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Anomalies and Curvature of W Manifolds

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Abstract. We study the holomorphic structure of certain complex manifolds associated with W_∞ algebras, namely, the flag manifolds W_∞/T_∞ and $W_{1+\infty}/T_{1+\infty}$, and the spaces $W_\infty/SL(\infty,R)$ and $W_{1+\infty}/GL(\infty,R)$, where T_∞ and $T_{1+\infty}$ are the maximal tori in W_∞ and $W_{1+\infty}$. We compute their Ricci curvature and show how the results are related to the anomaly-freedom conditions for W_∞ and $W_{1+\infty}$. We discuss the relation of these manifolds with extensions of universal Teichmüller space.

1. Introduction

An important problem in string theory is the search for a better understanding of its geometrical underpinnings, in the spirit of the beautiful interpretation of general relativity in terms of Riemannian geometry. It has been argued that a natural arena for addressing such geometrical issues is provided by the study of the manifold $\mathscr{M} \equiv \mathrm{diff}(S^1)/S^1$ of complex structures on loop space related by reparametrisations [1]. This remarkable manifold proves to possess a natural Kähler structure [1], and it has been found that many statements concerning the consistency of string theory can be reformulated in terms of geometric data for $\mathscr{M}$ or related structures [1–5]. For example, the condition of nilpotency for the BRST charge Q (required for quantisation in the BRST formalism) is replaced by the requirement that a certain vector bundle over $\mathscr{M}$ have vanishing Ricci curvature [1,2].

In this paper, we show that this geometrical formalism admits very natural extensions when one replaces the algebra $\mathrm{diff}(S^1)$ (essentially the centreless Virasoro algebra) by certain higher-spin extended algebras which have been

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explicitly constructed recently. As obtained in [6,7], the W_∞ and $W_{1+\infty}$ algebras involve (quasi-)conformal fields $V^i(z)$ of spin $(i+2)$, where the index i ranges over the integers $i \geqq 0$ and $i \geqq -1$, respectively. In either case, the commutation relations between Fourier modes V^i_m take the form

$$[V^i_m, V^j_n] = \sum_{r\geqq 0} g^{ij}_{2r}(m,n)\, V^{i+j-2r}_{m+n} + c_i(m)\delta^{ij}\delta_{m+n,0}, \tag{1}$$

where the form of the structure constants $g^{ij}_{2r}(m,n)$ guarantees that the summation only involves generators in the algebra, and the central terms take the form

$$c_i(m) = m(m^2-1)\cdots(m^2-(i+1)^2)c_i, \tag{2}$$

where the central charges c_i are all fixed up to an overall scale, determined by the Virasoro central charge $c=12c_0$. Explicit expressions for the structure constants and central charges can be found in [6] for W_∞ and [7] for $W_{1+\infty}$.

In the following section, we briefly review the results of recent investigations of the condition $Q^2=0$ for the W_∞ and $W_{1+\infty}$ algebras [8,9]. In Sect. 3, we introduce several manifolds which are the natural W-extensions of $\mathrm{diff}(S^1)/S^1$ and $\mathrm{diff}(S^1)/SL(2,R)$, discuss their Kähler structure, and compute their Ricci curvature. In the case of the Virasoro algebra, the Kähler metric on $\mathrm{diff}(S^1)/S^1$ has been shown to induce the Weil–Petersson metric on the universal Teichmüller space, parametrised by Beltrami differentials [10]. In Sect. 4, we show how some of these results may be extended to the higher-spin algebras.

2. Review of BRST

For an arbitrary Lie algebra G, with generators T^a satisfying the commutation relations $[T^a, T^b] = f^{ab}{}_c T^c$, one can define the BRST charge $Q = Q_T + Q_{\mathrm{gh}}$, where $Q_T = c_a T^a$ and $Q_{\mathrm{gh}} = -\frac{1}{2} f^{ab}{}_c c_a c_b b^c$, with c_a and b^a being anticommuting ghost operators satisfying the anticommutation relations $\{c_a, b^c\} = \delta^b_a$. For W_∞ or $W_{1+\infty}$, the BRST charge for the "matter" sector is given by

$$Q_T = \alpha_0 c^0_0 + \sum_{i,m} V^i_{-m} c^i_m, \tag{3}$$

allowing for a general intercept α_0, and for the ghost sector

$$Q_{\mathrm{gh}} = -\frac{1}{2} \sum_{\substack{i,j,r,\\ m,n}} g^{ij}_{2r}(m,n) : c^i_{-m} c^j_{-n} b^{i+j-2r}_{m+n} :. \tag{4}$$

Because the algebra is infinite dimensional, one has to adopt some appropriate normal-ordering convention for the ghost modes. The customary choice, extended to the W_∞ or $W_{1+\infty}$ algebras, is to define a vacuum state $|0\rangle$ by

$$\begin{aligned} c^i_m|0\rangle &= 0, \quad m \geqq 0, \\ b^i_m|0\rangle &= 0, \quad m > 0, \end{aligned} \tag{5}$$

and to normal order with respect to the corresponding decomposition into creation and annihilation operators.

The nilpotency condition $Q^2=0$ becomes $Q_T^2+Q_{\rm gh}^2=0$, where

$$Q_T^2=\sum_i\sum_{m>0}\{c_i(m)-\alpha_0 g_{2i}^{ii}(m,-m)\}c_{-m}^i c_m^i \tag{6}$$

and

$$Q_{\rm gh}^2=\sum_{i,k}\sum_{m>0}R^{ik}(m)c_{-m}^i c_m^k. \tag{7}$$

The coefficients $R^{ik}(m)$ in (7) are given by

$$R^{ik}(m)\equiv\sum_{r=0}^{(i+k)/2}\sum_{j=\max(0,2r-i)}^{\infty}\left\{\sum_{n=0}^{m-1}g_{2r}^{ij}(m,-n)g_{i+k-2r}^{k,j+i-2r}(-m,m-n)\right\} \tag{8}$$

when $i+k$ is even, and zero otherwise.

In the case of the Virasoro algebra, the normal-ordering procedure described above would be sufficient to render the expression for $Q_{\rm gh}^2$ finite. However, here the corresponding expression (8) involves a sum over j, reflecting the fact that there are contributions coming from anticommutators of ghost operators for all the higher-spin generators in the algebra. This summation over j is naïvely divergent. However, as was discussed in [8,9], one can introduce a regularisation of the divergent sum, by viewing it as being formally equivalent to a set of summations whose analytic continuations define generalised zeta functions. A priori, one would expect that such a re-interpretation would be fraught with ambiguities. Remarkably, however, as was shown in [9], there is a well-defined and easily specifiable prescription under which all the coefficients $R^{ik}(m)$ given by (8) are regularised in a self-consistent way to give

$$R^{ik}(m)=\delta^{ik}c_{i,\rm gh}(m), \tag{9}$$

where $c_{i,\rm gh}$ is precisely the central term of (1), (2) corresponding to central charge $c_{\rm gh}=2$ for W_∞, and $c_{\rm gh}=0$ for $W_{1+\infty}$. Such a result must necessarily occur if the regularisation procedure is a sensible one[1]. This follows from the fact that the operators $V_m^i(\rm gh)$ defined by $V_m^i(\rm gh)=\{Q,b_m^i\}$ yield a ghostly realisation of the algebra. Since $Q_{\rm gh}$ can be written as $\frac{1}{2}c_m^i V_m^i(gh)$, it follows that (9) must be of the same form as (2), since the form of the central terms in W_∞ or $W_{1+\infty}$ algebras are uniquely determined, up to one overall scale factor, by the Jacobi identities [6]. Thus requiring $Q^2=0$ leads to the anomaly-freedom conditions for the central charge in the matter sector $c=-2$ and $c=0$, for W_∞ and $W_{1+\infty}$ respectively. The ground-state energy α_0 is determined to vanish in both cases.

As we shall now show, the above BRST-anomaly expressions admit an elegant reinterpretation, in which they can be viewed as Ricci curvatures of certain Kähler manifolds. These results generalise ones that were obtained previously for the Virasoro algebra.

[1] One might hope that the existence of a simple, consistent regularisation scheme indicates that there is some underlying deeper structure to the algebras, such as a higher-dimensional interpretation, from which standpoint the regularisation would be seen to be natural [9]

3. Curvature Calculations

In this section we compute the Ricci curvature for the Kähler manifold W_∞/H, where H is either the maximal torus T_∞ generated by the V^i_0, or the subalgebra $SL(\infty,R)$ generated by the V^i_m for $|m|\leqq i+1$. (The calculations for the manifolds $W_{1+\infty}/T_{1+\infty}$ and $W_{1+\infty}/GL(\infty,R)$ proceed quite similarly, so for those cases we shall only give the final results.) For convenience, we define the function $\sigma(i)$ by

$$\sigma(i)\equiv\begin{cases}0, & H=T_\infty;\\ i+1, & H=SL(\infty,R);\end{cases} \tag{10}$$

this will allow us to treat both choices of H simultaneously throughout the calculations.

The W_∞ algebra naturally decomposes into the direct sum of the three subalgebras

$$W_\infty=W_+\oplus W_-\oplus H, \tag{11}$$

where W_+ is generated by the V^i_m for $m>\sigma(i)$, W_- is generated by the V^i_m for $m<-\sigma(i)$, and H is generated by the V^i_m for $|m|\leqq\sigma(i)$. The coset manifold W_∞/H is Kähler, and one can think of the generators in W_+ as spanning the space of (1,0) forms, whilst the generators in W_- span the space of (0,1) forms. There is a natural Kähler metric on W_∞/H, given by

$$g(V^i_m,V^j_n)=c_i(m)\delta^{ij}\delta_{m+n,0}, \tag{12}$$

where $c_i(m)$ is as in (2).

Following [11,1], we introduce the Toeplitz operator

$$\varphi(X)\equiv\nabla_X-\mathscr{L}_X \tag{13}$$

for each generator $X\in W_\infty$, where $\mathscr{L}_X$ is the Lie derivative with respect to X. As in [11,1], one finds

$$\varphi(X)\equiv\begin{cases}\mathrm{ad}\,X, & X\in H;\\ \pi_+\circ\mathrm{ad}\,X, & X\in W_-;\\ -\varphi(\bar{X})^\dagger, & X\in W_+,\end{cases} \tag{14}$$

where π_+ is the projector onto W_+, and the operator $\mathrm{ad}\,X$ acting on any vector field Y is defined by $(\mathrm{ad}X)Y=[X,Y]$. The Riemann curvature operator is then given by

$$R(X,Y)\equiv[\varphi(X),\varphi(Y)]-\varphi([X,Y]). \tag{15}$$

In order to calculate specific components of the curvature tensor, we first apply (14) to determine the explicit action of the Toeplitz operator $\varphi(V^i_m)$ upon V^j_n, for any given W_∞ generators V^i_m and V^j_n. If V^i_m is a generator of H, (14) is simply the adjoint action of the algebra as determined by (1) (without the central terms):

$$\varphi(V^i_m)V^j_n=\sum_{r\geqq0}g^{ij}_{2r}(m,n)V^{i+j-2r}_{m+n}. \tag{16}$$

When V^i_{-m} is a generator of W_-, according to (14) one must project its adjoint action onto the subspace W_+. This may be written as

$$\varphi(V^i_{-m})V^j_n=\sum_{r\geqq0}\theta(n-m-\sigma(i+j-2r))g^{ij}_{2r}(-m,n)V^{i+j-2r}_{n-m}, \tag{17}$$

since $\theta(m-\sigma(i))$ is 1 for $V^i_m \in W_+$, and 0 otherwise. (We define $\theta(x)=1$ for $x>0$, $\theta(x)=0$ otherwise.)

In order to determine the action of $\varphi(V^i_m)$ when V^i_m is a generator of W_+, we require consistency with the Kähler metric (12). This is done by imposing the condition

$$\begin{aligned} 0 &= \varphi(V^i_m)g(V^k_p, V^j_n) \\ &= g((-\varphi(V^i_{-m})V^k_{-p})^\dagger, V^j_n) + g(V^k_p, \varphi(V^i_m)V^j_n) \end{aligned} \tag{18}$$

for arbitrary V^j_n and V^k_p. The first term on the last line may be evaluated using (17), while the second term is essentially the "projection" of $\varphi(V^i_m)V^j_n$ onto V^k_p. Since (18) must hold for arbitrary V^k_p, this allows us to solve completely for $\varphi(V^i_m)V^j_n$. The result is

$$\varphi(V^i_m)V^j_n = \sum_{r\geqq 0} g^{i,j-i+2r}_{2r}(-m,m+n)\frac{c_j(n)}{c_{j-i+2r}(m+n)}V^{j-i+2r}_{m+n}. \tag{19}$$

Note that unlike (16) and (17), which terminate due to the properties of the structure constants $g^{ij}_{2r}(m,n)$, this expression contains an indefinite number of terms for given V^i and V^j. However, for any particular values of the Fourier indices m and n, after a certain point (specifically, for $2r>m+n+i-j+1$) the $c_{j-i+2r}(m+n)$ in the denominator of (19) and the $g^{i,j-i+2r}_{2r}(-m,m+n)$ in the numerator both vanish (the former statement is obvious from the form of the $c_i(m)$; the latter is easy to demonstrate from the forms of the structure constants given in [6, 7].) This means that the coefficient of V^{j-i+2r}_{m+n} in $\varphi(V^i_m)V^j_n$ is not determined by (18). As we shall see (after Eq. (26)), the correct definition of $\varphi(V^i_m)V^j_n$ is to restrict the summation (19) to $2r<m+n+i-j+2$, for which the denominator of the summand is well-defined.

For studying the complex geometry of the manifold W_∞/H, one considers the Riemann curvature operator $R(X,Y)$ with $X=V^i_{-m}\in W_-$ and $Y=V^j_n\in W_+$ (that is, $m>\sigma(i)$ and $n>\sigma(j)$). The definition (15) becomes

$$\begin{aligned} R(V^i_{-m}, V^j_n) \equiv{} & \varphi(V^i_{-m})\varphi(V^j_n) - \varphi(V^j_n)\varphi(V^i_{-m}) \\ & - \sum_{r\geqq 0} g^{ij}_{2r}(-m,n)\varphi(V^{i+j-2r}_{n-m}), \end{aligned} \tag{20}$$

where specific components $R_{i\bar{m},jn,kp}{}^{lq}$ of the curvature are defined by the action of R upon an arbitrary generator V^k_p, according to

$$R(V^i_{-m}, V^j_n)V^k_p \equiv R_{i\bar{m},jn,kp}{}^{lq}V^l_{-q}. \tag{21}$$

We are interested in computing the Ricci curvature, obtained by tracing over the indices (jn) and (lq) of the Riemann tensor:

$$R_{i\bar{m},kp} \equiv R_{i\bar{m},jn,kp}{}^{jn}. \tag{22}$$

Given the explicit actions (16, 17, 19) of the Toeplitz operators, it is straightforward to write down a formal expression for the Ricci curvature. The first term in (20) gives, upon taking the trace (22),

$$\delta_{mp}\sum_{r,r',j,n} g^{j,k-j+2r}_{2r}(-n,m+n)\frac{c_k(m)}{c_{k-j+2r}(m+n)}\theta(n-\sigma(j))g^{i,k-j+2r}_{2r'}(-m,m+n), \tag{23}$$

where the sum runs over $r,r' \geqq 0$ with $2r-2r'=2j-i-k$. Here and in the sequel, the summation over n runs over $n>\sigma(j)$, so the factor $\theta(n-\sigma(j))$ in (23) is identically 1. The second term gives

$$-\delta_{mp}\sum_{r,r',j,n}\theta(-\sigma(i+k-2r))g^{ik}_{2r}(-m,m)g^{ij}_{2r'}(-n,n)\frac{c_{i+k-2r}(0)}{c_j(n)}, \tag{24}$$

where $2r'-2r=2j-i-k$; one immediately sees that (24) vanishes identically. For the third term, one must distinguish two cases: for $n-m\leqq\sigma(i+j-2r)$ one obtains

$$-\delta_{mp}\sum_{r,r',n,j}g^{ij}_{2r}(-m,n)g^{i+j-2r,k}_{2r'}(n-m,m) \tag{25}$$

with $2r+2r'=i+k$, while for $n-m>\sigma(i+j-2r)$ one has

$$-\delta_{mp}\sum_{r,r',n,j}g^{ij}_{2r}(-m,n)g^{i+j-2r,j}_{2r'}(m-n,n)\frac{c_k(m)}{c_j(n)} \tag{26}$$

with $2r+2r'=2j+i-k$.

By relabelling the summation variables, it is straightforward to see that the summand in (23) precisely cancels that appearing in (26). Furthermore, one can verify that the ranges of summation then coincide; in the case $\sigma(i)=i+1$ (that is, $H=SL(\infty,R)$), this fact relies crucially upon the proper definition of the Toeplitz action (19) with the proper range of summation. Thus, for both choices for H, the Ricci tensor is given by simply (25), where n is summed over $\sigma(j)<n\leqq m+\sigma(i+j-2r)$. In fact, the expressions given here all involve divergent summations. We expect that any sensible regularisation should respect the formal manipulations described here.

With the explicit ranges of all summations displayed, the result reads

$$R_{i\bar{m},kp}=\sum_{r=0}^{(i+k)/2}\;\sum_{j=\max(0,2r-i)}^{\infty}\left\{\sum_{j=1+\sigma(j)}^{m+\sigma(i+j-2r)}g^{ij}_{2r}(m,-n)g^{k,j+i-2r}_{i+k-2r}(-m,m-n)\right\}, \tag{27}$$

for $i+k$ even, and zero otherwise. When $H=T_\infty$ (in which case $\sigma(i)=0$), (27) is precisely

$$R_{i\bar{m},kp}=\delta_{mp}R^{ik}(m), \tag{28}$$

where $R^{ik}(m)$ is the result (9) of the BRST calculation.

For $H=SL(\infty,R)$ ($\sigma(i)=i+1$), the expression (27) is in fact the result of a modified BRST calculation in which the normal ordering of the ghost modes corresponds to the $SL(\infty,R)$ vacuum $|0\rangle$ defined by

$$\begin{aligned} c^i_m|0\rangle&=0, \quad m\geqq-\sigma(i),\\ b^i_m|0\rangle&=0, \quad m>\sigma(i), \end{aligned} \tag{29}$$

extending the $SL(2,R)$ vacuum discussed in [2,4]. Upon regularising the summations over j as described in [9], one in fact obtains the same result,

$$R_{i\bar{m},kp}=\delta_{mp}\delta^{ik}c_i(m) \tag{30}$$

with $c=-2$, for both choices of H; this fact corresponds to the observation in

[9] that the vacuum is automatically $SL(\infty,R)$-invariant without the need of shifting the ground-state energy.

For $W_{1+\infty}/T_{1+\infty}$ and $W_{1+\infty}/GL(\infty,R)$ the corresponding (regularised) Ricci curvatures vanish identically. These results are a reflection of the fact, first found in [9], that the regularised ghost central charge for $W_{1+\infty}$ is zero.

4. The Weil–Petersson Metric and Teichmüller Space

In [10] it was shown that the manifold $\mathrm{diff}(S^1)/SL(2,R)$ can be naturally embedded in the classical universal Teichmüller space and, furthermore, that the homogeneous Kähler metric on $\mathrm{diff}(S^1)/SL(2,R)$ induces the Weil–Petersson metric on Teichmüller space. We shall briefly review these results and then discuss how they may be generalised to the case of $W_\infty/SL(\infty,R)$.

The Kähler metric (12) for the special case $\mathrm{diff}(S^1)/SL(2,R)$ takes the form

$$g(X,Y)=\sum_{m=2}^{\infty} X_m Y_{-m}(m^3-m)+\text{c.c.}, \tag{31}$$

where X and Y are two vectors of the form $X=\sum_m X_m L_m$ and $Y=\sum_m Y_m L_m$. The complex conjugate of X_m is given by $\bar{X}_m=X_{-m}$. Under the Virasoro transformation generated by L_n, we have $\delta_n X=[L_n,X]$, and thus

$$\delta_{(n)}X_m=(2n-m)X_{m-n}. \tag{32}$$

One can easily check that the Kähler metric (31) is invariant under the $SL(2,R)$ subalgebra generated by L_{-1}, L_0 and L_1.

On the other hand the Weil–Petersson metric on the Teichmüller space $T(G)$ of a Riemann surface is given by

$$\text{W–P}(\mu,\nu)=\int_{\Delta/G} d^2z\int_{\Delta} d^2\zeta\,\frac{\mu(z)\bar{\nu}(\zeta)}{(1-z\bar{\zeta})^4}, \tag{33}$$

where Δ is a unit disk; G is the Fuchsian group that characterises the Riemann surface; and $\mu(z)$ and $\nu(\zeta)$ are Beltrami differentials. The universal Teichmüller space $T(1)$ corresponds to the case when G is taken to be identity group. One can understand the Weil–Petersson metric as the pairing of the holomorphic quadratic differential $\varphi[\nu](z)$ and the Beltrami differential $\mu(z)$,

$$\langle\mu,\varphi\rangle\equiv\int_{\Delta/G} d^2z\,\mu(z)\varphi[\nu](z), \tag{34}$$

where the $\varphi[\nu](z)$ is given by

$$\varphi[\nu](z)\equiv\int_{\Delta} d^2\zeta\,\frac{\bar{\nu}(\zeta)}{(1-z\bar{\zeta})^4}. \tag{35}$$

The Beltrami differential parametrises the general class of metrics $\sim|dz+\mu d\bar{z}|^2$ and satisfies the Beltrami equation $w_{\bar{z}}=\mu w_z$. Thus one can think of μ as a two-index tensor of the form $\mu_{\bar{z}}^{z}$. Since $(1-z\bar{\zeta})^{-2}$ is a metric on the upper half plane, it follows from (35) that $\varphi(z)$ is a tensor of the form φ_{zz}. One can easily check that the Weil–Petersson metric (33) is invariant under the fractional linear group (i.e. under $SL(2,R)$).

The connection between the two metrics (31) and (33) can be established by defining

$$X_m = \frac{i}{\pi} \int_\Delta d^2 z \bar{\mu}(z) \bar{z}^{m-2}. \tag{36}$$

In [10], it was shown that the X_m parametrise vector fields on the unit circle (the boundary of Δ) describing the diffeomorphism induced by the Beltrami parameter $\mu(z)$. It follows that (31) can be interpreted as the Weil–Petersson metric (33) on the universal Teichmüller space $T(1)$.

Much of the above discussion can be extended to the higher-spin case. Defining components X^i_m for a vector $X = \sum_{i,m} X^i_m V^i_m$, the Kähler metric (12) on $W_\infty/SL(\infty,R)$ can be written as

$$g(X, Y) = \sum_{i \geqq 0} \sum_{m \geqq i+2} X^i_m Y^i_{-m} c_i(m) + \text{c.c.}, \tag{37}$$

where $c_i(m)$ are the central-charge terms appearing in (2). Under V^j_n transformations, we have $\delta_{(jn)} X = [V^j_n, X]$. This co-adjoint transformation on X^i_m is somewhat complicated, and we shall not give the result explicitly here; it may be found in [12]. This metric is invariant under the $SL(\infty,R)$ subalgebra of W_∞. Under the $SL(2,R)$ subalgebra generated by V^0_{-1}, V^0_0 and V^0_1, the transformations of X^i_m take the simple form

$$\delta_{(0n)} X^i_m = [(i+2)n - m] X^i_{m-n}, \tag{38}$$

where $n = -1$, 0 or 1.

The metric (37) may be cast into the Weil–Petersson form by generalising (36) and writing

$$X^j_m = \frac{i}{\pi} \int_\Delta d^2 z \bar{\mu}^j(z) \bar{z}^{m-j-2}. \tag{39}$$

Here, the quantities $\mu^j(z)$ are higher-spin generalisations of the Beltrami differentials. From the transformation rules for the X^j_m, it follows that these generalised Beltrami differentials transform just like the spin $(j+2)$ gauge fields of W_∞ gravity, which were introduced in [12]. In fact, the Beltrami differential μ^j_m has the tensorial structure $\mu_{\bar{z}}^{z \cdots z}$, with $(j+1)$ contravariant holomorphic indices, and one can think of it as parametrising deformations of a $(j+2)$-index traceless symmetric tensor $\delta A_{\bar{z} \cdots \bar{z}} = g_{z\bar{z}} \cdots g_{z\bar{z}} \mu_{\bar{z}}^{z \cdots z}$. Substituting (39) into the metric (37), and using the identity that follows from differentiating $\sum_{m \geqq 0} x^m = (1-x)^{-1}$ repeatedly, one obtains the corresponding analogue of the Weil–Petersson metric,

$$\text{W–P}(\mu^i, \nu^j) = \delta^{ij} \int_{\Delta/G} d^2 z \int_\Delta d^2 \zeta \frac{\mu^i(z) \bar{\nu}(\zeta)}{(1 - z\bar{\zeta})^{2i+4}}. \tag{40}$$

One can think of this as giving a measure on the space of deformations of higher-spin gauge fields.

5. Discussion

In this paper we have shown how some of the geometrical ideas introduced in [1] may be generalised to the case of the W_∞ and $W_{1+\infty}$ algebras. In particular, we have shown how the Ricci curvatures of the W manifolds, obtained as cosets of W_∞ or $W_{1+\infty}$ factored by certain subalgebras, are related to the regularised anomalies of the corresponding ghostly BRST algebras.

We have also shown how one may relate the Kähler metrics on $W_\infty/SL(\infty,R)$ and $W_{1+\infty}/GL(\infty,R)$ to generalisations of the Weil–Petersson metric on the modulus space of higher-spin gauge fields for W_∞ or $W_{1+\infty}$ gravity. What is lacking in our understanding so far is some geometrical interpretation of the higher-spin symmetries. For example, one can interpret the Virasoro symmetry as diffeomorphisms on the S^1 boundary of the disk Δ discussed in Sect. 4. We do not yet have an equivalent interpretation of the higher-spin generators of the W_∞ and $W_{1+\infty}$ algebras. On the other hand, we know that the contraction of $W_{1+\infty}$ to $w_{1+\infty}$ can be viewed as the algebra of area-preserving diffeomorphisms of an infinite cylinder, which suggests that, at least in this limit, a higher-dimensional interpretation would be appropriate[2]. Presumably the uncontracted $W_{1+\infty}$ algebra could also be expected to admit a higher-dimensional interpretation, and one may hope that this may provide the natural arena for understanding the geometric issues raised in this paper.

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[2] In some related recent work, curvature calculations for the algebra of area-preserving transformations of a finite-length cylinder have been carried out [13]