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ON THE RATE OF GROWTH OF LÉVY PROCESSES WITH NO POSITIVE JUMPS CONDITIONED TO STAY POSITIVE

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Abstract

In this note, we study the asymptotic behaviour of Lévy processes with no positive jumps conditioned to stay positive and some related processes. In particular, we establish an integral test for the lower envelope at 0 and at $+\infty$ and an analogue of Khintchin's law of the iterated logarithm at 0 and at $+\infty$, for the upper envelope of the reflected process at its future infimum.

1 Introduction and main results.

Let $\mathscr{D}$ denote the Skorokhod space of càdlàg paths with real values and defined on the positive real half-line $[0,\infty)$ and $\mathbb{P}$ a probability measure defined on $\mathscr{D}$ under which ξ will be a real-valued Lévy process with no positive jumps starting from 0 and unbounded variation (the latter assumption is to exclude the case when ξ is the difference of a constant drift and a subordinator). It is known (see for instance [2]), that ξ has finite exponential moments of arbitrary positive order. In particular, we have $\mathbb{E}(\exp\{\lambda\xi_t\}) = \exp\{t\psi(\lambda)\}$, for $\lambda \geq 0$, where ψ is given by the celebrated Lévy-Khintchine formula.
According to Bertoin [2], the mapping $\psi : [0,\infty) \to (-\infty,\infty)$ is convex and ultimately increasing and we denote its right-inverse on $[0,\infty)$ by Φ. Let us introduce the first passage time of ξ by $T_x = \inf\{s : \xi_s \geq x\}$, for $x \geq 0$. From Theorem VII.1 in [2], we know that, under $\mathbb{P}$, the process $T = (T_x, x \geq 0)$ is a subordinator, killed at an independent exponential time when ξ drifts towards $-\infty$ and with Laplace exponent given by Φ. In order to study the case when ξ drifts towards $-\infty$, we define the probability measure,

$$\mathbb{P}^{\natural}(A) = \mathbb{E}\Big(\exp\{\Phi(0)\xi_t\}\mathbb{1}_A\Big), \qquad A \in \mathscr{F}_t,$$

where $\mathscr{F}_t$ is the $\mathbb{P}$-complete sigma-field generated by $(\xi_s, s \leq t)$. Note that under $\mathbb{P}^{\natural}$, the process ξ is a Lévy process with no positive jumps which drifts towards $+\infty$ and whose Laplace exponent

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is given by $\psi^{\natural}(\lambda) = \psi(\Phi(0) + \lambda)$, for $\lambda \geq 0$. Moreover the first passage process T is still a subordinator with Laplace exponent $\Phi^{\natural}(\lambda) = \Phi(\lambda) - \Phi(0)$.
The scale function W give us the probability that ξ first exits $[-x, y]$ at the upper boundary point through the formula

$$\mathbb{P}\left(\inf_{0\leq t\leq T_y} \xi_t \geq -x\right) = \frac{W(x)}{W(x+y)}.$$

Furthermore, $W : [0,\infty) \to [0,\infty)$ is the unique absolutely continuous increasing function whose Laplace transform is $1/\psi$.
Using the Doob's theory of h-transforms, we construct a new Markov process by an h-transform of the law of the Lévy process killed at time $R = \inf\{t \geq 0 : \xi_t < 0\}$ with the harmonic function W (see for instance Chapter VII in Bertoin [2] or Chaumont and Doney [6]), and its semigroup is given by

$$\mathbb{P}^{\uparrow}_x(\xi_t \in \mathrm{d}y) = \frac{W(y)}{W(x)}\mathbb{P}_x(\xi_t \in \mathrm{d}y, t < R) \quad \text{for} \quad x > 0,$$

where $\mathbb{P}_x$ denotes the law of ξ starting from $x > 0$. Under $\mathbb{P}^{\uparrow}_x$, ξ is a process taking values in $(0,\infty)$. It will be referred to as the Lévy process started at x and conditioned to stay positive. We also point out that there are two path constructions of such process, the Doney-Tanaka construction and Bertoin's construction (see for instance Chapter 8 in [8]). Note that when ξ drifts towards $-\infty$, we have that $\mathbb{P}^{\uparrow}_x = \mathbb{P}^{\natural\uparrow}_x$, for all $x > 0$. Hence the study of this case is reduced to the study of the processes which drift towards $+\infty$. Another important property of Lévy processes conditioned to stay positive is given in Lemma VII.12 in [2], which says that

$$\lim_{t\to\infty} \xi_t = \infty, \qquad \mathbb{P}^{\uparrow}_x\text{-a.s.} \quad \text{for every } x > 0.$$

Bertoin proved in [2] the existence of a measure $\mathbb{P}^{\uparrow}_0$ under which the process starts at 0 and stays positive. In fact, the author in [2] proved that the probability measures $\mathbb{P}^{\uparrow}_x$ converge as x goes to 0^+ in the sense of finite-dimensional distributions to $\mathbb{P}^{\uparrow}_0 := \mathbb{P}^{\uparrow}$ and noted that this convergence also holds in the sense of Skorokhod. In particular we have that, under $\mathbb{P}^{\uparrow}$, the process ξ drifts to ∞.
One of the starting points of this note is a remarkable result on the right-continuous inverse of subordinators of Friested and Pruitt [10] which in particular give us the following law of iterated logarithm (or LIL) for Levy process with no negative jumps,

$$\limsup_{t\to 0} \frac{\xi_t \Phi(t^{-1}\log|\log t|)}{\log|\log t|} = c \qquad \mathbb{P}\text{-a.s.,} \tag{1}$$

where c is a positive constant (see also Bertoin [1] for its proof).
Several partial results on the upper envelope of $(\xi, \mathbb{P}^{\uparrow})$ have been established before, the more general of which is due to Bertoin [1], where he proved that there exists a constant $k \in [c, 3c]$, such that

$$\limsup_{t\to 0} \frac{\xi_t \Phi(t^{-1}\log|\log t|)}{\log|\log t|} = k \qquad \mathbb{P}^{\uparrow}\text{-a.s.} \tag{2}$$

In fact, the above result is established in terms of the rate of growth at a local minimum of Lévy process with no positive jumps, but it is noted in its proof that the sample path behaviour of a Lévy process with no positive jumps immediately after a local minimum is the same as that of its conditioned version. It is important to note that it is not known when the constants c and k may be different. In the particular case, when ξ is the stable process with no positive jumps of index

$\alpha \in (1,2]$, c and k are equal to some constant $c(\alpha)$ (see Monrad and Silverstain [13]) which only depends on the index α.
Recently the author noted in [15], when ξ is the stable process, that $(\xi,\mathbb{P}^{\uparrow})$, its future infimum and the process $(\xi,\mathbb{P}^{\uparrow})$ reflected at its future infimum satisfy the same LIL (see Corollary 4 in [15]).
To the best of our knowledge the lower envelope has never been studied in full generality. In [7], it is obtained an integral tests at 0 and at $+\infty$, for the lower envelope of stable processes with no positive jumps conditioned to stay positive (see Theorem 2 and example 2 in [7]).
Our first result consists in LIL at 0 and at ∞ for the upper envelope of $(\xi,\mathbb{P}^{\uparrow})$. Our arguments show, in particular, that the constants c and k in (1) and (2) are always the same. That is to say that the rates of growth of the Lévy process $(\xi,\mathbb{P})$ and its conditioned version $(\xi,\mathbb{P}^{\uparrow})$ are exactly the same at 0.

Theorem 1. *Let c as above, then*

$$\limsup_{t\to 0} \frac{\xi_t \Phi(t^{-1}\log|\log t|)}{\log|\log t|} = c, \qquad \mathbb{P}^{\uparrow} - a.s.$$

There exist a positive constant c' such that

$$\limsup_{t\to +\infty} \frac{\xi_t \Phi(t^{-1}\log\log t)}{\log\log t} = c', \qquad \mathbb{P}^{\uparrow} - a.s.$$

We point out that when the Lévy process $(\xi,\mathbb{P})$ does not drift to $-\infty$, its rate of growth coincides with that of its conditioned version $(\xi,\mathbb{P}^{\uparrow})$ at ∞. We also note that when ψ is regularly varying at ∞ with some index $\alpha > 1$, we have and explicit expression for the constant c which is given by

$$c = (1/\alpha)^{-1/\alpha}(1-1/\alpha)^{\frac{1-\alpha}{\alpha}}.$$

Let $J_t = \inf_{s\geq t}\xi_s$, be the future infimum of ξ and define $((\xi_t - J_t, t\geq 0),\mathbb{P}^{\uparrow})$ the Lévy process conditioned to stay positive reflected at its future infimum. Let us suppose that for all $\beta < 1$

$$(\mathbf{H}_1) \quad \limsup_{x\to 0}\frac{W(\beta x)}{W(x)} < 1 \quad \text{and} \quad (\mathbf{H}_2) \quad \limsup_{x\to +\infty}\frac{W(\beta x)}{W(x)} < 1.$$

Theorem 2. *i) Under the hypothesis $(\mathbf{H}_1)$, we have that*

$$\limsup_{t\to 0} \frac{(\xi_t - J_t)\Phi(t^{-1}\log|\log t|)}{\log|\log t|} = c, \qquad \mathbb{P}^{\uparrow} - a.s.$$

ii) If $(\mathbf{H}_2)$ is satisfied, then

$$\limsup_{t\to +\infty} \frac{(\xi_t - J_t)\Phi(t^{-1}\log\log t)}{\log\log t} = c', \qquad \mathbb{P}^{\uparrow} - a.s.$$

We remark that conditions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ are satisfied, in particular, when ψ is regularly varying at $+\infty$ and at 0, respectively.
The lower envelope of $(\xi,\mathbb{P}^{\uparrow})$ and its future infimum at 0 and at $+\infty$ is determined as follows,

Theorem 3. *i) Let $f : [0,\infty) \to [0,\infty)$ be an increasing function such that $t \to f(t)/t$ decreases, then one has*

$$\liminf_{t\to 0}\frac{\xi_t}{f(t)} = \liminf_{t\to 0}\frac{J_t}{f(t)} = 0 \quad \mathbb{P}^{\uparrow}\text{-a.s.} \quad \textit{if and only if} \quad \int_0 f(x)\nu(\mathrm{d}x) = \infty,$$

where ν is the Lévy measure of the subordinator $(T,\mathbb{P})$. Moreover,

$$\text{if} \qquad \int_0 f(x)\nu(\mathrm{d}x)<\infty \qquad \text{then} \qquad \lim_{t\to 0}\frac{\xi_t}{f(t)}=\liminf_{t\to 0}\frac{J_t}{f(t)}=\infty \quad \mathbb{P}^{\uparrow}\text{-a.s.}$$

ii) If ξ drifts to $+\infty$ one has

$$\lim_{t\to+\infty}\frac{J_t}{t}=\frac{1}{\mathbb{E}(T_1)} \qquad \mathbb{P}^{\uparrow}\text{-a.s.}$$

iii) The lower envelope at $+\infty$ is determined as follows: if ξ does not drift towards ∞ and the function $f:[0,\infty)\to[0,\infty)$ is increasing such that $t\to f(t)/t$ decreases, one has

$$\liminf_{t\to+\infty}\frac{\xi_t}{f(t)}=\liminf_{t\to+\infty}\frac{J_t}{f(t)}=0 \quad \mathbb{P}^{\uparrow}\text{-a.s.} \quad \text{if and only if} \quad \int^{+\infty} f(x)\nu(\mathrm{d}x)=\infty.$$

Moreover,

$$\text{if} \qquad \int^{+\infty} f(x)\nu(\mathrm{d}x)<\infty \qquad \text{then} \qquad \lim_{t\to+\infty}\frac{\xi_t}{f(t)}=\liminf_{t\to+\infty}\frac{J_t}{f(t)}=\infty \qquad \mathbb{P}^{\uparrow}\text{-a.s.}$$

Let $A(x)$ denotes the total time spent by ξ below the level $x\geq 0$, i.e.

$$A(x)=\int_0^{\infty}\mathbb{1}_{\{\xi_s\leq x\}}\mathrm{d}s,$$

also known as the occupation time of the process ξ below the level x. Our last result describes the lower envelope of the occupation times of $(\xi,\mathbb{P}^{\uparrow})$ at 0 and at ∞ which extends the LIL for the occupation times of Bessel processes of dimension 3 due to Biggins [4].

Theorem 4. *Suppose that ψ is regularly varying at $+\infty$ (or at 0) with index $\alpha>1$. Then the occupation times of $(\xi,\mathbb{P}^{\uparrow})$ satisfies the following LIL,*

$$\liminf_{x\to 0(or\ \infty)}\frac{A(x)\psi\left(x^{-1}\log|\log x|\right)}{\log|\log x|}=\frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{\alpha-1}, \qquad \mathbb{P}^{\uparrow}\text{-a.s.} \tag{3}$$

The rest of this note consist of two sections, the first of which provides the proof of Theorem 4 which in turn relies on the lower envelope of the first and the last passage times. Finally, Section 3 is devoted to the proofs of Theorems 1,2 and 3.

2 First and last passage: the proof of Theorem 4.

We first recall the definition of the first and last passage times of ξ,

$$T_x=\inf\Big\{t\geq 0:\xi_t\geq x\Big\} \quad \text{and} \quad U_x=\sup\Big\{t\geq 0:\xi_t\leq x\Big\} \quad \text{for} \quad x\geq 0,$$

and note that $T_x\leq A(x)\leq U_x$, for all $x\geq 0$. Our arguments consist in study the lower envelope of the first and last passage times of $(\xi,\mathbb{P}^{\uparrow})$, under the assumption that ψ is regularly varying at ∞ (or at 0) with index $\alpha>1$, and then use the previous inequality.
We first prove the upper bound of (3). Applying Theorem VII.18 and Corollary VII.19 in [2], we deduce that the last passage time process $U:=(U_x,x\geq 0)$, under $\mathbb{P}^{\uparrow}$, is a subordinator and that

its law is the same as that of $T=(T_x, x\geq 0)$, under $\mathbb{P}$. In particular, we have that the Laplace exponent of $(U,\mathbb{P}^{\uparrow})$ is given by $\Phi(\lambda)$. From Theorems III.11 and III.14 in [2], the process $(U,\mathbb{P}^{\uparrow})$ satisfies

$$\liminf_{x\to 0 (\text{or } \infty)} \frac{U_x\psi\big(x^{-1}\log|\log x|\big)}{\log|\log x|} = \frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{\alpha-1}, \qquad \mathbb{P}^{\uparrow}\text{-a.s.,} \tag{4}$$

and the upper bound of (3) follows.
The proof of the lower bound of (3) needs a more detailed analysis, since T is not a subordinator, under $\mathbb{P}^{\uparrow}$. The later observation follows from the fact that $(\xi,\mathbb{P}^{\uparrow})$ is not a Lévy process anymore. Nevertheless, the Markov property of $(\xi,\mathbb{P}^{\uparrow})$ and the absence of positive jumps implies that $(T,\mathbb{P}^{\uparrow})$ is increasing and has independent increments which are nice properties for studying its lower envelope.

Proposition 1. *Suppose that ψ is regularly varying at $+\infty$ (or at 0) with index $\alpha>1$. Then the first passage time process satisfies the following LIL,*

$$\liminf_{x\to 0 (or\ \infty)} \frac{T_x\psi\big(x^{-1}\log|\log x|\big)}{\log|\log x|} = \frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{\alpha-1}, \qquad \mathbb{P}^{\uparrow}\text{-}a.s.$$

Proof: We will only prove the result for small times since the proof for large times is very similar. For all $x\geq 0$, recall that $T_x\leq U_x$, $\mathbb{P}^{\uparrow}$-a.s. Then from (4) we get

$$\liminf_{x\to 0} \frac{T_x\psi\big(x^{-1}\log|\log x|\big)}{\log|\log x|} \leq \liminf_{x\to 0} \frac{U_x\psi\big(x^{-1}\log|\log x|\big)}{\log|\log x|} = \frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)^{\alpha-1}, \quad \mathbb{P}^{\uparrow}\text{-a.s.}$$

The lower bound needs the following sharp estimation for the distribution of $(T,\mathbb{P}^{\uparrow})$.

Lemma 1. *Assume that ψ is regularly varying at $+\infty$ with index $\alpha>1$. Then for every constant $c_1>0$, we have*

$$-\log\mathbb{P}^{\uparrow}\Big(T_x\leq c_1 g(x)\Big)\sim k_{\alpha,c_1}\log|\log x| \quad as \quad x\to 0,$$

where

$$g(x)=\frac{\log|\log x|}{\psi\big(x^{-1}\log|\log x|\big)} \qquad and \qquad k_{\alpha,c_1}=\left(1-\frac{1}{\alpha}\right)\left(\frac{1}{c_1\alpha}\right)^{1/(\alpha-1)}.$$

Proof of Lemma 1: Recall, from Lemma III.12 in [2], that

$$-\log\mathbb{P}^{\uparrow}\Big(U_x\leq c_1 g(x)\Big)\sim k_{\alpha,c_1}\log|\log x| \quad \text{as} \quad x\to 0.$$

This clearly implies that

$$\limsup_{x\to 0}\Big(-\log\mathbb{P}^{\uparrow}\Big(T_x\leq c_1 g(x)\Big)/\log|\log x|\Big)\leq k_{\alpha,c_1}.$$

For the lower bound, let us first define the supremum process $S=(S_t, t\geq 0)$ by $S_t=\sup_{0\leq s\leq t}\xi_s$. Next, we fix $\epsilon\in(0,1)$, then by the Markov property

$$\begin{aligned}\mathbb{P}^{\uparrow}\big(J_{c_1g(x)}>(1-\epsilon)x\big) &\geq \mathbb{P}^{\uparrow}\big(S_{c_1g(x)}>x, J_{c_1g(x)}>(1-\epsilon)x\big)\\ &= \int_0^{c_1g(x)}\mathbb{P}^{\uparrow}\Big(T_x\in \mathrm{d}t\Big)\mathbb{P}^{\uparrow}_x\big(J_{c_1g(x)-t}>(1-\epsilon)x\big) \qquad (5)\\ &\geq \mathbb{P}^{\uparrow}\Big(T_x<c_1g(x)\Big)\mathbb{P}^{\uparrow}_x\big(J_0>(1-\epsilon)x\big).\end{aligned}$$

From the definition of the future infimum process, it is clear that J_0 is the absolute minimum of $(\xi, \mathbb{P}^{\uparrow}_x)$. Then by Lemma VII.12 in [2] we have

$$\mathbb{P}^{\uparrow}_x\big(J_0 > (1-\epsilon)x\big) = \frac{W(\epsilon x)}{W(x)}.$$

On the other hand, since the Laplace transform of W is the inverse of ψ an application of the Tauberian and Monotone density theorems (see for instance Bingham et al [5]) gives

$$W(x) \sim \frac{\alpha}{\Gamma(1+\alpha)} \frac{1}{x\psi(1/x)} \qquad \text{as} \quad x \to 0,$$

and therefore,

$$\mathbb{P}^{\uparrow}_x\big(J_0 > (1-\epsilon)x\big) \to \epsilon^{(\alpha-1)} \quad \text{as} \quad x \to 0. \tag{6}$$

Now, since the last passage times process is the right-continuous inverse of the future infimum process, we have that $\mathbb{P}^{\uparrow}(J_{c_1 g(x)} > (1-\epsilon)x) = \mathbb{P}^{\uparrow}(U_{(1-\epsilon)x} < c_1 g(x))$.
Similar arguments as those used in the proof of the lower envelope in Lemma III.12 in [2] and the above facts, give us

$$(1-\epsilon)^{\frac{\alpha}{\alpha-1}} k_{\alpha,c_1} \leq \liminf_{x\to 0} \frac{-\log \mathbb{P}^{\uparrow}\Big(U_{(1-\epsilon)x} \leq c_1 g(x)\Big)}{\log|\log x|} \leq \liminf_{x\to 0} \frac{-\log \mathbb{P}^{\uparrow}\Big(T_x \leq c_1 g(x)\Big)}{\log|\log x|},$$

and since ϵ can be chosen arbitrarily small, the lemma is proved. ■

Now, take $r < 1$ and $0 < c_1 < c_2 < \alpha^{-\alpha}(\alpha-1)^{\alpha-1}$. Since ψ is regularly varying at ∞ with index α and we can chose r close enough to 1, we see that for n sufficiently large

$$\mathbb{P}^{\uparrow}\Big(T_{r^{n+1}} < c_1 g(r^n)\Big) \leq \mathbb{P}^{\uparrow}\Big(T_{r^{n+1}} < c_2 g(r^{n+1})\Big).$$

Now, observe that

$$\begin{aligned}\sum_n \mathbb{P}^{\uparrow}\Big(T_{r^{n+1}} < c_2 g(r^{n+1})\Big) &\leq \sum_n \int_n^{n+1} \mathbb{P}^{\uparrow}\Big(T_{r^u} \leq c_2 g\big(r^u\big)\Big)\mathrm{d}u \\ &= \frac{1}{\log r^{-1}} \sum_n \int_{r^{n+1}}^{r^n} \mathbb{P}^{\uparrow}\Big(T_x \leq c_2 g\big(x\big)\Big)\frac{\mathrm{d}x}{x} \leq \frac{1}{\log r^{-1}} \int_0^1 \mathbb{P}^{\uparrow}\Big(T_x \leq c_2 g\big(x\big)\Big)\frac{\mathrm{d}x}{x},\end{aligned}$$

so from Lemma 1 this last integral is finite since $(1-1/\alpha)(c_2\alpha)^{-1/(\alpha-1)} > 1$, and the series $\sum_n \mathbb{P}^{\uparrow}(T_{r^{n+1}} < c_1 g(r^n))$ converges. According to the the first Borel-Cantelli's Lemma, we have $T_{r^{n+1}} \geq c_1 g(r^n)$ for all n large enough $\mathbb{P}^{\uparrow}$-a.s. Since the function g and the process T are increasing in a neighbourhood of 0, we have $\mathbb{P}^{\uparrow}$-a.s.

$$T_x \geq c_1 g(x) \qquad \text{for} \qquad r^{n+1} \leq x \leq r^n,$$

with this we finish the proof. ■

3 Proofs of Theorems 1,2 and 3.

Recall that $(U, \mathbb{P}^{\uparrow})$ is a subordinator with Laplace exponent Φ and that the future infimum of $(\xi, \mathbb{P}^{\uparrow})$ corresponds to the first passage of $(U, \mathbb{P}^{\uparrow})$. The same arguments as those used in the proof of Theorem 1 in [1] give us the following LIL for $(J, \mathbb{P}^{\uparrow})$, which is crucial for what follows.

Lemma 2. *Let c as in (1), then*

$$\limsup_{t\to 0} \frac{J_t \Phi(t^{-1}\log|\log t|)}{\log|\log t|} = c, \qquad \mathbb{P}^{\uparrow} - a.s.,$$

and

$$\limsup_{t\to +\infty} \frac{J_t \Phi(t^{-1}\log\log t)}{\log\log t} = c', \qquad \mathbb{P}^{\uparrow} - a.s.$$

where c' is a positive constant.

For simplicity, we introduce the notation

$$h(t) = \frac{\log|\log t|}{\Phi(t^{-1}\log|\log t|)}.$$

Proof of Theorem 1: We first prove the LIL for large times. The lower bound is easy to deduce from Lemma 2. Hence

$$c' = \limsup_{t\to\infty} \frac{J_t \Phi(t^{-1}\log\log t)}{\log\log t} \le \limsup_{t\to\infty} \frac{\xi_t \Phi(t^{-1}\log\log t)}{\log\log t} \qquad \mathbb{P}^{\uparrow} - a.s.$$

Now, we prove the upper bound. Let $r > 1$ and recall that S is the supremum process of ξ. We define the events $A_n = \{S_{r^n} > \eta c' h(r^{n-1})\}$, where $(c')^{-1}(2+r) = \eta$. From the first Borel-Cantelli's Lemma, if $\sum_n \mathbb{P}^{\uparrow}(A_n) < \infty$, it follows that $S_{r^n} \le \eta c' h(r^{n-1})$ for all n large enough, $\mathbb{P}^{\uparrow}$-a.s. Since the function h and the process S are increasing in a neighbourhood of $+\infty$, we have

$$S_t \le \eta c' h(t) \qquad \text{for} \quad r^{n-1} \le t \le r^n, \quad \text{under } \mathbb{P}^{\uparrow}.$$

Then, it is enough to prove that $\sum_n \mathbb{P}^{\uparrow}(A_n) < \infty$. To this end, we need the following two lemmas.

Lemma 3. *For every $\beta > 1$, we have that*

$$\beta^2 \psi(\theta) \ge \psi(\beta\theta), \qquad \textit{for all} \quad \theta \ge 0.$$

Proof of Lemma 3: Recall that the Laplace exponent ψ satisfies the so called Lévy-Khintchine formula, that is to say

$$\psi(\theta) = a\theta + \frac{\sigma^2}{2}\theta^2 + \int_{(-\infty,0)} (e^{\theta x} - 1 - x\theta)\Pi(\mathrm{d}x),$$

where $a \in \mathbb{R}$, $\sigma \ge 0$ and Π is a measure on $(-\infty, 0)$ satisfying $\int_{(-\infty,0)}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$.
Now, define the function $\phi(\theta) = \theta^{-1}\psi(\theta)$, for $\theta > 0$, and

$$\phi(0) = \lim_{\theta\downarrow 0} \frac{\psi(\theta)}{\theta} = a.$$

Since

$$\left|\frac{e^{\theta x}(e^{-\theta x}-1+\theta x)}{\theta^2}\right| \leq 1\wedge x^2,$$

for all $\theta > 0$ and $x \in (-\infty, 0)$, we deduce that

$$\phi'(\theta) = \frac{\sigma^2}{2} + \int_{(-\infty,0)} \frac{e^{\theta x}(e^{-\theta x}-1+\theta x)}{\theta^2}\Pi(\mathrm{d}x), \qquad \theta > 0.$$

On the other hand, from the definition of ϕ and the above identity we have

$$\phi'(0^+) = \frac{\sigma^2}{2} + \int_{(-\infty,0)} x^2\Pi(\mathrm{d}x) = \lim_{\theta\downarrow 0}\phi'(\theta).$$

From its form, it is clear that $\phi'(\cdot)$ is a decreasing function. Hence for $\beta > 1$, we have that $\phi'(\theta) \geq \phi'(\beta\theta)$, for all $\theta \geq 0$, which clearly implies that $\beta\phi(\theta) \geq \phi(\beta\theta)$. ■

Lemma 4. *Let* $0 < \epsilon < 1$ *and* $r > 1$*, then there exists a positive real number* K *such that*

$$\mathbb{P}^{\uparrow}\big(J_{r^n} > (1-\epsilon)\eta c'h(r^{n-1})\big) \geq \epsilon K^2\mathbb{P}^{\uparrow}\big(A_n\big), \qquad n \geq 2. \tag{7}$$

Proof of Lemma 4: From the inequality (5), we have that

$$\mathbb{P}^{\uparrow}\big(J_{r^n} > (1-\epsilon)\eta c'h(r^{n-1})\big) \geq \mathbb{P}^{\uparrow}_{\eta c'h(r^{n-1})}\big(J_0 > (1-\epsilon)\eta c'h(r^{n-1})\big)\mathbb{P}^{\uparrow}\big(S_{r^n} > \eta c'h(r^{n-1})\big),$$

and since, under $\mathbb{P}^{\uparrow}_{\eta c'h(r^{n-1})}$, J_0 is the absolute minimum of $(\xi, \mathbb{P}^{\uparrow}_{\eta c'h(r^{n-1})})$ then by Lemma VII.12 in [2], we have

$$\mathbb{P}^{\uparrow}_{\eta c'h(r^{n-1})}\big(J_0 > (1-\epsilon)\eta c'h(r^{n-1})\big) = \frac{W\big(\epsilon\eta c'h(r^{n-1})\big)}{W\big(\eta c'h(r^{n-1})\big)}.$$

On the other hand, an application of Proposition III.1 in [2] gives that there exist a positive real number K such that

$$K\frac{1}{x\psi(1/x)} \leq W(x) \leq K^{-1}\frac{1}{x\psi(1/x)}, \qquad \text{for all} \quad x > 0, \tag{8}$$

then it is clear that

$$\frac{W\big(\epsilon\eta c'h(r^{n-1})\big)}{W\big(\eta c'h(r^{n-1})\big)} \geq K^2\epsilon^{-1}\frac{\psi\big(1/\eta c'h(r^{n-1})\big)}{\psi\big(\epsilon^{-1}/\eta c'h(r^{n-1})\big)}.$$

From this inequality and lemma 3, we have that

$$\mathbb{P}^{\uparrow}\big(J_{r^n} > (1-\epsilon)\eta c'h(r^{n-1})\big) \geq \epsilon K^2\mathbb{P}^{\uparrow}\big(S_{r^n} > \eta c'h(r^{n-1})\big),$$

which proves our result. ■

Now, we prove the upper bound for the LIL of $(\xi, \mathbb{P}^{\uparrow})$. Fix $0 < \epsilon < 1/(2+r)$ and recall that $\mathbb{P}^{\uparrow}\big(J_{r^n} > (1-\epsilon)\eta c'h(r^{n-1})\big) = \mathbb{P}^{\uparrow}\big(U_{(1-\epsilon)\eta c'h(r^{n-1})} < r^n\big)$. This probability is bounded from above by

$$\exp\{\lambda r^n\}\mathbb{E}^{\uparrow}\Big(\exp\big\{-\lambda U_{(1-\epsilon)\eta c'h(r^{n-1})}\big\}\Big) = \exp\big\{\lambda r^n - (1-\epsilon)\eta c'h(r^{n-1})\Phi(\lambda)\big\},$$

for $\lambda \geq 0$. We choose $\lambda = r^{-(n-1)} \log\log r^{n-1}$, then

$$\mathbb{P}^{\uparrow}\big(J_{r^n} > (1-\epsilon)\eta c' h(r^{n-1})\big) \leq \exp\Big\{-\big((1-\epsilon)\eta c' - r\big)\log\log r^{n-1}\Big\},$$

hence from the above inequality and Lemma 4, we have that

$$\sum_n \mathbb{P}^{\uparrow}(A_n) \leq K^{-2}\epsilon^{-1}\sum_n \big((n-1)\log r\big)^{-(1-\epsilon)\eta c'+r} < +\infty,$$

since $(1-\epsilon)\eta c' - r > 1$. Therefore,

$$\limsup_{t\to+\infty} \frac{S_t}{h(t)} \leq 3, \qquad \mathbb{P}^{\uparrow}\text{-a.s.},$$

since we can choose r close enough to 1.
The two preceding parts show that

$$\limsup_{t\to\infty} \frac{\xi_t}{h(t)} \in [c', 3], \qquad \mathbb{P}^{\uparrow}\text{-a.s.}$$

By the Blumenthal zero-one law, it must be a constant number k', $\mathbb{P}^{\uparrow}-$a.s.
Now, we prove that the constant k' is equal to c'. Note that we only need to prove that $c' \geq k'$. Fix $\epsilon \in (0, 1/2)$ and define

$$R_n = \inf\left\{s \geq n : \frac{\xi_s}{k' h(s)} \geq (1-\epsilon)\right\}.$$

It is clear that $n \leq R_n < \infty$ and that R_n diverges $\mathbb{P}^{\uparrow}$-a.s., as n goes to $+\infty$. From Lemma VII.12 in [2] and since $(\xi, \mathbb{P}^{\uparrow})$ is a strong Markov process with no positive jumps, we have that

$$\begin{aligned}
\mathbb{P}^{\uparrow}\left(\frac{J_{R_n}}{k' h(R_n)} \geq (1-2\epsilon)\right) &= \mathbb{P}^{\uparrow}\left(J_{R_n} \geq \frac{(1-2\epsilon)\xi_{R_n}}{(1-\epsilon)}\right) \\
&= \mathbb{E}^{\uparrow}\left(\mathbb{P}^{\uparrow}\left(J_{R_n} \geq \frac{(1-2\epsilon)\xi_{R_n}}{(1-\epsilon)}\middle| \xi_{R_n}\right)\right) \\
&= \mathbb{E}^{\uparrow}\left(\frac{W(\ell(\epsilon)\xi_{R_n})}{W(\xi_{R_n})}\right),
\end{aligned}$$

where $\ell(\epsilon) = \epsilon/(1-\epsilon)$. Applying (8) and lemma 3, we get

$$\mathbb{E}^{\uparrow}\left(\frac{W(\ell(\epsilon)\xi_{R_n})}{W(\xi_{R_n})}\right) \geq K^2 \ell(\epsilon),$$

which implies that

$$\lim_{n\to+\infty} \mathbb{P}^{\uparrow}\left(\frac{J_{R_n}}{k' h(R_n)} \geq (1-2\epsilon)\right) > 0.$$

Since $R_n \geq n$,

$$\mathbb{P}^{\uparrow}\left(\frac{J_t}{k' h(t)} \geq (1-2\epsilon), \text{ for some } t \geq n\right) \geq \mathbb{P}^{\uparrow}\left(\frac{J_{R_n}}{k' h(R_n)} \geq (1-2\epsilon)\right).$$

Therefore, for all $\epsilon \in (0, 1/2)$

$$\mathbb{P}^{\uparrow}\left(\frac{J_t}{k'h(t)} \geq (1-2\epsilon), \text{ i.o., as } t \to +\infty\right) \geq \lim_{n\to+\infty} \mathbb{P}^{\uparrow}\left(\frac{J_{R_n}}{k'h(R_n)} \geq (1-2\epsilon)\right) > 0.$$

The event on the left hand side is in the upper-tail sigma-field of $(\xi, \mathbb{P}^{\uparrow})$ which is trivial from Bertoin's construction (see for instance Section 8.5.2 in [8]). Hence

$$\limsup_{t\to+\infty} \frac{J_t}{h(t)} \geq k'(1-2\epsilon), \qquad \mathbb{P}^{\uparrow} - \text{a.s.},$$

and since ϵ can be chosen arbitrarily small, we deduce that $c' \geq k'$.
Finally, we show that the constant k in (2) is equal to the constant c in (1). Our arguments are very similar to those presented above, for this reason we briefly explain the idea of the proof. Again, we fix $\epsilon \in (0, 1/2)$ and define the following stopping time

$$R_n = \inf\left\{\frac{1}{n} < s : \frac{\xi_s}{kh(s)} \geq (1-\epsilon)\right\}.$$

First note that for n sufficiently large $1/n < R_n < \infty$, $\mathbb{P}^{\uparrow}$-a.s. Moreover, since (2) holds we have that R_n converge to 0 as n goes to ∞, $\mathbb{P}^{\uparrow}$-a.s. Similar computations as above allow us to deduce that for n sufficiently large

$$\mathbb{P}^{\uparrow}\left(\frac{J_{R_n}}{kh(R_n)} \geq (1-2\epsilon)\right) > 0.$$

Next, we note that

$$\mathbb{P}^{\uparrow}\left(\frac{J_{R_p}}{kh(R_p)} \geq (1-2\epsilon), \text{ for some } p \geq n\right) \geq \mathbb{P}^{\uparrow}\left(\frac{J_{R_n}}{kh(R_n)} \geq (1-2\epsilon)\right).$$

Since R_n converge to 0 as n goes to ∞, $\mathbb{P}^{\uparrow}$-a.s., it is enough to take limits in both sides of the above inequality and use lemma 2 to get the result. ∎

Proof of Theorem 2: Here, we will follow similar arguments as those used in the last part of Theorem 1. Assume that the hypothesis $(\mathbf{H}_2)$ is satisfied. From Theorem 1, it is clear that

$$\limsup_{t\to+\infty} \frac{\xi_t - J_t}{h(t)} \leq \limsup_{t\to+\infty} \frac{\xi_t}{h(t)} = c' \qquad \mathbb{P}^{\uparrow}\text{-a.s.}$$

Fix $\epsilon \in (0, 1/2)$ and define

$$R_n = \inf\left\{s \geq n : \frac{\xi_s^{\uparrow}}{c'h(s)} \geq (1-\epsilon)\right\}.$$

From Lemma VII.12 in [2] and since $(\xi, \mathbb{P}^{\uparrow})$ is a strong Markov process with no positive jumps, we have that

$$\begin{aligned}
\mathbb{P}^{\uparrow}\left(\frac{\xi_{R_n} - J_{R_n}}{c'h(R_n)} \geq (1-2\epsilon)\right) &= \mathbb{P}^{\uparrow}\left(J_{R_n} \leq \frac{\epsilon}{(1-\epsilon)}\xi_{R_n}\right) \\
&= \mathbb{E}^{\uparrow}\left(\mathbb{P}^{\uparrow}\left(J_{R_n} \leq \frac{\epsilon}{(1-\epsilon)}\xi_{R_n}\Big|\xi_{R_n}\right)\right) \\
&= 1 - \mathbb{E}^{\uparrow}\left(\frac{W\big(\ell(\epsilon)\xi_{R_n}\big)}{W(\xi_{R_n})}\right),
\end{aligned}$$

where $\ell(\epsilon) = (1-2\epsilon)/(1-\epsilon)$. Since the hypothesis $(\mathbf{H}_2)$ is satisfied, an application of the Fatou-Lebesgue Theorem shows that

$$\limsup_{n\to+\infty} \mathbb{E}^{\uparrow}\left(\frac{W\big(\ell(\epsilon)\xi_{R_n}\big)}{W(\xi_{R_n})}\right) \le \mathbb{E}^{\uparrow}\left(\limsup_{n\to+\infty} \frac{W\big(\ell(\epsilon)\xi_{R_n}\big)}{W(\xi_{R_n})}\right) < 1,$$

which implies that

$$\lim_{n\to+\infty} \mathbb{P}^{\uparrow}\left(\frac{\xi_{R_n} - J_{R_n}}{c'h(R_n)} \ge (1-2\epsilon)\right) > 0.$$

Again, since $R_n \ge n$,

$$\mathbb{P}^{\uparrow}\left(\frac{\xi_t - J_t}{c'h(t)} \ge (1-2\epsilon), \text{ for some } t \ge n\right) \ge \mathbb{P}^{\uparrow}\left(\frac{\xi_{R_n} - J_{R_n}}{c'h(R_n)} \ge (1-2\epsilon)\right).$$

Therefore, for all $\epsilon \in (0, 1/2)$

$$\mathbb{P}^{\uparrow}\left(\frac{\xi_t - J_t}{c'h(t)} \ge (1-2\epsilon), \text{ i.o., as } t \to +\infty\right) \ge \lim_{n\to+\infty} \mathbb{P}^{\uparrow}\left(\frac{\xi_{R_n} - J_{R_n}}{c'h(R_n)} \ge (1-2\epsilon)\right) > 0.$$

The event on the left hand side is in the upper-tail sigma-field of $(\xi, \mathbb{P}^{\uparrow})$ which is trivial, then

$$\limsup_{t\to+\infty} \frac{\xi_t - J_t}{h(t)} \ge c'(1-2\epsilon), \qquad \mathbb{P}^{\uparrow} - \text{a.s.},$$

and since ϵ can be chosen arbitrarily small, the result for large times is proved.
Similarly, we can prove the result for small times using the following stopping time

$$R_n = \inf\left\{\frac{1}{n} < s : \frac{\xi_s}{ch(s)} \ge (1-\epsilon)\right\}.$$

Following same argument as above and assuming that $(\mathbf{H}_1)$ is satisfied, we get that for a fixed $\epsilon \in (0, 1/2)$ and n sufficiently large

$$\mathbb{P}^{\uparrow}\left(\frac{\xi_{R_n} - J_{R_n}}{ch(R_n)} \ge (1-2\epsilon)\right) > 0.$$

Next, we note that

$$\mathbb{P}^{\uparrow}\left(\frac{\xi_{R_p} - J_{R_p}}{ch(R_p)} \ge (1-2\epsilon), \text{ for some } p \ge n\right) \ge \mathbb{P}^{\uparrow}\left(\frac{\xi_{R_n} - J_{R_n}}{ch(R_n)} \ge (1-2\epsilon)\right).$$

Again, since R_n converge to 0 as n goes to ∞, $\mathbb{P}^{\uparrow}$-a.s., the conclusion follows taking the limit when n goes to $+\infty$. ■

Proof of Theorem 3: Let (x_n) be a decreasing sequence such that $\lim x_n = 0$. We define the events $A_n = \{$ There exist $t \in [U_{x_{n+1}}, U_{x_n}]$ such that $\xi_t < f(t)\}$. Since U_{x_n} tends to 0 as n goes to $+\infty$, , $\mathbb{P}^{\uparrow}$-a.s., we have

$$\mathbb{P}^{\uparrow}\left(\left\{\xi_t^{\uparrow} < f(t), \text{ i.o., as } t \to 0\right\}\right) = \mathbb{P}^{\uparrow}\left(\limsup_{n\to+\infty} A_n\right).$$

Let us chose $x_n = r^n$, for $r < 1$. Since f is increasing the following inequalities hold

$$\mathbb{P}^{\uparrow}(A_n) \le \mathbb{P}^{\uparrow}\Big(\text{There exist } t \in [r^{n+1}, r^n] \text{ such that } tr < f\big(U_t\big)\Big),$$

and

$$\mathbb{P}^{\uparrow}\Big(\text{There exist } t \in [r^{n+1}, r^n] \text{ such that } tr^{-1} < f\big(U_t\big)\Big) \le \mathbb{P}^{\uparrow}(A_n).$$

Then we prove the convergent part. Let us suppose that f satisfies that $\int_0 f(x)\nu(\mathrm{d}x)$ converges. Hence from Theorem VI.3.2 in [11] and the fact that $(U, \mathbb{P}^{\uparrow})$ is a subordinator, we have that

$$\mathbb{P}^{\uparrow}\Big(tr < f\big(U_t\big), \text{ i.o., as } t \to 0\Big) = 0,$$

which implies that

$$\lim_{t\to 0} \frac{\xi_t}{f(t)} = \infty \qquad \mathbb{P}^{\uparrow}\text{-a.s.,}$$

since we can replace f by cf, for any $c > 1$.
Similarly, if f satisfies that $\int_0 f(x)\nu(\mathrm{d}x)$ diverges; again from Theorem VI.3.2 in [11], we have that

$$\mathbb{P}^{\uparrow}\Big(tr^{-1} < f\big(U_t\big), \text{ i.o., as } t \to 0\Big) = 1,$$

which implies that

$$\liminf_{t\to 0} \frac{\xi_t}{f(t)} = 0 \qquad \mathbb{P}^{\uparrow}\text{-a.s.,}$$

since we can replace f by cf, for any $c < 1$.
The integral test at $+\infty$ is very similar to this of small times, it is enough to take $x_n = r^n$, for $r > 1$ and follows the same arguments as in the proof for small times. The proof of parts (i) and (iii) for the future infimum follows from the above arguments, it is enough to note that we can replace ξ by J in the sets A_n. The proof of part (ii) follows from Proposition 4.4 in [3]. ■

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