum_{k=1}^{n}e^{-z/b_k} = \frac{dz}{z}\sum_{k=1}^{n}\exp\left\{-z\ \sin^2\left(\frac{\pi}{2}\frac{2k-1}{2n}\right)\right\}. \tag{3.13}$$

Moreover $\frac{1}{\left(\prod_{k=1}^{n}(1+b_kx)\right)^{1/n}}$, for $n\to\infty$, converges to the Laplace transform of a GGC with Thorin measure density:

$$\mu_-(z) \quad=\quad \mu_+(z). \tag{3.14}$$

We now express the above results in terms of the Laplace transforms φ_m and $\tilde{\varphi}_m$. Using the following result from [8], p.390, formula (193):

$$\begin{aligned}E\left[\exp\left(-x\Gamma_t\left(\mathbb{G}_{1/2}\right)\right)\right] &= \exp\left\{-t\int_0^\infty\frac{dz}{z}\left(1-e^{-xz}\right)E\left[\exp\left(-z\mathbb{G}_{1/2}\right)\right]\right\}\\ &= \frac{1}{\left(\sqrt{1+x}+\sqrt{x}\right)^{2t}}\end{aligned} \tag{3.15}$$

with $2t = m = \frac{\pi}{2c\varepsilon}$, with c a positive constant, together with (3.12) and (3.14), we obtain (1.18). □

Remark 3.2. *The natural question that arises now is whether the results of Proposition 1.2 could be generalized for every $m > 0$ (not necessarily an integer), in other words wether $\varphi_m(x) = \frac{2}{(G_+(x))^m + (G_-(x))^m}$ is the Laplace transform of a generalized Gamma convolution (GGC, see [4] or [8]), that is:*

$$\varphi_m(x) = E\left[e^{-x\Gamma_m}\right], \tag{3.16}$$

with

$$\Gamma_m \overset{(law)}{=} \int_0^\infty f_m(s) d\gamma_s, \tag{3.17}$$

where $f_m : \mathbb{R}_+ \to \mathbb{R}_+$ and γ_s is a gamma process.
This conjecture will be investigated in future work.

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