

Decay Rates of Solutions of an Anisotropic Inhomogeneous n -Dimensional Viscoelastic Equation with Polynomially Decaying Kernels

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Abstract: We consider the anisotropic and inhomogeneous viscoelastic equation and we prove that the first and second order energy decay polynomially as time goes to infinity when the relaxation function also decays polynomially to zero. That is, if the kernel G_{ijkl} satisfies

$$\dot{G}_{ijkl} \leq -c_0 G_{ijkl}^{1+\frac{1}{p}}; \text{ and } G_{ijkl}, G_{ijkl}^{1+\frac{1}{p}} \in L^1(\mathbb{R}) \text{ for } p > 2 \text{ such that } 2^m - 1 < p,$$

then the first and second order energy decay as $\frac{1}{(1+t)^q}$ with $q = 2^m - 1$.

1. Introduction

Several authors have studied the asymptotic stability of the solutions in viscoelasticity. Thanks to the works [1–5, 8, 9, 11] among others, it is well known that the stability holds for inhomogeneous and anisotropic n -dimensional materials and also for one-dimensional nonlinear equations. The question now is about the uniform rate of decay of the solution as time goes to infinity. Somehow, the way that the solution goes to zero depends on the decay of the kernel as time goes to infinity. We may ask, under what conditions on the kernel does the solution decay to zero exponentially or at least polynomially? To fix ideas, let us consider the simplest homogeneous isotropic n -dimensional viscoelastic equation with density $\rho = 1$,

$$u_{tt} - \mu \Delta u - (\mu + \lambda) \nabla \operatorname{div} u + \int_0^t g(t - \tau) [\mu \Delta u - (\mu + \lambda) \nabla \operatorname{div} u] d\tau = 0, \quad (1.1)$$

where λ and μ stand for Lamé’s constant and by g we denote the relaxation function. The kernel “ g ” plays an important role in the study of the asymptotic behaviour of the solutions. To see this, let us cite a few results about the uniform rate of decay. For example, in the work of Hrusa [8] the author showed, among others, properties

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that the solution of the one dimensional viscoelastic equation goes to zero exponentially when the kernel is given by

$$g(t) = c_0 e^{-\gamma t}.$$

Dassios and Zafiropoulos [6] showed, for the same kernel, that the solution of Eq. (1.1) decays as $t^{-3/2}$ for materials which occupy the whole 3-dimensional space, with a rate of decay that can be improved depending on the symmetry of the initial data. The solution of the viscoelastic equation decays uniformly (exponentially in bounded domains and algebraically in \mathbb{R}^n), when the kernel is an exponential function. A serious restriction in this point is that Eq. (1.1) does not behave as a genuine integro differential equation, because in this particular case the integral term can be removed so the resulting equation is a partial differential equation. To see this let us differentiate Eq. (1.1) with respect to time. Using $g'(t) = -\gamma g(t)$, we can write, in the resulting equation, the first three terms given in Eq. (1.1) instead of the integral term. So the new equation does not have the integral term; it has turned into a third order (in time) partial differential equation. To know how this transformation is useful let us consider the 3-dimensional Cauchy problem (1.1). Using the Fourier transform we get a third order ordinary differential equation with coefficients depending on the Fourier parameter ξ . To know the asymptotic behaviour of the solution we have to study the behaviour of the real part of the roots of the characteristic polynomial associated to the ordinary differential equation. This method works for any ordinary differential equations, with constant coefficients which do not depend on time, but unfortunately does not work for "genuine" integro-differential equations.

In [10] one of the authors of this paper improved the result due to Dassios and Zafiropoulos. In that paper we proved that, when the kernel decays exponentially to zero, the solution also decays exponentially for bounded domains, while for materials which occupy the whole \mathbb{R}^n space, the solution decays as $t^{-n/2}$, with rates which can be improved depending on the regularity of the initial data. For kernels which *decay* exponentially, the asymptotic behavior of the solution is known for bounded and unbounded domains. But what about non exponential kernels? More precisely, what can we say when the kernel decays like $(1+t)^{-p}$? Does the solution decay exponentially? Does the solution decay algebraically, for example, like $(1+t)^{-q}$? If this is the case, then what is the relation between p and q ? It seems to us that there is no result in the literature concerning these questions, so to fill this gap, we will study these points here.

The main result of this paper is to show that a uniform rate of decay also holds for kernels which decay like $(1+t)^{-p}$. We will prove that the solution decay to zero as $(1+t)^{-q}$, where $q = 2^m - 1$ and $p > 2^m - 1$, for some natural number m . The rate of decay also depends on the L^p -regularity of the kernel. This means in our framework that the kernel must satisfy $\int_0^\infty |G_{ijkl}| d\tau < \infty$ and $\int_0^\infty |G_{ijkl}|^{1-\frac{1}{p}} d\tau < \infty$ and also $\dot{G}_{ijkl} \leq -CG_{ijkl}^{1+\frac{1}{p}}$ for $2^m - 1 < p$ and $C > 0$.

We study in Sect. 2 the asymptotic behaviour of inhomogeneous and anisotropic bounded material, while in Sect. 3 we study the case of materials which occupy the whole n -dimensional space. In this later case, to use our approach we consider isotropics and homogeneous materials, because our method uses the Fourier transform of the solution. The hypotheses we consider here are simpler than others use in previous works (see [2-5, 8, 9, 11]). Our method is different and explores the dissipation given by the memory effect as well as the L^p -regularity of the kernel,

to construct a Liapunov functional whose derivative is negative proportional to a power of itself.

We finish this section giving some notations and introducing the hypotheses we need to prove our result. For spatial derivatives we use the following notation:

$$\mathcal{H}_{i_1, \dots, i_k, j_1, \dots, j_s} := \frac{\partial^s \mathcal{H}_{i_1, \dots, i_k}}{\partial x_{j_1} \dots \partial x_{j_s}},$$

while for time derivatives we use:

$$\dot{w} := \frac{\partial w}{\partial t}, \quad \ddot{w} := \frac{\partial^2 w}{\partial t^2} \quad \text{and} \quad \overset{(k)}{w} = \frac{\partial^k w}{\partial t^k}.$$

The viscoelastic equation we study in this paper is:

$$\rho \ddot{u}_i = \{C_{ijkl} u_{k,l}\}_{,j} - \left\{ \int_0^t G_{ijkl}(\cdot, t - \tau) u_{k,l}(\cdot, \tau) d\tau \right\}_{,j} \quad \text{in } \Omega \times]0, T[, \quad (1.2)$$

with initial data

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x),$$

and supporting the Dirichlet boundary condition,

$$u = 0 \quad \text{in } \Gamma \times]0, +\infty[. \quad (1.3)$$

Our result is also valid for Newman boundary conditions with normalized initial data. The proof is essentially the same. Further, we suppose the following conditions are valid:

$$C_{iljk} = C_{jkil}, \quad G_{ijkl} = G_{klij} \quad \text{and} \quad C_{iljk} \in C^1, \quad G_{ijkl} \in C^3. \quad (1.4)$$

Also there exist positive constants α, β, κ, C and v_0 such that

$$C_{iljk} w_{il} w_{jk} \geq \alpha w_{il} w_{il}, \quad G_{ijkl}(\cdot, 0) w_{il} w_{jk} \geq \beta w_{il} w_{il}, \quad (1.5)$$

$$\dot{G}_{ijkl} \leq -\kappa [G_{ijkl}]^{1+\frac{1}{p}}; \quad \ddot{G}_{ijkl} \leq C [G_{ijkl}]^{1+\frac{1}{p}}, \quad \text{for } p > 1, \quad (1.6)$$

$$\int_{\Omega} \left\{ C_{ijkl} - \int_0^{\infty} G_{ijkl} d\tau \right\} u_{i,j} u_{k,l} dx \geq v_0 \int_{\Omega} u_{i,j} u_{i,j} dx. \quad (1.7)$$

Finally we suppose that the following integrability conditions hold:

$$\int_0^{\infty} [G_{ijkl}]^{1-\frac{1}{p}} < \infty \quad (1.8)$$

for $p > 1$.

Remark 1.1. Condition (1.6) says that the kernel is like the function

$$t \mapsto \frac{1}{(1+t)^p}.$$

If we take $p > 2$ then hypothesis (1.8) is always true.

In the next section we will prove that the solution of (1.2) decays uniformly to zero as time goes to infinity, for bounded domains.

2. Asymptotic Behaviour

To simplify our analysis we introduce the following notation:

$$G \square \partial v := \int_0^t G_{ijkl}(\cdot, t - \tau) \{v_{k,l}(\cdot, t) - v_{k,l}(\cdot, \tau)\} \{v_{i,j}(\cdot, t) - v_{i,j}(\cdot, \tau)\} .$$

To prove the polynomial rate of decay we will use the following lemma.

Lemma 2.1. *Let us suppose that G satisfies hypothesis (1.4) and $G_{ijkl} \in C^1$. Then the following identity holds:*

$$\begin{aligned} & \int_0^t G_{ijkl}(\cdot, t - \tau) v_{k,l}(\cdot, \tau) d\tau \dot{v}_{i,j} \\ &= -\frac{1}{2} \frac{d}{dt} G \square \partial v + \frac{1}{2} \dot{G} \square \partial v + \frac{1}{2} \frac{d}{dt} \left\{ \int_0^t G_{ijkl} d\tau v_{k,l} v_{i,j} \right\} - G_{ijkl}(\cdot, t) v_{k,l} v_{i,j} . \end{aligned}$$

Proof. It is easy to see that

$$\begin{aligned} \frac{d}{dt} G \square \partial v &= \frac{d}{dt} \int_0^t G_{ijkl}(\cdot, t - \tau) \{v_{k,l}(\cdot, t) - v_{k,l}(\cdot, \tau)\} \{v_{i,j}(\cdot, t) - v_{i,j}(\cdot, \tau)\} \\ &= \dot{G} \square \partial v + 2 \int_0^t G_{ijkl}(\cdot, t - \tau) \{v_{k,l}(\cdot, t) - v_{k,l}(\cdot, \tau)\} d\tau v_{i,j} \\ &= \dot{G} \square \partial v + 2 \int_0^t G_{ijkl} d\tau v_{k,l} \dot{v}_{i,j} - 2 \int_0^t G_{ijkl}(\cdot, t - \tau) v_{k,l}(\cdot, \tau) d\tau \dot{v}_{i,j} . \end{aligned}$$

From the above it follows that

$$\begin{aligned} \frac{d}{dt} G \square \partial v &= \dot{G} \square \partial v + \frac{d}{dt} \left\{ \int_0^t G_{ijkl} d\tau v_{k,l} v_{i,j} \right\} - G_{ijkl} v_{k,l} v_{i,j} \\ &\quad - 2 \int_0^t G_{ijkl}(\cdot, t - \tau) v_{k,l}(\cdot, \tau) d\tau \dot{v}_{i,j} , \end{aligned}$$

which proves the lemma. \square

The following lemma plays an important role in the rest of this paper:

Lemma 2.2. *Assume that hypotheses of Lemma 2.1 holds. Then for any $w, v \in C([0, T]; L^2(\Omega))$, the following inequality is valid:*

$$\begin{aligned} & \int_{\Omega} \int_0^t G_{ijkl}(\cdot, t - \tau) \{w_{k,l}(\cdot, t) - w_{k,l}(\cdot, \tau)\} d\tau v_{i,j} dx \\ & \leq c_0 \left\{ \int_{\Omega} G^{1+\frac{1}{p}} \square \partial w dx \right\}^{\frac{1}{2}} \left\{ \int_{\Omega} v_{i,j} v_{i,j} dx \right\}^{\frac{1}{2}} . \end{aligned}$$

Proof. Using Holder’s inequality we get

$$\begin{aligned} & \int_{\Omega} \int_0^t G_{ijkl}(\cdot, t - \tau) \{w_{k,l}(\cdot, t) - w_{k,l}(\cdot, \tau)\} d\tau v_{i,j} dx \\ &= \int_{\Omega} \int_0^t G_{ijkl}^{\frac{1}{2}-\frac{1}{2p}} G_{ijkl}^{\frac{1}{2}+\frac{1}{2p}}(\cdot, t - \tau) \{w_{k,l}(\cdot, t) - w_{k,l}(\cdot, \tau)\} d\tau v_{i,j} dx \end{aligned}$$

$$\begin{aligned} &\leq \left\{ \int_0^t G_{ijkl}^{1-\frac{1}{p}} d\tau \right\}^{\frac{1}{2}} \int_{\Omega} \{G^{1+\frac{1}{p}} \square \partial w\}^{\frac{1}{2}} |v_{i,j}| dx \\ &\leq \left\{ \int_0^t G_{ijkl}^{1-\frac{1}{p}} d\tau \right\}^{\frac{1}{2}} \left\{ \int_{\Omega} G^{1+\frac{1}{p}} \square \partial w dx \right\}^{\frac{1}{2}} \left\{ \int_{\Omega} v_{i,j} v_{i,j} dx \right\}^{\frac{1}{2}} . \end{aligned}$$

From (1.8) we have that the integral:

$$\int_0^{\infty} |G_{ijkl}(\cdot, \tau)|^{1-\frac{1}{p}} d\tau ,$$

is bounded. Taking $c_0 = \{ \int_0^t G_{ijkl}^{1-\frac{1}{p}} d\tau \}^{\frac{1}{2}}$ our result follows. \square

By $E(t; v)$ we denote

$$E(t; v) := \frac{1}{2} \left\{ \int_{\Omega} \dot{v}_i \dot{v}_i + G \square \partial v dx + \int_{\Omega} \left\{ C_{ijkl} - \int_0^t G_{ijkl} d\tau \right\} v_{i,j} v_{k,l} dx \right\} .$$

Lemma 2.3. *Under the above conditions we have*

$$\begin{aligned} \frac{d}{dt} E(t; u) &= -\frac{1}{2} \int_{\Omega} G_{ijkl} u_{k,l} u_{i,j} dx + \frac{1}{2} \int_{\Omega} \dot{G} \square \partial u dx , \\ \frac{d}{dt} E(t; \dot{u}) &= -\frac{1}{2} \int_{\Omega} G_{ijkl} \dot{u}_{k,l} \dot{u}_{i,j} dx + \frac{1}{2} \int_{\Omega} \dot{G} \square \partial \dot{u} dx - \int_{\Omega} [G_{ijkl}(\cdot, t) u_{k,l}(\cdot, 0)]_{,j} \dot{u} dx . \end{aligned}$$

Proof. Let us multiply Eq. (1.2) by \dot{u}_i to get

$$\frac{1}{2} \frac{d}{dt} \left\{ \int_{\Omega} \dot{u}_i \dot{u}_i + C_{ijkl} u_{i,j} u_{k,l} dx \right\} = \int_{\Omega} \int_0^t G_{ijkl}(\cdot, t - \tau) u_{k,l}(\cdot, \tau) d\tau \dot{u}_{i,j} dx .$$

Using Lemma 2.1 our first identity holds. To show the second assertion, let us take the time derivative of Eq. (1.2) to get

$$\overset{(3)}{u}_i - \{C_{ijkl} \dot{u}_{k,l}\}_{,j} + \{G_{ijkl}(\cdot, 0) u_{k,l}\}_{,j} + \left\{ \int_0^t \dot{G}_{ijkl}(\cdot, t - \tau) u_{k,l}(\cdot, \tau) d\tau \right\}_{,j} = 0 . \tag{2.1}$$

Since $\dot{G}_{ijkl} = -\frac{\partial}{\partial t} G_{ijkl}$, we have

$$\begin{aligned} \left\{ \int_0^t \dot{G}_{ijkl}(\cdot, t - \tau) u_{k,l}(\cdot, \tau) d\tau \right\}_{,j} &= -\{G_{ijkl}(\cdot, 0) u_{k,l}\}_{,j} + \{G_{ijkl}(\cdot, t) u_{k,l}(\cdot, 0)\}_{,j} \\ &\quad + \left\{ \int_0^t G_{ijkl}(\cdot, t - \tau) \dot{u}_{k,l}(\cdot, \tau) d\tau \right\}_{,j} . \end{aligned}$$

Substitution of this identity in (2.1) yields:

$$\overset{(3)}{u}_i - \{C_{ijkl} \dot{u}_{k,l}\}_{,j} - \left\{ \int_0^t G_{ijkl}(\cdot, t - \tau) \dot{u}_{k,l}(\cdot, \tau) d\tau \right\}_{,j} = \{G_{ijkl}(\cdot, t) u_{k,l}(\cdot, 0)\}_{,j} . \tag{2.2}$$

Multiplying Eq. (2.2) by \ddot{u}_i and using the same reasoning as above, we prove the second part of this theorem. \square

Remark 2.1. In particular Lemma 2.1 implies that the first and second order energy are bounded, that is:

$$E(t; u) \leq E(0; u); \quad E(t; u_t) \leq cE(0; u_t). \quad \square$$

Let us define the functionals $K(t; v)$ and $I(t; v)$ as

$$\begin{aligned} K(t; v) &:= \frac{1}{2} \int_{\Omega} \ddot{v}_i \ddot{v}_i \, dx + \frac{1}{2} \int_{\Omega} C_{ijkl} \dot{v}_{k,l} \dot{v}_{i,j} \, dx - \int_{\Omega} G_{ijkl}(\cdot, 0) v_{k,l} \dot{v}_{i,j} \, dx \\ &\quad + \gamma \int_{\Omega} C_{ijkl} v_{k,l} \dot{v}_{i,j} \, dx - \int_{\Omega} \int_0^t F_{ijkl}(\cdot, t - \tau) v_{k,l}(\cdot, \tau) \, d\tau \dot{v}_{i,j} \, dx, \\ I(t; v) &:= \int_{\Omega} \ddot{v}_i \dot{v}_i \, dx - \frac{1}{2} \int_{\Omega} G_{ijkl}(\cdot, 0) v_{k,l} v_{i,j} \, dx - \int_{\Omega} \int_0^t \dot{G}_{ijkl} \, d\tau v_{k,l} v_{i,j} \, dx + \dot{G} \square \partial v, \end{aligned}$$

where

$$F_{ijkl} := \gamma G_{ijkl} + \dot{G}_{ijkl}. \tag{2.3}$$

with these conditions we get the following lemma

Lemma 2.4. *Under the above notations there exist constants $C_2, c, \gamma > 0$ satisfying*

$$\begin{aligned} \frac{d}{dt} \{K(t; u) + (\gamma - c)I(t; u)\} &\leq g(t)cE(0) - c \int_{\Omega} \ddot{u}_i \ddot{u}_i \, dx - \frac{\beta}{2} \int_{\Omega} \dot{u}_{i,j} \dot{u}_{i,j} \, dx \\ &\quad - \frac{\gamma}{2} \kappa \int_{\Omega} G_{ijkl}(\cdot, t) u_{k,l} u_{i,j} \, dx + C_2 \int_{\Omega} G^{1+\frac{1}{p}} \square \partial u \, dx. \end{aligned}$$

Proof. Using the identity

$$\{C_{ijkl} u_{k,l}\}_{,j} = \ddot{u}_i + \left\{ \int_0^t G_{ijkl}(\cdot, t - \tau) u_{k,l}(\cdot, \tau) \, d\tau \right\}_{,j},$$

we get that Eq. (2.1) implies:

$$\begin{aligned} &\overset{(3)}{u}_i - \{C_{ijkl} \dot{u}_{k,l}\}_{,j} + \gamma \ddot{u}_i + \{G_{ijkl}(\cdot, 0) u_{k,l}\}_{,j} + \gamma \{C_{ijkl} u_{k,l}\}_{,j} \\ &= - \left\{ \int_0^t F_{ijkl}(\cdot, t - \tau) u_{k,l}(\cdot, \tau) \, d\tau \right\}_{,j}, \end{aligned} \tag{2.4}$$

where F_{ijkl} is given by (2.3). Multiplying (2.4) by \ddot{u}_i , we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\{ \int_{\Omega} \ddot{u}_i \ddot{u}_i \, dx + \int_{\Omega} C_{ijkl} \dot{u}_{k,l} \dot{u}_{i,j} \, dx \right\} &= -\gamma \int_{\Omega} \ddot{u}_i \ddot{u}_i \, dx + \int_{\Omega} G_{ijkl}(\cdot, 0) u_{k,l} \dot{u}_{i,j} \, dx \\ &\quad - \gamma \int_{\Omega} C_{ijkl} u_{k,l} \ddot{u}_{i,j} \, dx + \int_{\Omega} \int_0^t F_{ijkl}(\cdot, t - \tau) u_{k,l}(\cdot, \tau) \, d\tau \ddot{u}_{i,j} \, dx. \end{aligned}$$

Using the identities:

$$\int_{\Omega} G_{ijkl}(\cdot, 0) u_{k,l} \ddot{u}_{i,j} dx = \frac{d}{dt} \int_{\Omega} G_{ijkl}(\cdot, 0) u_{k,l} \dot{u}_{i,j} dx - \int_{\Omega} G_{ijkl}(\cdot, 0) \dot{u}_{k,l} \dot{u}_{i,j} dx,$$

$$-\gamma \int_{\Omega} C_{ijkl} u_{k,l} \ddot{u}_{i,j} dx = -\gamma \frac{d}{dt} \int_{\Omega} C_{ijkl} u_{k,l} \dot{u}_{i,j} dx + \gamma \int_{\Omega} C_{ijkl} \dot{u}_{k,l} \dot{u}_{i,j} dx,$$

$$\int_{\Omega} \int_0^t F_{ijkl}(\cdot, t - \tau) u_{k,l}(\cdot, \tau) d\tau \ddot{u}_{i,j} dx = \frac{d}{dt} \int_{\Omega} \int_0^t F_{ijkl}(\cdot, t - \tau) u_{k,l}(\cdot, \tau) d\tau \dot{u}_{i,j} dx$$

$$- \int_{\Omega} F_{ijkl}(\cdot, 0) u_{k,l} \ddot{u}_{i,j} dx - \int_{\Omega} \int_0^t F_{ijkl}(\cdot, t - \tau) u_{k,l}(\cdot, \tau) d\tau \ddot{u}_{i,j} dx,$$

we get

$$\begin{aligned} \frac{d}{dt} K(t; u) &= -\gamma \int_{\Omega} \ddot{u}_i \ddot{u}_i dx - \int_{\Omega} G_{ijkl}(\cdot, 0) \dot{u}_{k,l} \dot{u}_{i,j} dx + \gamma \int_{\Omega} C_{ijkl} \dot{u}_{k,l} \dot{u}_{i,j} dx \\ &\quad - \int_0^t \dot{F}_{ijkl}(\cdot, t - \tau) \{u_{k,l}(\cdot, t) - u_{k,l}(\cdot, \tau)\} d\tau \dot{u}_{i,j} \\ &\quad - \int_{\Omega} F_{ijkl}(\cdot, t) u_{k,l} \dot{u}_{i,j} dx. \end{aligned} \tag{2.5}$$

Let us multiply Eq. (2.1) by \dot{u} to get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \ddot{u}_i \dot{u}_i dx &= \int_{\Omega} \ddot{u}_i \ddot{u}_i dx - \int_{\Omega} C_{ijkl} \dot{u}_{k,l} \dot{u}_{i,j} dx + \int_{\Omega} G_{ijkl}(\cdot, 0) u_{k,l} \dot{u}_{i,j} dx \\ &\quad + \int_{\Omega} \int_0^t \dot{G}_{ijkl}(\cdot, t - \tau) u_{k,l}(\cdot, \tau) d\tau \dot{u}_{i,j} dx. \end{aligned}$$

From Lemma 2.1 and recalling the definition of I we obtain:

$$\frac{d}{dt} I(t; u) = \int_{\Omega} \ddot{u}_i \ddot{u}_i dx - \int_{\Omega} C_{ijkl} \dot{u}_{k,l} \dot{u}_{i,j} dx + \int_{\Omega} \dot{G}_{ijkl}(\cdot, t) u_{k,l} u_{i,j} dx + \int_{\Omega} \ddot{G} \square \partial u dx, \tag{2.6}$$

which together with relations (2.5) and (2.6) yields:

$$\begin{aligned} \frac{d}{dt} \{K(t; u) + (\gamma - c)I(t; u)\} &= -c \int_{\Omega} \ddot{u}_i \ddot{u}_i dx + c \int_{\Omega} C_{ijkl} \dot{u}_{k,l} \dot{u}_{i,j} dx \\ &\quad - \int_{\Omega} G_{ijkl}(\cdot, 0) \dot{u}_{k,l} \dot{u}_{i,j} dx + (\gamma - c) \int_{\Omega} \dot{G}_{ijkl}(\cdot, t) u_{k,l} u_{i,j} dx + (\gamma - c) \int_{\Omega} \ddot{G} \square \partial u dx \\ &\quad - \int_{\Omega} F_{ijkl}(\cdot, t) u_{k,l} \dot{u}_{i,j} dx - \int_0^t \dot{F}_{ijkl}(\cdot, t - \tau) \{u_{k,l}(\cdot, t) - u_{k,l}(\cdot, \tau)\} d\tau \dot{u}_{i,j}. \end{aligned} \tag{2.7}$$

Since C_{ijkl} is continuous, there exists a positive constant c_0 such that

$$\int_{\Omega} C_{ijkl} \dot{u}_{k,l} \dot{u}_{i,j} dx \leq c_0 \dot{u}_{i,j} \dot{u}_{i,j} .$$

From hypothesis (1.5) and taking $c \leq \frac{\beta}{2c_0}$ we conclude

$$c \int_{\Omega} C_{ijkl} \dot{u}_{k,l} \dot{u}_{i,j} dx - \int_{\Omega} G_{ijkl}(\cdot, 0) \dot{u}_{k,l} \dot{u}_{i,j} dx \leq -\frac{\beta}{2} \int_{\Omega} \dot{u}_{i,j} \dot{u}_{k,l} dx . \tag{2.8}$$

Using hypothesis (1.6) we get

$$(\gamma - c) \int_{\Omega} \dot{G}_{ijkl}(\cdot, t) u_{k,l} u_{i,j} dx \leq -(\gamma - c) \kappa \int_{\Omega} G_{ijkl}^{1+\frac{1}{p}}(\cdot, t) u_{k,l} u_{i,j} dx ; \tag{2.9}$$

$$\begin{aligned} (\gamma - c) \int_{\Omega} \ddot{G} \square \partial u dx - \int_{\Omega} F_{ijkl}(\cdot, t) u_{k,l} \dot{u}_{i,j} dx &\leq (\gamma - c) C \int_{\Omega} G^{1+\frac{1}{p}} \square \partial u dx \\ + c_1 \left\{ \int_{\Omega} G_{ijkl}(\cdot, t) u_{k,l} u_{i,j} dx \right\}^{\frac{1}{2}} &\left\{ \int_{\Omega} G_{ijkl}(\cdot, t) \dot{u}_{k,l} \dot{u}_{i,j} dx \right\}^{\frac{1}{2}} ; \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\int_0^t \dot{F}_{ijkl}(\cdot, t - \tau) \{u_{k,l}(\cdot, t) - u_{k,l}(\cdot, \tau)\} d\tau \dot{u}_{i,j} \\ &\leq c_2 \left\{ \int_{\Omega} G^{1+\frac{1}{p}} \square \partial u dx \right\}^{\frac{1}{2}} \left\{ \int_{\Omega} \dot{u}_i \dot{u}_i dx \right\}^{\frac{1}{2}} . \end{aligned} \tag{2.11}$$

From (2.7)–(2.11) and taking γ big enough, our conclusion follows. The proof is now complete. \square

To get our final result we will use the following lemma.

Lemma 2.5. *Let us suppose that f is a nonnegative C^1 -function satisfying*

$$f'(t) \leq -k_0 [f(t)]^{1+\frac{1}{p}} + \frac{k_1}{(1+t)^p} ,$$

then we have that

$$f(t) \leq \frac{p^p f(0) + 2p^{p-1}c}{(p + \sqrt[p]{(g(0)c_1 t)^p}}$$

for $c_1 := \kappa_0 \min\{1, \frac{p}{2\kappa_0} (\frac{p}{\kappa_1})^{1/p}\}$.

Proof. Let us denote by $h(t)$ and $g(t)$ the functions

$$h(t) := \frac{2\kappa_1}{p(1+t)^p}, \quad g(t) := f(t) + h(t) .$$

We have

$$\begin{aligned} g'(t) &= f'(t) - \frac{2\kappa_1}{(1+t)^{p+1}} \leq -\kappa_0[f(t)]^{1+\frac{1}{p}} - \frac{\kappa_1}{(1+t)^{p+1}} \\ &\leq -\kappa_0 \left\{ [f(t)]^{1+\frac{1}{p}} + \left[\frac{\kappa_1}{\kappa_0(1+t)^p} \right]^{1+\frac{1}{p}} \right\} \\ &= -\kappa_0 \left\{ [f(t)]^{1+\frac{1}{p}} + \frac{p^{1+\frac{1}{p}}}{2\kappa_0\kappa_1^{\frac{1}{p}}} [h(t)]^{1+\frac{1}{p}} \right\}. \end{aligned}$$

It follows that there exists a positive constant κ_1 for which we have:

$$g'(t) \leq -\kappa_1 \{ [f(t)]^{1+\frac{1}{p}} + [h(t)]^{1+\frac{1}{p}} \} \leq -\kappa_1 [g(t)]^{1+\frac{1}{p}}.$$

Our result follows. \square

Lemma 2.6. *Let us suppose that the function G satisfies:*

$$\int_0^\infty G^{1-r} dt < \infty, \tag{2.12}$$

for $r = \frac{2^m-1}{p} < 1$. Then the following inequality hold:

$$G \square \partial u \leq \{G^{1-r} \square \partial u\}^{\frac{1}{2^m}} \{G^{1+\frac{1}{p}} \square \partial u\}^{1-\frac{1}{2^m}}.$$

Proof. Using Holder’s inequality we get:

$$G^s \square \partial u \leq \{G^{2s-1-\frac{1}{p}} \square \partial u\}^{\frac{1}{2}} \{G^{1+\frac{1}{p}} \square \partial u\}^{\frac{1}{2}}, \tag{2.13}$$

for any $s \geq 1$, from (2.13) we get:

$$\begin{aligned} \underbrace{G}_{s=1} \square \partial u &\leq \underbrace{\{G^{1-\frac{1}{p}} \square \partial u\}^{\frac{1}{2}} \{G^{1+\frac{1}{p}} \square \partial u\}^{\frac{1}{2}}}_{s=1-\frac{1}{p}} \leq \underbrace{\{G^{1-\frac{3}{p}} \square \partial u\}^{\frac{1}{4}} \{G^{1+\frac{1}{p}} \square \partial u\}^{\frac{1}{2}+\frac{1}{4}}}_{s=1-\frac{3}{p}} \\ &\leq \{G^{1-\frac{7}{p}} \square \partial u\}^{\frac{1}{8}} \{G^{1+\frac{1}{p}} \square \partial u\}^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}, \end{aligned}$$

obvious induction shows our result. \square

Theorem 2.1. *Under assumptions of Lemma 2.6 and hypotheses (1.4)–(1.6), for $p > 2$ with $2^m - 1 < p$, we have that*

$$E(t; u) + E(t; \dot{u}) \leq c\{E(0; u) + E(0; \dot{u})\} \frac{1}{(1+t)^q},$$

where $q = 2^m - 1$.

Proof. Multiplying Eq. (1.2) by u_i we get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \dot{u}_i u_i dx &= \int_{\Omega} \dot{u}_i \dot{u}_i dx - \int_{\Omega} C_{ijkl} u_{k,l} u_{i,j} dx \\ &\quad + \int_{\Omega} \int_0^t G_{ijkl}(\cdot, t - \tau) u_{k,l}(\cdot, \tau) d\tau u_{i,j} dx \\ &= \int_{\Omega} \dot{u}_i \dot{u}_i dx - \int_{\Omega} \{C_{ijkl} - \int_0^t G_{ijkl} d\tau\} u_{i,j} u_{k,l} dx \\ &\quad + \int_{\Omega} \int_0^t G_{ijkl}(\cdot, t - \tau) \{u_{k,l}(\cdot, t) - u_{k,l}(\cdot, \tau)\} d\tau u_{i,j} . \end{aligned} \tag{2.14}$$

On the other hand, Lemma 2.6 implies

$$\begin{aligned} &\int_0^t \int_{\Omega} G_{ijkl}(\cdot, t - \tau) \{u_{k,l}(\cdot, t) - u_{k,l}(\cdot, \tau)\} d\tau u_{i,j} \\ &\leq \frac{C^2}{v_0} \int_{\Omega} G^{1+\frac{1}{p}} \square \partial u dx + \frac{v_0}{4} \int_{\Omega} u_{i,j} u_{i,j} dx . \end{aligned} \tag{2.15}$$

From (2.14) and (2.15) we get

$$\frac{d}{dt} \int_{\Omega} \dot{u}_i u_i dx \leq \int_{\Omega} \dot{u}_i \dot{u}_i dx - \frac{3v_0}{4} \int_{\Omega} u_{i,j} u_{i,j} dx + C \int_{\Omega} G^{1+\frac{1}{p}} \square \partial u dx . \tag{2.16}$$

From Lemmas 2.1, 2.3, 2.4 and inequality (2.16) positive constants N_1 and N_2 exist such that the function

$$\mathcal{L}(t; u) = N_1 E(t; u) + N_2 E(t; \dot{u}) + K(t; u) + (\gamma - c) I(t; u) + \frac{\beta}{8} c_p \int_{\Omega} \dot{u}_i u_i dx ,$$

satisfies

$$\begin{aligned} &\frac{d}{dt} \mathcal{L}(t; u) \leq cg(t) \\ &-C_0 \left\{ \underbrace{\int_{\Omega} u_{i,j} u_{i,j} dx + \int_{\Omega} \dot{u}_{i,j} \dot{u}_{i,j} dx + \int_{\Omega} \ddot{u}_i \ddot{u}_i dx}_{:=R(t)} + \underbrace{\int_{\Omega} G^{1+\frac{1}{p}} \square \partial u dx + \int_{\Omega} G^{1+\frac{1}{p}} \square \partial \dot{u} dx}_{:=S(t)} \right\} , \end{aligned}$$

where c_p denotes Poincaré’s constant. Using Lemma 2.6, it is not difficult to see that

$$S(t) \geq c \left\{ \int_{\Omega} G \square \partial u dx + \int_{\Omega} G \square \dot{u} dx \right\}^{\frac{2^m}{2^m-1}} ; \quad R(t) \geq R(t)^{\frac{2^m}{2^m-1}} .$$

So we have

$$\frac{d}{dt} \mathcal{L}(t; u) \leq cg(t) + c_5 \mathcal{L}(t; u)^{\frac{2^m}{2^m-1}} .$$

Using Lemma 2.5 our result follows. \square

3. Rates of Decay in \mathbb{R}^n

In this section we extend the results due to Dassios and Zafiropoulos [6]. We study the asymptotic behaviour of the energy of the following Cauchy problem:

$$\begin{aligned}
 &u_{tt} - \mu \Delta u - (\mu + \gamma) \nabla \operatorname{div} u + \int_0^t g(t - \tau) \Delta u(\cdot, \tau) d\tau \\
 &+ \int_0^t h(t - \tau) \nabla \operatorname{div} u(\cdot, \tau) d\tau = 0 \quad \text{in } \mathbb{R}^n, \tag{3.17} \\
 &u(\cdot, 0) = u_0(x); \quad u_t(\cdot, 0) = u_1(x).
 \end{aligned}$$

To do this we prove that the displacement vector field can be decomposed into two parts: solenoidal and irrotational, both having the same regularity as u , whose corresponding energies decay uniformly to zero with the rate depending on the regularity of the initial data and the rate of decay of g and h . The regularity of the decompositions is similar to the regularity of the solution for the equation:

$$\Delta u = f \quad \text{in } \mathbb{R}^n, \tag{3.18}$$

whose existence and regularity results are given in the following lemma

Lemma 3.1. *Let f be a C^2 -function satisfying*

$$f \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2) \quad \text{and} \quad x \mapsto |x|^\alpha |f(x)| \in L^1(\mathbb{R}) \quad n = 2,$$

for $p \geq 2$ and $\alpha > 0$. Then there exists a function

$$u \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \quad \text{with} \quad \nabla u \in [H^1(\mathbb{R}^2)]^2$$

satisfying (3.18). While for $n \geq 3$, $p > \frac{n}{n-2}$, and f satisfying

$$f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n),$$

there exist a solution of (3.18) satisfying:

$$u \in L^p(\mathbb{R}^n); \quad \nabla u \in [L^q(\mathbb{R}^n)]^n; \quad \text{for any } q > \frac{n}{n-1}.$$

In particular we get $\nabla u \in [H^1(\mathbb{R}^n)]^n$. Finally if $p > n > 2$ we get

$$u \in L^p(\mathbb{R}^n) \cap C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2).$$

Proof. Let us denote by U the function:

$$U(x) = \begin{cases} \frac{1}{2\pi} \ln(|x|) & \text{if } n = 2 \\ \frac{1}{(n-2)\sigma_n |x|^{n-2}} & \text{if } n > 2 \end{cases},$$

where σ_n is the area of the unit ball of \mathbb{R}^n . It is well known that the solution of Eq. (3.18) is given by:

$$u(x) = \int_{\mathbb{R}^n} U(x - \xi) f(\xi) d\xi.$$

Without loss of generality we can suppose that f has bounded support. To prove the regularity result we will use the convolution:

$$f_v = \rho_v * f ,$$

where ρ_v is the mollifier satisfying

$$\rho_v(-x) = \rho_v(x); \quad \int_{\mathbb{R}^n} \rho_v(\xi) d\xi = 1 \quad \text{and} \quad \rho_v(x) = 0, \quad \text{if } |x| \geq \frac{1}{v} .$$

It is well known that f_v converges to f strong in $L^r(\mathbb{R}^n)$ when $f \in L^r(\mathbb{R}^n)$. Let us denote by u_v the solution of (3.18) for $f = f_v$, then it follows that

$$u_v(x) = \int_{\mathbb{R}^n} U(x - \xi) f_v(\xi) d\xi . \tag{3.19}$$

From (3.19) it follows that

$$\int_{\mathbb{R}^n} |\nabla u_v(x)|^2 dx \leq \int_{\mathbb{R}^n} |u_v| |f_v| dx , \tag{3.20}$$

$$\int_{\mathbb{R}^n} |\Delta u_v(x) - \Delta u_m(x)|^2 dx \leq \int_{\mathbb{R}^n} |f_v(x) - f_m(x)|^2 dx . \tag{3.21}$$

First, we prove that u_v is bounded in $H^1(\mathbb{R}^n)$, then that (u_v) is a Cauchy sequence in $C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Using (3.19) we have:

$$\frac{\partial u_v}{\partial x_i} = \int_{\mathbb{R}^n} \frac{\partial U}{\partial x_i}(x - \xi) f_v(\xi) d\xi .$$

Denoting by χ and χ_c the characteristic functions over the unit ball and its complementary set respectively, we get

$$U(\xi) = \chi U(\xi) + \chi_c U(\xi) .$$

It is easy to see that

$$\frac{\partial U}{\partial x_i}(\xi) = \begin{cases} \frac{1}{2\pi} \frac{\xi_i}{|\xi|^2} & \text{if } n = 2 \\ \frac{-\xi_i}{2\sigma_n |\xi|^n} & \text{if } n > 2 \end{cases} ,$$

and also

$$\chi \frac{\partial U}{\partial x_i} \in L^1(\mathbb{R}^n) \quad \text{and} \quad \chi_c \frac{\partial U}{\partial x_i} \in L^r(\mathbb{R}^n) \quad \text{for any } r > \frac{n}{n-1} .$$

Using Young's inequality we get:

$$\chi \frac{\partial U}{\partial x_i} * f_v \in L^r(\mathbb{R}^n) \quad \forall 1 \leq r \leq p; \quad \chi_c \frac{\partial U}{\partial x_i} * f_v \in L^r(\mathbb{R}^n) \quad \forall r > \frac{n}{n-1} .$$

Also,

$$\left\{ \int_{\mathbb{R}^n} \left| \chi \frac{\partial U}{\partial x_i} * f_v \right|^r d\xi \right\}^{\frac{1}{r}} \leq \int_{\mathbb{R}^n} \left| \chi \frac{\partial U}{\partial x_i} \right| d\xi \left\{ \int_{\mathbb{R}^n} |f_v|^r d\xi \right\}^{\frac{1}{r}} ,$$

$$\left\{ \int_{\mathbb{R}^n} \left| \chi_c \frac{\partial U}{\partial x_i} * f_v \right|^p d\xi \right\}^{\frac{1}{p}} \leq \left\{ \int_{\mathbb{R}^n} \left| \chi_c \frac{\partial U}{\partial x_i} \right|^p d\xi \right\}^{\frac{1}{p}} \int_{\mathbb{R}^n} |f_v| d\xi .$$

Under these conditions, for $p = q = r$, we get that

$$\int_{\mathbb{R}^n} \left| \frac{\partial u_v}{\partial x_i} \right|^q dx \leq c \int_{\mathbb{R}^n} \left| \chi \frac{\partial U}{\partial x_i} \right| d\xi \int_{\mathbb{R}^n} |f_v|^q d\xi + c \int_{\mathbb{R}^n} \left| \chi_c \frac{\partial U}{\partial x_i} \right|^q d\xi \left\{ \int_{\mathbb{R}^n} |f_v| d\xi \right\}^q,$$

which implies that

$$\left\| \frac{\partial u_v}{\partial x_i} \right\|_q \leq c \{ \|f_v\|_q + \|f_v\|_1 \} \quad \text{for } i = 1, \dots, n, \tag{3.22}$$

for $q > n/(n - 1)$. To prove that u_v is a Cauchy sequence in $C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, let us consider

$$\begin{aligned} |u_v| &= \left| \int_{\mathbb{R}^n} U(x - \xi) f_v(\xi) d\xi \right| = \left| \frac{1}{\sigma_2} \int_{\mathbb{R}^n} \ln|x - \xi| f_v(\xi) d\xi \right| \\ &\leq c_\alpha \int_{|\xi| \geq |x|} |x - \xi|^\alpha |f_v(\xi)| d\xi + \frac{1}{\sigma_2} \left| \int_{|\xi| \leq |x|} \frac{\ln|x - \xi|}{|x|^\alpha} |\xi|^\alpha f_v(\xi) d\xi \right| \\ &\leq \frac{1}{\sigma_2} c_\alpha \left| \int_{|\xi| \geq |x|} |x - \xi|^\alpha |f_v(\xi)| d\xi \right| + \int_{|\xi| \leq |x|} \underbrace{\frac{|x - \xi|^\alpha}{|x|^\alpha}}_{\leq 2^\alpha} |\xi|^\alpha |f_v(\xi)| d\xi \\ &\leq C_\alpha \int_{\mathbb{R}^n} |\xi|^\alpha |f_v(\xi)| d\xi. \end{aligned}$$

Repeating the above reasoning for $u_v - u_m$ instead of u_v , we conclude that u_v is a Cauchy sequence in $L^\infty(\mathbb{R}^2)$. Since f has bounded support we can apply Lebesgue’s Convergence Theorem to show that:

$$|u| \leq C_\alpha \int_{\mathbb{R}^n} |\xi|^\alpha |f(\xi)| d\xi.$$

Using (3.20), (3.21) and the fact that $f \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ we conclude that ∇u_v is bounded in $H^1(\mathbb{R}^n)$. So, our result follows for case $n = 2$. For $n > 2$ using (3.22) we get that ∇u_v is bounded in $[H^2(\mathbb{R}^n)]^n$. Moreover we have

$$\chi U \in L^1(\mathbb{R}^n), \quad \chi_c U \in L^p(\mathbb{R}^n) \quad \text{for } p > \frac{n}{n - 2}.$$

From hypotheses on f we get

$$u_v \in L^q(\mathbb{R}^n) \cap L^1(\mathbb{R}^n),$$

and also that

$$\left\{ \int_{\mathbb{R}^n} |u_v|^q dx \right\}^{\frac{1}{q}} \leq c \{ \|f_v\|_q + \|f_v\|_1 \},$$

where by $\|\cdot\|_p$ we denote the norm of the space L^p . It remains only to prove the regularity of u when $p > n$. To do this we will use the following lemma:

Lemma 3.2. *Let us suppose that v is a continuous function satisfying*

$$u \in L^q(\mathbb{R}^n), \quad \nabla u \in [L^p(\mathbb{R}^n)]^n,$$

for any $q \geq 1$ and any $p > n$. Then the following inequality holds:

$$|v(x)| \leq \frac{1}{\sqrt[q]{\sigma_n}} \|v\|_q + \frac{p}{(p-n)\sqrt[q]{\sigma_n}} \|\nabla v\|_p.$$

Using Lemma 3.2 we conclude that u_v satisfies

$$|u_v(x)| \leq c \left\{ \left\{ \int_{\mathbb{R}^n} |u_v|^q dx \right\}^{\frac{1}{q}} + \left\{ \int_{\mathbb{R}^n} |\nabla u_v|^p dx \right\}^{\frac{1}{p}} \right\} \leq C \{ \|f_v\|_1 + \|f_v\|_q + \|f_v\|_p \}$$

and our result follows. \square

Proof of Lemma 3.2. Let us consider the identity:

$$v(\xi) - v(x) = \int_0^1 \nabla v(t\xi + (1-t)x) \cdot [\xi - x] dt.$$

Integration over the ball B of center x and radius $r = 1$ yields

$$\begin{aligned} \left| \int_B v(\xi) d\xi - \sigma_n v(x) \right| &\leq \int_B \int_0^1 |\nabla v(t\xi + (1-t)x) \cdot [\xi - x]| dt d\xi \\ &\leq \int_B \int_0^1 |\nabla v(t\xi + (1-t)x)| dt d\xi. \end{aligned}$$

Making a change of variable we get

$$\left| \int_B v(\xi) d\xi - \sigma_n v(x) \right| \leq \int_0^1 \frac{1}{t^n} \int_D |\nabla v(\xi)| d\xi dt,$$

where D is the disk of center in the origin of coordinates and the radius equals to t . Since $t \leq 1$ and $tD \subset D$, it follows that

$$\int_D |\nabla v(\xi)| d\xi dt \leq \sqrt[p]{\sigma_n t^n} \left\{ \int_D |\nabla v(\xi)|^p d\xi \right\}^{\frac{1}{p}},$$

so we have that:

$$|v(x)| \leq \frac{1}{\sigma_n} \left| \int_B v(\xi) d\xi \right| + \frac{1}{\sqrt[q]{\sigma_n}} \frac{p}{p-n} \left\{ \int_D |\nabla v(\xi)|^p d\xi \right\}^{\frac{1}{p}}.$$

Using the Hölder inequality our conclusion follows. \square

Now we are in a position to prove the decomposition of the displacement vector field.

Lemma 3.3. *Let f be a vector field in $[H^k(\mathbb{R}^n)]^n$ such that the divergence of F ($\operatorname{div} F = f$) satisfies the conditions of Lemma 3.1. Then, we can decompose F in two parts, both in $[H^k(\mathbb{R}^n)]^n$, one of them a gradient and the other a solenoidal function (that is with null divergence).*

Proof. From Lemma 3.1 there exists a function p such that $\nabla p \in [H^1(\mathbb{R}^n)]^n$ satisfies:

$$\Delta p = \operatorname{div} F, \quad \text{in } \mathbb{R}^n.$$

Since $\Delta p = \operatorname{div} F \in H^{k-1}(\mathbb{R}^n)$, then $\nabla p \in H^k(\mathbb{R}^n)$. From the identity:

$$F = \nabla p + (F - \nabla p),$$

we obtain the desired decomposition. \square

Using Lemma 3.3 we can decompose the initial data of Eq. (3.17) in the following way:

$$u_0 = u_0^I + u_0^S; \quad u_1 = u_1^I + u_1^S,$$

where u_i^I and u_i^S are the irrotational and the solenoidal part of the function u_i for $i = 1, 2$ given in Lemma 3.3. Let us denote by u^S by u^I the solution of the equations:

$$u_{tt}^S - \mu \Delta u^S + \mu \int_0^t g(t - \tau) \Delta u^S(\cdot, \tau) d\tau = 0 \quad \text{in } \mathbb{R}^n, \tag{3.23}$$

$$u(x, 0) = u_0^S(x), \quad u_t(x, 0) = u_1^S(x),$$

$$u_{tt}^I - (2\mu + \gamma) \Delta u^I + (2\mu + \gamma) \int_0^t h(t - \tau) \Delta u^I(\cdot, \tau) d\tau = 0 \quad \text{in } \mathbb{R}^n, \tag{3.24}$$

$$u(x, 0) = u_0^I(x); \quad u_t(x, 0) = u_1^I(x),$$

respectively. Since $\operatorname{div} u_i^S = 0$ for $i = 1, 2$, the solution u^S also satisfies: $\operatorname{div} u^S = 0$. Similarly, since the initial data u_i^I is such that

$$\frac{\partial u_0^{i,I}}{\partial x_j} = \frac{\partial u_0^{j,I}}{\partial x_i}, \quad \frac{\partial u_1^{i,I}}{\partial x_j} = \frac{\partial u_1^{j,I}}{\partial x_i},$$

then the solution of (3.23) also satisfies:

$$\frac{\partial u^{i,I}}{\partial x_j} = \frac{\partial u^{j,I}}{\partial x_i}.$$

Using these properties we conclude that the sum $u^S + u^I$ is the solution of Eq. (3.17). Our next goal is to prove that the solution of Eq. (3.23) and (3.24) decay algebraically as time goes to infinity. Without loss of generality we consider the equation:

$$u_{tt} - \mu \Delta u + \mu \int_0^t g(t - \tau) \Delta u(\cdot, \tau) d\tau = 0 \quad \text{in } \mathbb{R}^n, \tag{3.25}$$

$$u(x, 0) = u_0(x); \quad u_t(x, 0) = u_1(x).$$

From now on by $g \square v$ we denote:

$$g \square v = \int_0^t g(t - \tau) |v(t) - v(\tau)|^2 d\tau.$$

Let us define the following functionals:

$$E(t; v) := \frac{1}{2} \int_{\mathbb{R}^n} \left[|\dot{v}(t)|^2 + \mu \left(1 - \int_0^t g d\tau \right) \sum_{i=1}^n |\nabla v_i|^2 + \frac{\mu}{2} \sum_{i=1}^n g \square \nabla v_i \right] dx.$$

Also we define the density energy function:

$$\mathcal{E}(\xi, t) := \frac{1}{2}|\hat{v}_i(t)|^2 + \mu \left(1 - \int_0^t g d\tau \right) |\xi|^2 |\hat{v}_i|^2 + \frac{\mu}{2} |\xi|^2 g \square \hat{v}_i.$$

To get the rate of decay of the energy function we will use the following lemmas:

Lemma 3.4. *Let v be a function in $L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ for which there exists $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ satisfying*

$$v = \partial^\alpha f \quad \text{where } |\alpha| = m.$$

Then we have that

$$|\hat{v}(\xi)| \leq [2\pi]^{\frac{n}{2}} |\xi^\alpha| \int_{\mathbb{R}^n} |f(x)| dx \quad \forall \xi \in \mathbb{R}^n.$$

Proof. The proof is immediate. \square

Lemma 3.5. *Let $m \geq 1$ and $s \geq 0$ be two natural numbers. Then we have that*

$$\int_0^1 \frac{\sigma^s d\sigma}{(1 + a\sigma)^m} \leq \begin{cases} c(m, s) a^{-m} & \text{if } m \leq s \\ c(m, s) a^{-(s+1)} & \text{if } m > s + 1 \\ c(m, s, \eta) a^{-s+\eta} & \text{if } m = s + 1 \end{cases}$$

for $0 < \eta < 1$, where $c(m, s, \eta) \rightarrow \infty$ as $\eta \rightarrow 0$.

Proof. Let us denote

$$I(s, m) := \int_0^1 \frac{\sigma^s d\sigma}{(1 + a\sigma)^m}.$$

Integration by parts shows:

$$\begin{aligned} \int_0^1 \frac{\sigma^s d\sigma}{(1 + a\sigma)^m} &= \frac{1}{a(-m + 1)} \int_0^1 \sigma^s d(1 + a\sigma)^{-m+1} \\ &= \frac{1}{a(-m + 1)} [\sigma^s (1 + a\sigma)^{-m+1}]_{\sigma=0}^{\sigma=1} + \frac{s}{a(m - 1)} \int_0^1 \frac{\sigma^{s-1} d\sigma}{(1 + a\sigma)^{m-1}} \\ &= \frac{1}{a(m - 1)} \int_0^1 \frac{\sigma^{s-1} d\sigma}{(1 + a\sigma)^{m-1}}, \end{aligned}$$

so we have:

$$I(s, m) \leq \frac{s}{a(m - 1)} I(s - 1, m - 1).$$

Note that a simple calculation shows that

$$I(s, 1) \leq c(s) a^{-1} \quad \text{for } s \geq 1, \quad I(0, m) \leq \frac{1}{m - 1} a^{-1} \quad \text{for } m > 1,$$

$$I(0, 1) = \frac{\ln(1 + a)}{a}.$$

Using the last three inequalities our conclusion follows. \square

In this condition we are able to prove the uniform rate of decay of the energy associated to the viscoelastic equation for the whole space \mathbb{R}^n . Our main result is summarized in the following theorem:

Theorem 3.1. *Let us suppose that the functions g and h satisfy the following hypotheses:*

$$g(t) \geq 0; \quad g'(t) \leq -cg^{1+\frac{1}{p}}(t), \quad g''(t) \leq cg^{1+\frac{1}{p}}(t), \quad (3.26)$$

$$h(t) \geq 0; \quad h'(t) \leq -ch^{1+\frac{1}{p}}(t), \quad h''(t) \leq ch^{1+\frac{1}{p}}(t), \quad (3.27)$$

for $p > 2$ such that $2^m - 1 < p$ and the couple (u_0, u_1) is a vector field in $[H^2(\mathbb{R}^n) \cup L^1(\mathbb{R}^n)]^n \times [H^1(\mathbb{R}^n) \cup L^1(\mathbb{R}^n)]^n$, such that $\operatorname{div} u_0$ and $\operatorname{div} u_1$ satisfy hypotheses of Lemma 3.3. Then the displacement vector field can be decomposed into two parts, one of them a solenoidal and the other a gradient of a real function whose energy decays as:

$$E(t; u) + E(t; u_t) \leq c\{E(0; u) + E(0; u_t)\} \frac{1}{(1+t)^\gamma},$$

where

$$\gamma = \begin{cases} 2^m - 1 & \text{if } 2^{m+1} \leq n - 1 \\ \min \left\{ 2^m - 1, \left(\frac{1}{2} - \frac{1}{2^{m+1}} \right) n \right\} & \text{if } 2^{m+1} > n - 1 \\ \min \left\{ 2^m - 1, \left(\frac{1}{2} - \frac{1-\eta}{2^{m+1}} \right) (n - 1) \right\} & \text{if } 2^{m+1} = n. \end{cases}$$

Moreover if there exist functions $f_0^k, f_1^k \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ such that:

$$\partial^{\alpha_k} f_0^k = u_0^k, \quad \text{and} \quad \partial^{\beta_k} f_1^k = u_1^k. \quad (3.28)$$

Then

$$E(t; u) + E(t; u_t) \leq c\{E(0; u) + E(0; u_t)\} \frac{1}{(1+t)^\mu},$$

where

$$\mu = \begin{cases} 2^m - 1 & \text{if } 2^{m+1} \leq n - 1 \\ \min \left\{ 2^m - 1, \frac{2^m - 1}{2^{m+1} + \beta} (n + \beta) \right\} & \text{if } 2^{m+1} > n - 1 \\ \min \left\{ 2^m - 1, \frac{2^m - 1 + \beta + \eta}{2^{m+1} + \beta} (2^m - 1) \right\} & \text{if } 2^{m+1} = n \end{cases}$$

with $\beta = \min\{|\alpha_k| + 1, |\beta_k|; k = 1, \dots, n\}$.

Proof. From Eq. (3.25) we have that the Fourier transform of u satisfies:

$$\hat{u}_{tt} + |\xi|^2 \hat{u} - |\xi|^2 \int_0^t g(t - \tau) \hat{u}(\cdot, \tau) d\tau = 0. \quad (3.29)$$

Reasoning as in Theorem 2.1 we can prove that:

$$\frac{d}{dt} \mathcal{E}(\xi, t, \hat{u}) = \frac{1}{2} |\xi|^2 g' \square \hat{u} - \frac{1}{2} g(t) |\hat{u}|^2.$$

Differentiating Eq. (3.29) we get

$$\hat{u}_{ttt} + |\xi|^2 \hat{u}_t - g(0) |\xi|^2 \hat{u} - |\xi|^2 \int_0^t g'(t - \tau) \hat{u}(\cdot, \tau) d\tau = 0.$$

Substituting \hat{u} given by Eq. (3.29) we get:

$$\hat{u}_{tt} + |\xi|^2 \hat{u}_t + g(0)\hat{u}_{tt} - |\xi|^2 \int_0^t \underbrace{[g'(t-\tau) + g(0)g(t-\tau)]}_{=f(t-\tau)} \hat{u}(\cdot, \tau) d\tau = 0 .$$

Multiplying the above equation by \hat{u}_{tt} we get:

$$\frac{1}{2} \frac{d}{dt} \{ |\hat{u}_{tt}|^2 + |\xi|^2 |\hat{u}_t|^2 \} = -g(0)|\hat{u}_{tt}|^2 + \underbrace{|\xi|^2 \int_0^t f(t-\tau) \hat{u}(\cdot, \tau) d\tau}_{=I} \hat{u}_{tt} ,$$

so we have:

$$\begin{aligned} I &= |\xi|^2 \int_0^t f d\tau \hat{u}_{tt} |\xi|^2 \left(g(t) - g(0) + g(0) \int_0^t g d\tau \right) \hat{u}_{tt} \\ &= -|\xi|^2 g(t) \hat{u}_{tt} + |\xi|^2 g(0) \left(1 - \int_0^t g d\tau \right) \hat{u}_{tt} . \end{aligned}$$

From this identity we get:

$$\begin{aligned} I &= -|\xi|^2 g(t) \hat{u}_{tt} - g(0) |\xi|^2 \left(1 - \int_0^t g d\tau \right) |\hat{u}_t|^2 + g(t)g(0) |\xi|^2 \hat{u}_{tt} \\ &\quad + \frac{d}{dt} \left\{ |\xi|^2 g(0) \left(1 - \int_0^t g d\tau \right) \hat{u}_{tt} \right\} \\ &\leq \frac{g(0)}{4} |\hat{u}_{tt}|^2 + g(t) |\xi|^4 |\hat{u}|^2 + \frac{g(t)g(0)^2}{2\alpha} |\xi|^2 |\hat{u}|^2 - \frac{g(0)\alpha}{2} |\hat{u}_t|^2 \\ &\quad + |\xi|^2 g(0) \frac{d}{dt} \left\{ \left(1 - \int_0^t g d\tau \right) \hat{u}_{tt} \right\} , \end{aligned}$$

where $\alpha = 1 - \int_0^\infty g d\tau$. So we conclude that there exist positive constants N_1, N_2 such that the function

$$\mathcal{L}(\xi, t) = N_1 \mathcal{E}(\xi, t; u) + N_2 |\xi|^2 \mathcal{E}(\xi, t; u) |\hat{u}_{tt}|^2 + |\xi|^2 |\hat{u}_t|^2 - g(0) |\xi|^2 \left(1 - \int_0^t g d\tau \right) \hat{u}_{tt}$$

satisfies

$$\frac{d}{dt} \mathcal{L}(\xi, t) \leq -c_0 \{ \mathcal{R}(\xi, t) + \mathcal{S}(\xi, t) \} ,$$

where

$$\mathcal{R}(\xi, t) := |\hat{u}_{tt}|^2 + |\xi|^4 |\hat{u}|^2 + |\xi|^2 |\hat{u}_t|^2, \quad \mathcal{S}(\xi, t) := |\xi|^2 g \square \hat{u} + |\xi|^4 g \square \hat{u} ,$$

and

$$\mathcal{L}(\xi, t) \leq c_1 \mathcal{R}(\xi, t) + \mathcal{S}(\xi, t) + c_2 r(\xi, t)$$

with

$$r(\xi, t) := |\xi|^2 |\hat{u}|^2 + |\hat{u}_t|^2 .$$

We will consider two cases. First we prove that the energy density decays algebraically as $\frac{1}{t^q}$ for $|\xi| \geq 1$, then that the density energy function decays as $\frac{1}{t^p}$ when $|\xi| \geq 1$. In fact, for $|\xi| \geq 1$ we get

$$\mathcal{L}(\xi, t) \leq c_3 \mathcal{M}(\xi, t) |\xi|^2 + |\xi|^2 g \square \hat{u} + |\xi|^4 g \square \hat{u}.$$

Since the energy is bounded, and from Lemma 2.6, a positive constant exists (depending on the initial data) satisfying:

$$\int_{|\xi| \geq 1} |\xi|^2 g \square \hat{u} + |\xi|^4 g \square \hat{u} d\xi \leq C_1 \left\{ \int_{|\xi| \geq 1} |\xi|^2 g^{1+\frac{1}{p}} \square \hat{u} + |\xi|^4 g^{1+\frac{1}{p}} \square \hat{u} d\xi \right\}^{1-\frac{1}{2^m}};$$

$$\int_{|\xi| \geq 1} \mathcal{L}(\xi, t) d\xi \leq c_1 \left\{ \int_{|\xi| \geq 1} \mathcal{R}(\xi, t) + \mathcal{S}(\xi, t) d\xi \right\}^{1-\frac{1}{2^m}}.$$

We get:

$$\frac{d}{dt} \int_{|\xi| \geq 1} \mathcal{L}(\xi, t) d\xi \leq -c_0 \left\{ \int_{|\xi| \geq 1} \mathcal{L}(\xi, t) d\xi \right\}^{\frac{2^m}{2^m-1}},$$

which implies that

$$\int_{|\xi| \geq 1} \mathcal{L}(\xi, t) d\xi \leq \frac{\int_{|\xi| \geq 1} \mathcal{L}(\xi, 0) d\xi}{\left\{ q + c_0 \left(\int_{|\xi| \geq 1} \mathcal{L}(\xi, 0) d\xi \right)^{\frac{1}{p}} t \right\}^q},$$

so, we have

$$\int_{|\xi| \geq 1} \mathcal{L}(\xi, t) d\xi \leq c \int_{|\xi| \geq 1} \mathcal{L}(\xi, 0; \hat{u}) d\xi \frac{1}{(1+t)^q}. \tag{3.30}$$

Now we consider the case $|\xi| \leq 1$. Since

$$|\xi|^2 \mathcal{L}(\xi, t; \hat{u}) \leq c_3 \{ \mathcal{R}(\xi, t; \hat{u}) + \mathcal{S}(\xi, t) \}^{1-\frac{1}{2^m}} \quad \forall |\xi| \leq 1, \tag{3.31}$$

it follows that

$$\frac{d}{dt} \mathcal{L}(\xi, t) \leq -c_0 \{ |\xi|^2 \mathcal{L}(\xi, t) \}^{\frac{2^m}{2^m-1}},$$

which implies

$$\mathcal{L}(\xi, t; \hat{u}) \leq c \frac{1}{(\mathcal{L}(\xi, 0)^{-\frac{1}{q}} + |\xi|^q r t)^q}. \tag{3.32}$$

Since the initial data belongs to $L^1(\mathbb{R}^n)$, and \hat{u} is a continuous function, then we get

$$\mathcal{L}(0, \xi) \leq c \quad \forall |\xi| \leq 1.$$

From (3.32) a positive constant c exists satisfying:

$$\mathcal{L}(\xi, t; \hat{u}) \leq \frac{c}{(1 + |\xi|^q r t)^q},$$

where $r := 2 + \frac{2}{q} = \frac{2^{m+1}}{2^m - 1}$. It is easy to see that:

$$\mathcal{L}(\xi, t; \hat{u}) \leq \frac{c_0}{(1 + |\xi|t^{\frac{1}{r}})^{2^{m+1}}}.$$

Using spherical coordinates we conclude that

$$\int_{|\xi| \leq 1} \mathcal{L}(\xi, t; \hat{u}) d\xi \leq c \int_0^1 \frac{\sigma^{n-1} d\sigma}{(1 + t^{\frac{1}{r}} \sigma)^{2^{m+1}}},$$

where n is the space dimension. Using Lemma 3.5 our conclusion follows. Finally if hypothesis (3.28) holds, then a positive number β exists such that $\mathcal{L}(0, \xi) \leq c|\xi|^\beta$, so we get,

$$\mathcal{L}(\xi, t; \hat{u}) \leq c \frac{|\xi|^\beta}{(1 + |\xi|t^{\frac{1}{q}})^q} \leq \frac{|\xi|^\beta}{(1 + |\xi|t^\gamma)^{2^{m+1} + \beta}},$$

where $\gamma := \frac{q}{q^r + \beta} = \frac{2^m - 1}{2^{m+1} + \beta}$. Again using Lemma 3.5 our conclusion follows. The proof is now complete. \square

Remark 3.1. Theorems 2.1 and 3.1 are optimal in the sense that when m increases, the rate of decay of the energy also increases. In the limit case when $m = \infty$, ($p = \infty$) from the hypotheses we get that the kernel of convolution is exponential. In this case our result is the same as in [10] or in [6].

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