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A Discretization of p-Adic Quantum Mechanics

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Abstract. We show that some compact subgroups ($\mathscr{H}_{n,m}$) of the p-adic Heisenberg group act irreducibly on corresponding finite dimensional spaces of test-functions ($S_{m,n}$). Under certain conditions, a compact group ($\mathscr{A}_{m+n}$) of linear canonical transformations, isomorphic to $SL(2, \mathbf{Z}_p)$, can be represented unitarily on $S_{m,n}$ as a group of automorphisms of $\mathscr{H}_{n,m}$. The restriction to $S_{m,n}$ can be considered as a discretization because an invariant subgroup ($\mathscr{I}_{m,n}$) of A_{m+n} is represented trivially. It is possible to take a limit where $\mathscr{I}_{m,n}$ becomes an arbitrarily small neighborhood of the identity, while the dimension of $S_{m,n}$ becomes arbitrarily large. This is a possible definition of the "continuum limit" that we relate to other projective limits appearing naturally in the present context.

1. Introduction

In elementary quantum mechanics, the state of a system is represented by a *ray* in a complex vector space [14]. In other words, states differing by a phase (unimodular complex number) are not physically distinguishable. In ray space, the representations of the transformation groups of Hamiltonian mechanics are only required to be representations up to a phase, i.e. projective representations. For instance, translations in positions and momenta commute and form an abelian group; however, the interchange of their representatives in ray space may produce a phase. A well-known example of such a realization is the Heisenberg group.

Let us first consider the usual situation: the positions and momenta are real numbers and the algebra of infinitesimal transformations of the Heisenberg group are the usual commutation relations of quantum mechanics. We now summarize some of the results obtained by C. Itzykson [7]. One can construct a unitary realization of the group of linear canonical transformations as a group of automorphisms of the Heisenberg group. With an appropriate choice of phase, this unitary realization turns out to be a representation up to a sign. In the

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following, the linear canonical transformations over an A-valued phase space will be written LCT over A for brevity.

There are situations where the usual commutation relations of quantum mechanics cannot be implemented. For instance, if the dimension of the vector space is finite, we have $\mathrm{Tr}[A,B]=0$ and a commutator cannot be proportional to the identity. However, the unitary operators corresponding to the LCT over a finite ring can be constructed easily [5, 2]. In order to find the similarities among various constructions of this type, it is essential to identify the role played by the additive characters [3]. The definition of the Heisenberg group and the realization of the LCT as a group of automorphisms of the Heisenberg group given in Sect. 2 apply to arbitrary fields. The recent results [3, 8, 1, 13, 11, 9] concerning the LCT over the p-adic numbers $\mathbf{Q}_p$ and their quantization can be understood within this framework.

In the case where the phase space consists in a pair of p-adic conjugated variables, the group of LCT is $SL(2,\mathbf{Q}_p)$. In refs. 11 and 9, the additive parametrization ("time") of some compact abelian groups of $SL(2,\mathbf{Q}_p)$ and the eigenvalues of the corresponding unitary operators has been calculated using standard methods of harmonic analysis. These are new examples of non-trivial problems of quantum mechanics which are exactly solvable. In the process of these calculations, we noticed the existence of nested, finite dimensional spaces of test-functions (see ref. 4 for definitions), invariant under the unitary operators corresponding to these compact abelian subgroups [9]. Note that this approximation does not affect the eigenvalues or the eigenfunctions of these operators but only restricts the vector space to a finite number of eigenfunctions. The restriction to these spaces was then used to regularize some ill-defined traces and more recently to formulate a path integral representation of those unitary operators [10]. Similar patterns appear in the discretization of p-adic strings [15]. The aim of this paper is to give a systematic presentation, including elementary proofs, of the discretization of p-adic quantum mechanics used in refs. 9 and 10.

In Sect. 2, we present the p-adic Heisenberg group and its automorphisms corresponding to the LCT. Although this section is written in the case of $\mathbf{Q}_p$, most of the results apply to the general case. In Sect. 3, we define some compact subgroups ($\mathscr{H}_{n,m}$) of the p-adic Heisenberg group and we show that they act irreducibly on a corresponding finite dimensional space of test-functions ($S_{m,n}$). A normal abelian subgroup of $\mathscr{H}_{n,m}$ is realized trivially on $S_{m,n}$ and we are left with an irreducible representation of a *finite* Heisenberg group. As briefly discussed in ref. 10 this group appears in several problems with toroidal boundary conditions [6].

The notion of projective limit appears naturally in the definition of the p-adic integers $\mathbf{Z}_p$. Let ϕ_n be the obvious homomorphism of $\mathbf{Z}/p^n\mathbf{Z}$ onto $\mathbf{Z}/p^{n-1}\mathbf{Z}$. A p-adic integer can be defined [12] as a sequence x_n of elements of $\mathbf{Z}/p^n\mathbf{Z}$ such that $\phi_n(x_n)=x_{n-1}$ for $n\geqq 2$. The ring of p-adic integers $\mathbf{Z}_p$ is then the *projective limit* of the system $(\mathbf{Z}/p^n\mathbf{Z},\phi_n)$. In Sect. 4, we give a definition of $SL(2,\mathbf{Z}_p)$ as a projective limit. We show in Sect. 5 that under certain conditions, a compact group ($\mathscr{A}_{m+n}$) of linear canonical transformations, isomorphic to $SL(2,\mathbf{Z}_p)$, can be represented unitarily on $S_{m,n}$ as a group of automorphisms of $\mathscr{H}_{n,m}$. In addition, an invariant subgroup $\mathscr{I}_{m,n}$ of $\mathscr{A}_{m+n}$ is realized trivially on $S_{m,n}$. This leaves us with a projective representation of the finite group $\mathscr{A}_{m+n}/\mathscr{I}_{m,n}$ and the restriction to $S_{m,n}$ can be considered as a discretization. The continuum limit is obtained when the invariant subgroup $\mathscr{I}_{m,n}$ shrinks to an arbitrarily small neighborhood of the identity while the dimension of $S_{m,n}$ becomes arbitrarily large. The proofs given here only use elementary algebra, the non-Archimedean inequality and Schur's lemma. With a few exceptions they do not make use of a particular basis. A few considerations concerning the algebra generating the eigenfunctions of the unitary realization of some orthogonal groups are given in the conclusions.

2. Presentation of the p-Adic Heisenberg Group and some of Its Automorphisms

Let f be a continuous function from $\mathbf{Q}_p$ to $\mathbf{C}$, $\chi_p(x)$ an additive character of $\mathbf{Q}_p$ normalized as $e^{i2\pi x}$ and x, k, z, h some elements of $\mathbf{Q}_p$. We define the action of $H(x,k,z)$ on f as

$$[H(x,k,z)f](y) \equiv \chi_p\left(\frac{z+(x+y)k}{h}\right)f(x+y). \tag{2.1}$$

In the following h will be considered as a constant which parametrizes the additive character and plays a role analogous to the Planck constant in quantum mechanics. The parametrization of z as an element of $\mathbf{Q}_p$ is redundant since with our normalization, $\chi_p(x)=1$ if $|x|_p \leqq 1$. Consequently, $H(x,k,z)=H(x,k,z')$ if $|z-z'|_p \leqq |h|_p \equiv p^{-\operatorname{ord} h}$. We can obtain a 1–1 parametrization if z is chosen among representatives of $\mathbf{Q}_p/p^{\operatorname{ord} h}\mathbf{Z}_p$. Except for this restriction, the results of this section apply for an arbitrary field.

The composition of two transformations (2.1) reads

$$H(x,k,z)H(x',k',z') = H(x+x', k+k', z+z'-kx'). \tag{2.2}$$

We see that $H(0,0,0)$ is a representative of the identity and that $H(-x,-k,-z-kx)$ is a representative of the inverse of $H(x,k,z)$ and one easily checks that

$$\mathscr{H} \equiv \{H(x,k,z); x,k \in \mathbf{Q}_p \text{ and } z \in \mathbf{Q}_p/p^{\operatorname{ord} h}\mathbf{Z}_p\} \tag{23}$$

is a group for the product (2.2).

The center of $\mathscr{H}$ consists in the elements having the form $H(0,0,z)$. The interchange of two elements of $\mathscr{H}$ amounts to the multiplication by a central element, namely

$$H(x,k,z)H(x',k',z') = H(x',k',z')H(x,k,z)H(0,0,k'x-kx'). \tag{2.4}$$

This shows that (2.1) defines a projective representation of the abelian group $(\mathbf{Q}_p,+,0)^2$ or alternatively a linear representation of a central extension of this abelian group. However, since we only consider here a particular family of representations indexed by h, we shall call $\mathscr{H}$ the p-adic Heisenberg group, bearing in mind the h dependence. Note that if the group parameters x, k, z were real numbers, the infinitesimal version of (2.4) would be the commutation relations of quantum mechanics with h being the Planck constant.

We are now in position to introduce some automorphisms of $\mathscr{H}$ corresponding to the linear canonical transformations of classical mechanics (with a p-adic phase space). Let $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Q}_p)$ and $[\ \ ]_g$ be the mapping of $\mathscr{H}$ into itself defined by:

$$[H(x,k,z)]_g \equiv H\left(dx+bk, ak+cx, z-\frac{abk^2}{2}-\frac{cdx^2}{2}-bcxk\right). \tag{2.5}$$

Lemma 1. *$\forall H, H' \in \mathscr{H}$ and $g, g' \in SL(2,\mathbf{Q}_p)$,*

(a) $[HH']_g=[H]_g[H']_g$,
(b) $[[H]_g]_{g'}=[H]_{gg'}$,
(c) $[H]_g=H$ iff $g=\mathbf{1}$, (*$\mathbf{1}$ is the identity of $SL(2,\mathbf{Q}_p)$*).

The proofs are straightforward applications of (2.2), (2.5) and the multiplication of 2 by 2 matrices. □

Proposition 1. *$\{[\ \ ]_g, g \in SL(2,\mathbf{Q}_p)\}$ is a group of automorphisms of $\mathscr{H}$.*

Lemma 1 a) asserts that $[\ \ ]_g$ is an homomorphism of $\mathscr{H}$. But under this homomorphism, $H \in \mathscr{H}$ is the image of an unique element ($[H]_{g^{-1}}$) of $\mathscr{H}$ as a consequence of Lemma 1 b) and c). This shows that $[\ \ ]_g$ is an automorphism of $\mathscr{H}$ and the above set of transformations inherits the group properties of $SL(2,\mathbf{Q}_p)$ (Lemma 1 b)). □

3. Some Compact Subgroups of the p-Adic Heisenberg Group and Their Invariant Subspaces of Test Functions

In this section, f will be a test function [4], i.e. a locally constant function (from $\mathbf{Q}_p$ to $\mathbf{C}$) with compact support. As we now proceed to explain, it is important to realize that the transformations (2.1) are exactly what we need to "probe" the local constancy and the support of f. We can classify the test functions according to their invariance under some compact subsets of $\mathscr{H}$ defined as

$$\mathscr{H}_{m,n} \equiv \{H(x,k,z); |x|_p \leqq p^m, |k|_p \leqq |h|_p p^{-n}, |z|_p \leqq p^{m-n}|h|_p\}. \tag{3.1}$$

From (2.2) and the ultrametric inequality, we see that $\mathscr{H}_{m,n}$ is a subgroup of $\mathscr{H}$. These subgroups are nested according to the inclusion relations

$$\mathscr{H}_{m,n+1} \subset \mathscr{H}_{m,n} \subset \mathscr{H}_{m+1,n}. \tag{3.2}$$

From the discussion in Sect. 2, it is clear that z represents a class of equivalence of $Z_{n,m} \equiv p^{m-n+\operatorname{ord} h}\mathbf{Z}_p/p^{\operatorname{ord} h}\mathbf{Z}_p$. Note that if $m \leqq n$, $\mathscr{H}_{m,n}$ is abelian and (2.4) together with the ultrametric inequality implies that $\mathscr{H}_{m,n}$ is included in the center of $\mathscr{H}_{n,m}$ and therefore an invariant subgroup of $\mathscr{H}_{n,m}$. We can determine the quotient $\mathscr{H}_{n,m}/\mathscr{H}_{m,n}$. The transformations of $\mathscr{H}_{m,n}$ of the form $H(\delta x,0,0)$ can be used to reduce the parameter x (as in Eq. (2.1)) of a transformation of $\mathscr{H}_{n,m}$ to some representative of $X_{n,m} \equiv p^{-n}\mathbf{Z}_p/p^{-m}\mathbf{Z}_p$, i.e the classes of equivalence $\bar{x}$ with representatives x such that $|x|_p \leqq p^n$ and $\bar{x}=\bar{y}$ if $|x-y| \leqq p^m$. Similarly, we can reduce the parameter k to some representative of $K_{n,m} \equiv p^{m+\operatorname{ord} h}\mathbf{Z}_p/p^{n+\operatorname{ord} h}\mathbf{Z}_p$. Each of the three quotients of additive groups has order p^{n-m}. This makes clear the next proposition.

Proposition 2. *If $n \geqq m$, the quotient $\mathscr{H}_{n,m}/\mathscr{H}_{m,n}$ is a group with $p^{3(n-m)}$ elements (finite Heisenberg group).*

We now introduce the finite dimensional spaces of test functions:

$$S_{m,n} \equiv \{f : f(x) = 0 \text{ if } |x|_p > p^n \text{ and } f(x+\delta x) = f(x) \text{ if } |\delta x|_p \leqq p^{+m}\} \tag{3.3}$$

with $m \leqq n$ assumed (otherwise we only have the identically zero function). We have the obvious inclusion relations

$$S_{m,n-1} \subset S_{m,n} \subset S_{m-1,n}. \tag{3.4}$$

Note also that the mapping $f(x) \to f(px)$ is a bijection between $S_{m,n}$ and $S_{m-1,n-1}$.

It is sometimes convenient to use a particular basis of $S_{m,n}$, for instance the set of characteristic functions

$$f_{\bar{x}_0}(x) = \begin{cases} p^{-m/2} & \text{if } |x - x_0|_p \leqq p^m \\ 0 & \text{otherwise} \end{cases}, \tag{3.5}$$

where $\bar{x}_0 \in X_{n,m}$. The dimension of $S_{m,n}$ is thus p^{n-m}. In order to make contact with quantum mechanics, it is necessary to introduce the notion of unitarity. For this reason, we define the inner product of two functions of $S_{m,n}$ in the following way:

$$\langle f_1 | f_2 \rangle = \int_{\mathbf{Q}_p} dx\, f_1^*(x) f_2(x) = p^m \sum_{\bar{x} \in X_{n,m}} f_1^*(x) f_2(x). \tag{3.6}$$

It is clear that (3.5) is an orthonormal basis with respect to this inner product. In the following, the notions of adjointness or unitarity will always be understood with respect to (3.6).

Theorem 1. *If $m \leqq n$:*

(a) $f \in S_{m,n}$ iff $\forall H \in \mathscr{H}_{m,n}$, $Hf = f$,
(b) $\mathscr{H}_{n,m}$ *acts irreducibly on* $S_{m,n}$.

a) $\Rightarrow$ follows from (2.1) with the restrictions (3.1) and (3.2). a) $\Leftarrow$ is obtained by considering "maximal" elements: $H(x,0,0)$ with $|x|_p = p^m$ which control the local constancy and $H(0,k,0)$ with $|k|_p = p^{-n-\operatorname{ord} h}$ which control the support. The entire basis of Eq. (3.5) can be generated by acting iteratively with $H(p^{-n},0,0)$ on any $f_{\bar{x}_0}$ which proves b). □

Corollary 1. *The group of transformation $\mathscr{H}_{n,m}(m \leqq n)$ acting on $S_{m,n}$ provides an irreducible representation of the finite Heisenberg group $\mathscr{H}_{n,m}/\mathscr{H}_{m,n}$.*

It is shown in the appendix that it possible to combine linearly the elements of this finite group in order to obtain a basis of linear operators of $S_{m,n}$.

4. $SL(2, \mathbf{Z}_p)$ as a Projective Limit: A Discretization of the Classical Evolution

A starting point for the path integral formulation of ordinary quantum mechanics is the discretization of time. In the case of p-adic quantum mechanics, it has been pointed out recently [10] that the restriction to some nested $S_{m,n}$ induces a discretization of some compact group of evolution. Before considering the quantum evolution, we will present a discretization of the classical evolution, namely a definition of the compact subgroup of the LCT $SL(2,\mathbf{Z}_p)$ as a projective limit. As explained in the introduction, the ring of p-adic integers $\mathbf{Z}_p$ can be defined as the projective limit of the system $(\mathbf{Z}/p^n\mathbf{Z}, \phi_n)$, where ϕ_n is the obvious homomorphism of $\mathbf{Z}/p^n\mathbf{Z}$ into $\mathbf{Z}/p^{n-1}\mathbf{Z}$. We can also introduce a function ε_n from $\mathbf{Z}_p$ into $\mathbf{Z}/p^n\mathbf{Z}$ such that $\varepsilon_n(x)=x_n$ with the above notations. Since addition and multiplication of p-adic integers are defined by these operations within each of the $\mathbf{Z}/p^n\mathbf{Z}$, ε_n is a ring homomorphism with kernel $p^n\mathbf{Z}_p$.

Given an element g of $SL(2,\mathbf{Z}_p)$ the action of ε_n and ϕ_n on the four matrix elements defines the functions $\bar{\varepsilon}_n(g)=g_n$ and $\bar{\phi}_n(g_n)=g_{n-1}$. Note that $\bar{\varepsilon}_n$ is a group homomorphism with kernel

$$K_n \equiv \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}_p) : |a-1|_p \leqq p^{-n},\ |d-1|_p \leqq p^{-n},\ |b|_p \leqq p^{-n},\ |c|_p \leqq p^{-n}\right\} \tag{4.1}$$

and whose image is included in $SL(2,\mathbf{Z}/p^n\mathbf{Z})$. As in the case of the modular group, any element of $SL(2,\mathbf{Z}/p^n\mathbf{Z})$ is the image by $\bar{\varepsilon}_n$ of some elements of $SL(2,\mathbf{Z}_p)$. To each solution of $ad-bc=1 \bmod p^n$, corresponds an element of $SL(2,\mathbf{Z}/p^n\mathbf{Z})$. Clearly, out of the pairs (a,d) and (b,c) there is at least one whose elements are not zero mod p. Considering one of the elements of this pair as a variable, we can lift the solution mod p^n to a solution in $\mathbf{Z}_p$ using Hansel's lemma [12]. Consequently,

$$SL(2,\mathbf{Z}_p)/K_n \approx SL(2,\mathbf{Z}/p^n\mathbf{Z}) \tag{4.2}$$

and we can define $SL(2,\mathbf{Z}_p)$ as the projective limit of $(SL(2,\mathbf{Z}/p^n\mathbf{Z}), \bar{\phi}_n)$. In the next section, we will construct sequences of projective representations corresponding to the odd and the even subsequences of this projective limit.

5. Some Automorphisms of $\mathscr{H}_{m,n}$

In this section, we will study the subgroup of transformations (2.5) which map $\mathscr{H}_{m,n}$ into itself. We first define

$$\mathscr{A}_k = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Q}_p) : |a|_p \leqq 1,\ |d|_p \leqq 1,\ |b|_p \leqq p^k |h|_p^{-1},\ |c|_p \leqq p^{-k}|h|_p\right\} \tag{5.1}$$

Proposition 3. *$\mathscr{A}_{n+m}$ is a subgroup of $SL(2,\mathbf{Q}_p)$ isomorphic to $SL(2,\mathbf{Z}_p)$.*

Let $\lambda \in \mathbf{Q}_p[\sqrt{p}]$ such that $\lambda^2 \in \mathbf{Q}_p$ and $|\lambda^2|_p = p^{n+m}|h|_p^{-1}$. The conjugation

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} a & \lambda^2 b \\ \lambda^{-2} c & d \end{pmatrix} \tag{5.2}$$

is an isomorphism between $SL(2,\mathbf{Z}_p)$ and $\mathscr{A}_{m+n}$. $\square$

Proposition 4. *If $p^{m-n} \leqq |2|_p$, $\{[\ ]_g, g \in \mathscr{A}_{m+n}\}$ is a group of automorphism of $\mathscr{H}_{m,n}$.*

From Lemma 1 and Proposition 1, we only need to prove that if $g \in \mathscr{A}_{m+n}$, $[\ ]_g$ maps $\mathscr{H}_{m,n}$ into itself. This follows easily from (2.5), (3.1) and (4.1) and the non-Achimedean inequality. The special restriction for $p=2$ comes from the factor 1/2 in (2.5). □

Proposition 5. *If $p^{m-n} \leqq |2|_p$, $f \in S_{m,n}$ and $g \in \mathscr{A}_{m+n}$, then there exists a unitary transformation U_g such that for any $H \in \mathscr{H}_{n,m}$: $U_g f \in S_{m,n}$ and $[H]_g f = U_g^\dagger H U_g f$.*

Using the basis (3.5), we obtain U_g, up to an overall constant, such that $U_g[H]_g = HU_g$ for any $H \in \mathscr{H}_{n,m}$. The modulus of the constant can be fixed in such a way that unitarity is satisfied. This explicit calculation is reported in the appendix and provides us with alternative proofs of the following results. □

We are now ready to show that the restriction to $S_{m,n}$ induces a discretization of $\mathscr{A}_{m+n}$ in the sense that a neighborhood of the identity of $\mathscr{A}_{m+n}$ is realized trivially with U_g. We define

$$\mathscr{I}_{m,n} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{A}_{n+m} : |a-1|_p \leqq p^{m-n}, |d-1|_p \leqq p^{m-n}, |b|_p \leqq p^{2m}|h|_p^{-1}|2|_p, |c| \leqq p^{-2n}|h|_p|2|_p \right\}. \tag{5.3}$$

Proposition 6. *$\mathscr{I}_{m,n}$ is an invariant subgroup of $\mathscr{A}_{m+n}$. $\mathscr{A}_{m+n}/\mathscr{I}_{m,n}$ is isomorphic to $SL(2, \mathbf{Z}_p/p^{n-m}\mathbf{Z}_p)$ if $p \neq 2$ and to the subgroup of elements of $SL(2, \mathbf{Z}_2/2^{n-m}\mathbf{Z}_2)$ which satisfy $|b|_2 \leqq 2^{n-m-1}$ and $|c|_2 \leqq 2^{n-m-1}$ if $p=2$.*

This follows from the fact (see (4.1)) that K_{n-m} is an invariant subgroup of $SL(2, \mathbf{Z}_p)$ and that the isomorphism (5.2) induces an isomorphism between $\mathscr{I}_{m,n}$ and K_{n-m} if $p \neq 2$, and between $\mathscr{I}_{m,n}$ and the above invariant subgroup of $SL(2, \mathbf{Z}_2)$ intermediate between K_{n-m} and K_{n-m+1}, if $p=2$. □

Theorem 2. *If $p^{n-m} \leqq |2|_p$, $f \in S_{m,n}$ and $g, g' \in \mathscr{A}_{m+n}$, then, $U_g U_{g'} f = c(g, g') U_{gg'} f$ and $|c(g, g')|_{\mathbf{C}} = 1$.*

If $H \in \mathscr{H}_{n,m}$ and $g_1, g_2 \in \mathscr{A}_{m+n}$, then from Lemma 1 b), $U_{g_2}^\dagger U_{g_1}^\dagger H U_{g_1} U_{g_2} = U_{g_1 g_2}^\dagger H U_{g_1 g_2}$ (understood as an operator identity over $S_{m,n}$). Consequently, $U \equiv U_{g_1} U_{g_2} U_{g_1 g_2}^\dagger$ commutes with any such H. From the ireducibility of $\mathscr{H}_{n,m}$ (Theorem 1 b)) and Schur's lemma, we conclude that U is proportional to the identity. Since U is unitary (Proposition 5), c is unimodular. □

In ref. 9, we have shown that with appropriate choices of phases, the unitary realization turned out to be a representation up to a sign, generalizing the result for the reals [7]. In ref. 11, other phase choices were considered. However, the results presented here are independent of a particular choice.

Lemma 2. *If $H \in \mathscr{H}_{n,m}$ and $g \in \mathscr{I}_{m,n}$, then $H^{-1}[H]_g \in \mathscr{H}_{m,n}$.*

This is a straightforward application of (2.2), (2.5) and the non-Archimedean inequality. □

Theorem 3. *If $p^{m-n} \leqq |2|_p$, $g \in \mathscr{I}_{m,n}$ and $f \in S_{m,n}$, then $U_g f = \lambda(g) f$ and $|\lambda(g)|_{\mathbf{C}} = 1$.*

From Lemma 2 and Proposition 5, we find that if $H \in \mathscr{H}_{n,m}$ and $g \in \mathscr{I}_{m,n}$ then $(U_g H - HU_g) f = 0$ for any $f \in S_{m,n}$. Using Schur's lemma as in Theorem 2, we obtain the desired result. □

Corollary 2. *U_g acting on $S_{m,n}$ is a projective representation of $\mathscr{A}_{m+n}/\mathscr{I}_{m,n}$.*

The quotient group is clearly a finite group. A simple calculation shows that $\mathrm{Ord}\,(K_n/K_{n+1}) = p^3$ and $\mathrm{Ord}\,(\mathscr{A}_{m+n}/\mathscr{I}_{m,n})$ is $p^{3(n-m)}(1-p^{-2})$ if $p \neq 2$ and $3.2^{3(n-m)+2}$ if $p=2$. The restriction to $S_{m,n}$ is thus a discretization of $\mathscr{A}_{m+n}$.

We are now ready to define the continuum limit. If we increase $n-m$ with $n+m$ kept fixed, $\mathscr{A}_{m+n}$ is unchanged while $\mathscr{I}_{m,n}$ shrinks to smaller neighborhood of the identity. This defines a projective limit of $\mathscr{A}_{m+n}$ similar to the definition of $SL(2,\mathbf{Z}_p)$ given in Sect. 4, except that since $m+n$ is kept fixed, $m-n$ tends to infinity through odd or even values. Since $SL(2,\mathbf{Z}_p)$ is a continuous group for the natural (p-adic) topology, we call the corresponding sequence of representations a continuum limit.

6. Conclusions

We have shown that the restriction to some appropriate finite dimensional spaces of test-functions provides a consistent discretization of the quantum evolution corresponding to groups of LCT isomorphic to $SL(2,\mathbf{Z}_p)$. We proposed a continuum limit obtained as a sequence of unitary realizations related to the sequences of finite groups appearing in the definition of $SL(2,\mathbf{Z}_p)$ as a projective limit. The applications of this result can be found in refs. 9 and 10 and possibly in the context of refs. 6. In ordinary quantum mechanics, we can construct all the eigenfunctions of an harmonic oscillator by acting with the creation operator on the vacuum. This operator and its hermitian conjugate close in a very simple algebra (Heisenberg algebra). In p-adic quantum mechanics, we can also construct "creation operators" which map some $S_{m,n}$ into some larger $S_{m',n'}$. However, the study of the spectra [11,9] and the non-Archimedean inequality show that if an operator A is an eigenstate of $U_g^\dagger A U_g$ for g in some compact orthogonal group, then there is an upper limit on the spaces $S_{m',n'}$ which can be obtained by acting with A on the smallest possible of these spaces. The idea of getting the full spectrum with a single creator is clearly Archimedean! In the non-Archimedean case, the building blocks are finite dimensional spaces and the algebra generating the eigenstates is a finite dimensional representation of Lie algebra.

Appendix A

This appendix is devoted to the explicit results obtained in the basis (3.5). We first introduce the notations $|\bar{x}\rangle$ for the ket corresponding to $f_{\bar{x}}$ in (3.5). Obviously,

$$\langle \bar{x} | \bar{y} \rangle = \delta_{\bar{x},\bar{y}}. \tag{A1}$$

Through this appendix, we assume $n \geqq m$ and we use the notations $X_{n,m} \equiv p^{-n}\mathbf{Z}_p/p^{-m}\mathbf{Z}_p$ and $K_{n,m} \equiv p^{m+\mathrm{ord}\,h}\mathbf{Z}_p | p^{n+\mathrm{ord}\,h}\mathbf{Z}_p$ as in the text. An unbarred variable x should be understood as a representative of the equivalence class $\bar{x}$.

The Fourier transform of a function $f \in S_{m,n}$ reads

$$\tilde{f}(k) = \begin{cases} p^m \sum\limits_{\bar{x} \in X_{m,n}} f(x)\chi_p\left(\dfrac{kx}{h}\right); & \text{if } |k|_p \leqq p^{-(m+\operatorname{ord} h)} \\ 0; & \text{otherwise} \end{cases} \tag{A2}$$

We see that $\tilde{f}(k)$ is a function defined over $K_{n,m}$. The delta function can also be expressed as

$$\delta_{\bar{x},\bar{y}} = p^{n-m} \sum_{\bar{k} \in K_{n,m}} \chi_p\left(\frac{k(x-y)}{h}\right). \tag{A3}$$

The matrix elements of $H \in \mathscr{H}_{n,m}$ are given by

$$\langle \bar{y}_1 | H | \bar{y}_2 \rangle = \chi_p\left(\frac{z + y_2 k}{h}\right) \delta_{\bar{y}_2, \overline{y_1 + x}}. \tag{A4}$$

The operators

$$C(\bar{w}_1, \bar{w}_2) \equiv p^{n-m} \sum_{\bar{k} \in K_{n,m}} H(w_2 - w_1, k, -kw_2) \tag{A5}$$

form a basis of linear operators over $S_{m,n}$ since a short calculation shows that

$$\langle \bar{y}_1 | C(\bar{w}_1, \bar{w}_2) | \bar{y}_2 \rangle = \delta_{\bar{y}_1, \bar{w}_1} \delta_{\bar{y}_2, \bar{w}_2}. \tag{A6}$$

As announced in Proposition 5, the conditions $U_g[H]_g = HU_g$ for any $H \in \mathscr{H}_{n,m}$, $g \in \mathscr{A}_{m+n}$ and $p^{m-n} \leqq |2|_p$ fixes a unitary transformation U_g up to a phase ξ. The explicit expression of U_g in the basis (3.5) reads

a) If $|bh|_p > p^{2m}$ or $|bh|_p = p^{2m}$ if $p \neq 2$:

$$\langle \bar{x} | U_g | \bar{y} \rangle = \begin{cases} \xi p^m |bh|_p^{-1/2} \chi_p\left(\dfrac{1}{2bh}(ay^2 + dx^2 - 2xy)\right); & \text{if} \quad |dx - y|_p \leqq |bh|_p p^{-m} \\ 0; \quad \text{otherwise.} \end{cases} \tag{A7.1}$$

b) If $p = 2$ and $|bh|_2 = 2^{2m}$:

$$\langle \bar{x} | U_g | \bar{y} \rangle = \begin{cases} \xi \chi_2\left(\dfrac{1}{2bh}(ay^2 + dx^2 - 2xy)\right); & \text{if} \quad |dx - y|_2 = 2^{m+1} \\ 0; \quad \text{otherwise.} \end{cases} \tag{A7.2}$$

c) If $|bh|_p < p^{2m}$:

$$\langle \bar{x} | U_g | \bar{y} \rangle = \begin{cases} \xi \chi_p\left(\dfrac{1}{2h}(cdx^2)\right); & \text{if} \, |dx - y|_p \leqq p^m \\ 0; \quad \text{otherwise.} \end{cases} \tag{A7.3}$$

The check of unitarity does not involve quadratic sums and is easy.

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