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Deformations of Super-KMS Functionals*

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Abstract. We investigate the stability of the super-KMS property under deformations. We show that a family of continuous deformations of the super-derivation in the quantum algebra yields a continuous family of deformed super-KMS functionals. These functionals define a family of cohomologous, entire cocycles.

I. Introduction

In this paper we investigate the super-KMS (sKMS) property of functionals ω on a quantum algebra. Our interest in sKMS functionals was inspired by work of Kastler and by conversations with Alain Connes [C1, K, JLO2]. The sKMS construction relies on the existence of a super-derivation d acting on a dense subalgebra of a C^*-algebra $\mathscr{A}$. The square of d is the infinitesimal generator of a continuous, one-parameter automorphism group α_t of the quantum algebra. The usual KMS property relates the cyclicity of a state ω to the analytic continuation of a group α_t of automorphisms. The sKMS property also involves invariance under the super-derivation d whose square generates the automorphism group α_t.

It is known that an sKMS functional on a quantum algebra defines an entire cyclic cocycle τ. This is just the Chern character which Jaffe, Lesniewski, and Osterwalder defined in the context of supertrace functionals on a quantum algebra [JLO1]. The sKMS property ensures that the functional ω – and the cocycle τ which is derived from it – are invariant under this group action.

In this paper we study the stability of this structure under perturbations of d. We study only bounded perturbations which arise from the graded (super) commutator with an odd element q of the algebra $\mathscr{A}$. We show that such perturbations d_q of d can be used to define a deformation ω^q of ω which satisfies the sKMS property. Furthermore, the corresponding family of cocycles τ^q are cohomologous. Of course, more singular (unbounded) perturbations can lead to

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nontrivial perturbations of the cohomology, but we do not investigate this question here.

Much of our analysis has an analog in the theory of KMS states, see for example, [R, HKT, BKR, A, BR]. However, we carry through all the arguments here without using a positivity property for ω. In fact, we do not know a natural positivity property which would lead to a GNS construction representing ω as an invariant, vector state in a Hilbert space. Nor do we know whether a positivity property such as Osterwalder-Schrader positivity (or a spectral condition) holds.

Just as in statistical mechanics where a KMS state generalizes the notion of a Gibbs state, an sKMS functional generalizes the positive temperature supertrace functional. This allows us to deal with situations which occur in examples, both finite dimensional and infinite dimensional. One interesting class of infinite dimensional examples are supersymmetric field theories on a noncompact manifold. The Hamiltonian of such a theory is a Laplace-Beltrami operator on an infinite-dimensional, non-compact manifold. It will have continuum spectrum, so the heat kernel it generates will not be trace class. This is also characteristic of many other examples.

The line of several analytic arguments we use is the following: consider a function $f(a,t)$ for a in an algebra $\mathscr{A}$ and $t \in \mathbb{R}$. For a in a dense subalgebra $\mathscr{A}_\alpha \subset \mathscr{A}$, we know that $f(a,t)$ is the boundary value of an entire function $f(a,z)$ of the variable z. Moreover, for z on the boundary of a strip $0 \leqq \operatorname{Im} z \leqq 1$, we use the sKMS property to establish a bound of the form

$$|f(a,z)| \leqq \|a\| \exp(c|z|+c). \tag{I.1}$$

Using a Pfragmén-Lindelöf type theorem, we extend the bound (I.1) to the interior of the strip. Since (I.1) is uniform in $\|a\|$, we can also establish the analyticity of $f(a,z)$ and the bound (I.1) for all operators $a \in \mathscr{A}$.

For some examples, such as [JL, JLW 1–2], the sKMS functional ω is given by the supertrace of a trace-class heat kernel. In these examples, the related functional

$$\varrho(a) = \omega(\Gamma a)/\omega(\Gamma), \tag{I.2}$$

is a state. Here Γ denotes the grading operator. However, in general Γ is not an element of the algebra $\mathscr{A}$; thus $\omega(\Gamma)$ is not defined in general. This is the case if we consider the heat kernel for a noncompact manifold M, where the heat kernel has continuous spectrum and cannot be trace class. However, it may be the case that the functionals ω and ϱ can both be defined (e.g., as limits of functionals on a sequence of compact approximations M_n to M).

In this case, ω and ϱ will have characteristically different properties. The functional ϱ is normalized, $\varrho(\mathbf{1}) = 1$. On the other hand, $\omega(\mathbf{1})$ should be topological invariant of M. In a special case, where ω is the supertrace weighted by the heat kernel of a compact manifold, we can identify $\omega(\mathbf{1})$ as the index of a Dirac type operator on M. More generally, we define the abstract index of a super-derivation d on a quantum algebra, relative to an sKMS functional ω, by

$$i_\omega(d) = \omega(\mathbf{1}). \tag{I.3}$$

This is a natural notion, invariant under deformations, as we see in the final section.

II. Super-KMS Functionals on a Quantum Algebra

In [JLO2] we define a quantum algebra as a quadruple $(\mathscr{A}, \Gamma, \alpha_t, d)$, where $\mathscr{A}=\mathscr{A}_+\oplus\mathscr{A}_-$ is a $\mathbb{Z}_2$-graded C^*-algebra. Let $a=a_+ + a_-$ denote the decomposition of $a\in\mathscr{A}$ into homogeneous components under the grading. Define an involutive $*$-automorphism Γ of $\mathscr{A}$ by $a\to a^\Gamma=a_+ - a_-$. An automorphism of $\mathscr{A}$ which commutes with Γ is said to be *even.*

In [JLO2] we assumed that for t real, α_t is a continuous one-parameter group of even, $*$-automorphisms of $\mathscr{A}$. Here we only assume that α_t is a continuous, one parameter group of even, bounded automorphisms of $\mathscr{A}$. Continuity means that $\|\alpha_t(a)-a\|\to 0$, as $t\to 0$, for each $a\in\mathscr{A}$. Hence there exists a constant $c<\infty$ such that

$$\|\alpha_t\|\leqq e^{c|t|}, \qquad t\in\mathbb{R}. \tag{II.1}$$

If α_t is a $*$-automorphism, then we may take $c=0$.

Let $\mathscr{A}_\alpha$ denote the (norm dense) subalgebra of $\mathscr{A}$ such that $t\to\alpha_t(A)$ extends to an entire, $\mathscr{A}$-valued analytic function. Let

$$D=-i\frac{d}{dt}\alpha_t\bigg|_{t=0} \tag{II.2}$$

denote the derivation (with domain $\mathscr{A}_\alpha$) which is the infinitesimal generator of α_t. Finally, we assume that d is a super (odd) derivation of $\mathscr{A}$ with domain $\mathscr{D}(d)$ and range in $\mathscr{B}(\mathscr{H})$. We assume that $\mathscr{A}_\alpha$ is a core for d and that $d:\mathscr{A}_\alpha\to\mathscr{A}_\alpha$. A super-derivation satisfies

$$d^\Gamma=-d, \qquad d(ab)=(da)b+a^\Gamma db, \tag{II.3}$$

and that d is a square root of D, namely

$$D=d^2. \tag{II.4}$$

As a consequence, α_t and d commute,

$$\alpha_t\circ d=d\circ\alpha_t. \tag{II.5}$$

Definition II.1. A super-KMS (sKMS) functional ω on the quantum algebra $(\mathscr{A}, \Gamma, \alpha_t, d)$ is a continuous linear functional on $\mathscr{A}$ such that for all $a,b\in\mathscr{A}_\alpha$,

$$\omega(da)=0, \quad \text{and} \quad \omega(ab)=\omega(b^\Gamma\alpha_i(a)). \tag{II.6}$$

Composition of a functional ω on $\mathscr{A}$ with an automorphism σ of $\mathscr{A}$ defines a functional $\omega^\sigma=\omega\circ\sigma$. The functional ω is said to be invariant under σ if $\omega^\sigma=\omega$. An sKMS functional is *translation-invariant* and *even,*

$$\omega^{\alpha_t}=\omega, \quad \text{and} \quad \omega^\Gamma=\omega.$$

Another property of sKMS functionals is the *integration by parts identity*

$$\omega(a\cdot db)=\omega(da\cdot b^\Gamma)=-\omega(da^\Gamma\cdot b), \tag{II.7}$$

which follows from (II.3, 5) and $\omega^\Gamma=\omega$. We use the notation $a(t)=\alpha_{it}(a)$. As a consequence of (II.6–7), an sKMS functional satisfies

$$\omega(a\cdot db)=\omega(b\cdot da(1)). \tag{II.8}$$

Given an sKMS functional on the quantum algebra $(\mathscr{A}, \Gamma, \alpha_t, d)$, we define the function

$$\omega(a_0\alpha_{t_1}(a_1)\dots\alpha_{t_n}(a_n)), \tag{II.9}$$

for $a_j \in \mathscr{A}$ and $t_j \in \mathbb{R}$. Let $\mathscr{D}_n \subset \mathbb{C}^n$ denote the complex domain

$$\mathscr{D}_n = \{z \in \mathbb{C}^n : 0 \leqq \operatorname{Im} z_1 \leqq \operatorname{Im} z_2 \leqq \dots \leqq \operatorname{Im} z_n \leqq 1\}. \tag{II.10}$$

Theorem II.2 (Fundamental Theorem for sKMS Functionals). *Let ω be an* sKMS *functional for $(\mathscr{A}, \Gamma, \alpha_t, d)$ and let $a_0, \dots, a_n \in \mathscr{A}$. Then $\omega(a_0\alpha_{t_1}(a_1)\dots\alpha_{t_n}(a_n))$ is the boundary value of a function, which we denote $\omega(a_0\alpha_{z_1}(a_1)\dots\alpha_{z_n}(a_n))$, and which is holomorphic in the interior of $\mathscr{D}_n$ and continuous on the boundary. Furthermore, for $z \in \mathscr{D}_n$,*

$$|\omega(a_0\alpha_{z_1}(a_1)\dots\alpha_{z_n}(a_n))| \leqq |\omega|(\mathbf{1})e^{c\left(n + \sum_{j=1}^{n}|\operatorname{Re} z_j|\right)} \prod_{j=0}^{n} \|a_j\|, \tag{II.11}$$

where c is the constant in (II.1).

Remark 1. We do not wish to suggest by our notation that the operator $\alpha_t(a)$ is an analytic function of t. Rather the expectations $\omega(a\alpha_t(b))$, etc., are holomorphic. It is sufficient, however, to prove the theorem for $a_j \in \mathscr{A}_\alpha$, $j = 0, \dots, n$, where $\mathscr{A}_\alpha$ is the dense subalgebra of entire elements of $\mathscr{A}$. The estimate then extends by continuity to all $a_j \in \mathscr{A}$, since the upper bound (II.11) only depends on a_j through its norm. However, since for $a_j \in \mathscr{A}_\alpha$, (II.9) is an entire function of $t_1, \dots, t_n$, we infer from the Weierstrass approximation theorem that (II.9) is holomorphic in the interior of $\mathscr{D}_n$ for all $a_j \in \mathscr{A}$.

Remark 2. An sKMS functional is not necessarily positive. Hence for doing estimates we have to bound ω by the positive functional $|\omega|$, which according to the general theory, see e.g., [P], is defined as follows. For ω there is a unique decomposition

$$\omega = (\omega_{1+} - \omega_{1-}) + i(\omega_{2+} - \omega_{2-}) \tag{II.12}$$

with $\omega_{k\pm} \geqq 0$ and $\omega_{k+} \perp \omega_{k-}$. Then

$$|\omega| = \omega_{1+} + \omega_{1-} + \omega_{2+} + \omega_{2-}$$

is positive and satisfies

$$|\omega(a)| \leqq |\omega|(\mathbf{1})\,\|a\|. \tag{II.13}$$

Remark 3. As was shown in [JLO1], this estimate (II.11) follows immediately in case $\alpha_t(a) = \exp(itH)a\exp(-itH)$, where $0 \leqq H = H^*$, where e^{-tH} is trace class, and where ω is a supertrace.

Lemma II.3. *Let f be holomorphic in the interior of the strip $\mathscr{D}_\varepsilon = \{z = x + iy : 0 \leqq \varepsilon < y \leqq 1\}$ and continuous on the boundary. Assume there are constants C, c such that*

$$|f(z)| \leqq Ce^{c|x|}, \tag{II.12}$$

on the strip. If in addition the function f satisfies the estimates

$$|f(x+i\varepsilon)| \leqq Me^{c|x|} \quad \textit{and} \quad |f(x+i)| \leqq Me^{c|x|} \tag{II.13}$$

on the boundary of the strip, then for all $z \in \mathscr{D}_\varepsilon$,

$$|f(z)| \leqq M e^{c(|x|+1)}. \tag{II.14}$$

Proof. For $c=0$, this is a standard Pfragmén-Lindelöf type theorem. Applying this theorem to $f(z)(\cosh((\pi z)/4))^{-(4c/\pi)}$ yields the bound

$$|f(z)| \leqq 2^{(4c/\pi)} M e^{c|x|}.$$

Since $2^{4/\pi} < e$, we have the bound (II.14) as claimed.

Proof of Theorem II.2. We assume that $a_j \in \mathscr{A}_\alpha$, so the function $\omega(a_0 \alpha_{t_1}(a_1) \dots \alpha_{t_n}(a_n))$ extends to an entire function on $\mathbb{C}^n$. Thus we need only establish the bound (II.11). We proceed by induction on n. Let (i_n) denote the estimate (II.11). Clearly (i_0) holds, as

$$|\omega(a_0)| \leqq |\omega|(\mathbf{1}) \, \|a_0\|.$$

We now show that (i_{n+1}) is a consequence of (i_n). Let

$$f(z) = \omega(a_0 \alpha_{z_1}(a_1) \dots \alpha_{z_n}(a_n) \alpha_z(a_{n+1})).$$

We fix $z_1, \dots, z_n \in \mathscr{D}_n$ and let $(z_1, \dots, z_n, z) \in \mathscr{D}_{n+1}$. First we prove that the *a priori* bound on $f(z)$ in the strip $\mathscr{D}_{\mathrm{Im}\, z_n}$. Note that for $a_j \in \mathscr{A}_\alpha$,

$$\|\alpha_{z_j}(a_j)\| \leqq e^{c|\mathrm{Re}\, z_j|} \|\alpha_{i\, \mathrm{Im}\, z_j}(a_j)\|.$$

Furthermore, $0 \leqq \dots \leqq \mathrm{Im}\, z_{j-1} \leqq \mathrm{Im}\, z_j \leqq \dots \leqq 1$, so $|\mathrm{Im}\, z_j| \leqq 1$. We can define a norm $\| \cdot \|_\alpha$ on $\mathscr{A}_\alpha$ by

$$\|a\|_\alpha = \sum_{k=0}^{\infty} \frac{1}{k!} \|D^j a\|.$$

Then

$$\|\alpha_{i\, \mathrm{Im}\, z_j}(a_j)\| \leqq \|a_j\|_\alpha.$$

We thus have

$$|f(z)| \leqq |\omega|(\mathbf{1}) \exp\left(c \sum_{j=0}^{n} |\mathrm{Re}\, z_j|\right) \left(\prod_{j=0}^{n+1} \|a_j\|_\alpha\right) \exp(c|\mathrm{Re}\, z|) = C \exp(c|\mathrm{Re}\, z|),$$

where the constant C is given by

$$|\omega|(\mathbf{1}) \exp\left(c \sum_{j=0}^{n} |\mathrm{Re}\, z_j|\right) \prod_{j=0}^{n+1} \|a_j\|_\alpha.$$

Now we bound $f(z)$ on the boundary of the strip $\mathscr{D}_{\mathrm{Im}\, z_n}$. For $\mathrm{Im}\, z = \mathrm{Im}\, z_n$, we infer from the induction hypothesis (i_n) that

$$\begin{aligned} |f(\mathrm{Re}\, z + i\, \mathrm{Im}\, z_n)| &= |\omega(a_0 \alpha_{z_1}(a_1) \dots \alpha_{i\, \mathrm{Im}\, z_n}(\alpha_{\mathrm{Re}\, z_n}(a_n) \alpha_{\mathrm{Re}\, z_{n+1}}(a_{n+1})))| \\ &\leqq |\omega|(\mathbf{1}) \exp\left(c\left(n + \sum_{j=0}^{n-1} |\mathrm{Re}\, z_j|\right)\right) \left(\prod_{j=0}^{n-1} \|a_j\|\right) \|\alpha_{\mathrm{Re}\, z_n}(a_n) \alpha_{\mathrm{Re}\, z}(a_{n+1})\|. \end{aligned}$$

But

$$\|\alpha_{\mathrm{Re}\, z_n}(a_n) \alpha_{\mathrm{Re}\, z_{n+1}}(a_{n+1})\| \leqq \exp(c|\mathrm{Re}\, z_n| + c|\mathrm{Re}\, z|) \, \|a_n\| \, \|a_{n+1}\|,$$

so

$$|f(\mathrm{Re}\,z+i\,\mathrm{Im}\,z_n)|\leqq|\omega|(\mathbf{1})\exp\left(c\left(n+\sum_{j=0}^{n}|\mathrm{Re}\,z_j|\right)\right)\left(\prod_{j=0}^{n+1}\|a_j\|\right)\exp(c|\mathrm{Re}\,z|),$$

which is the desired bound.

Secondly, on the other boundary $\mathrm{Im}\,z=1$, we use the sKMS property to obtain the estimate: we have

$$\begin{aligned}f(\mathrm{Re}\,z+i)&=\omega(a_0\alpha_{z_1}(a_1)\dots\alpha_{z_n}(a_n)\alpha_i(\alpha_{\mathrm{Re}\,z}(a_{n+1})))\\&=\omega(\alpha_{\mathrm{Re}\,z}(a_{n+1}^{\Gamma})a_0\alpha_{z_1}(a_1)\dots\alpha_{z_n}(a_n)).\end{aligned}$$

Thus from (i_n) and

$$\|\alpha_{\mathrm{Re}\,z}(a_{n+1}^{\Gamma})\|\leqq\exp(c|\mathrm{Re}\,z|)\,\|a_{n+1}\|,$$

we infer

$$|f(\mathrm{Re}\,z+i)|\leqq|\omega|(\mathbf{1})\exp\left(c\left(n+\sum_{j=0}^{n}|\mathrm{Re}\,z_j|\right)\right)\left(\prod_{j=0}^{n+1}\|a_j\|\right)\exp(c|\mathrm{Re}\,z|).$$

We apply Lemma II.2 in the variable z to obtain (i_{n+1}), which completes the proof.

III. Perturbation Theory

In this section we consider the perturbation of the super-derivation d by an odd element q both in $\mathscr{A}$ and in the domain $\mathscr{D}(d)$ of d. Define the bounded super-derivation δ_q by

$$\delta_q(a)=qa+(aq)^{\Gamma}=qa-a^{\Gamma}q,\tag{III.1}$$

and the perturbed derivation d_q by

$$d_q=d+\delta_q.\tag{III.2}$$

Let $d^+a=(da^*)^{*\Gamma}$ be an adjoint super-derivation. With this definition, $d_q^+=d_q$ when $d=d^+$ and $q=q^*$. The square of d_q is the derivation

$$D_q=(d_q)^2=d^2+\mathrm{ad}(\Omega),\tag{III.3}$$

where

$$\Omega=\Omega_q=dq+q^2\tag{III.4}$$

is the curvature determined by q. Since q is bounded, the assumption that q is in the domain of d is equivalent to $\Omega\in\mathscr{A}_+$. Also $\Omega=\Omega^*$, if $d=d^+$ and $q=q^*$.

Let α_t denote the group of automorphisms of $\mathscr{A}$ generated by $D=d^2$. We define α_t^q as the group of automorphisms of $\mathscr{A}$ generated by D_q. Explicitly,

$$\alpha_t^q(a)=\exp(itD_q)(a).\tag{III.5}$$

We also use the group of transformations

$$\gamma_t^q(a)=\exp(it(D+\Omega))(a).\tag{III.6}$$

Standard perturbation theory then yields the following:

Theorem III.1. *With the assumption $q\in\mathscr{A}_-\cap\mathscr{D}(d)$, the perturbation series*

$$\alpha_t^q(a)=\sum_{n=0}^{\infty}(it)^n\int_{\sigma_n}\mathrm{ad}(\alpha_{ts_1}(\Omega))\dots\mathrm{ad}(\alpha_{ts_n}(\Omega))\alpha_t(a)ds_1\dots ds_n\,, \tag{III.7}$$

where σ_n denotes the simplex $0\leqq s_1\leqq\dots\leqq s_n\leqq 1$, and

$$\gamma_t^q(a)=\sum_{n=0}^{\infty}(it)^n\int_{\sigma_n}\alpha_{ts_1}(\Omega)\dots\alpha_{ts_n}(\Omega)\alpha_t(a)ds_1\dots ds_n \tag{III.8}$$

converge in norm to the groups α_t^q and γ_t^q respectively. They are bounded by

$$\|\alpha_t^q\|,\ \|\gamma_t^q\|\leqq\exp(M+M|t|)\,, \tag{III.9}$$

where

$$M=M(\Omega)=c+2e^c\|\Omega\|\,, \tag{III.10}$$

and where c is the constant in (II.1).

*If α_t is a *-automorphism and if Ω is self adjoint, then* $\|\alpha_t^q\|=\|\gamma_t^q\|=1$.

Finally, if q,q' are two such perturbations Ω, Ω', then

$$\|\alpha_t^q-\alpha_t^{q'}\|\,,\quad \|\gamma_t^q-\gamma_t^{q'}\|\leqq 2\|\Omega-\Omega'\|\,|t|e^{\tilde M+\tilde M|t|}\,, \tag{III.11}$$

where

$$\tilde M=M(\Omega)+M(\Omega')\,.$$

Proof. We study α_t^q in detail. The treatment of γ_t^q is similar. Let us first estimate the norm convergence of (III.7). We use

$$\|\mathrm{ad}(\alpha_t(\Omega))\|\leqq 2\|\Omega\|\exp(c|t|)\,.$$

Furthermore, we note that for $c\geqq 0$,

$$\int_{\sigma_n}\exp\left(c\sum_{j=1}^{n}s_j\right)ds_1\dots ds_n\leqq\frac{e^{cn}}{n!}\,. \tag{III.12}$$

Thus the n^{th} term in (III.7) is bounded in norm by

$$\|a\|\,(2|t|\,\|\Omega\|\,e^{c|t|})^n e^{c|t|}\frac{1}{n!}\,. \tag{III.13}$$

Summing over n yields the bound for $|t|\leqq 1$,

$$\|\alpha_t^q\|\leqq\exp(c+2\|\Omega\|e^c)\,.$$

For $|t|>1$, we can choose $n\in Z_+$ such that $|t/n|<1$, and $n\leqq|t|+1$. Thus

$$\|\alpha_t^q\|=\|(\alpha_{t/n}^q)^n\|\leqq\exp(c+2\|\Omega\|e^c)(|t|+1)\,,$$

which is the bound (III.9) as claimed for α_t^q.

Clearly, we can estimate the norm of $\alpha_t^q-\alpha_t^{q'}$ in a similar fashion. The factor $\|\Omega\|^n$ in (III.13) will be replaced by

$$\sum_{j=0}^{n-1}\|\Omega\|^j\|\Omega'\|^{n-1-j}\|\Omega-\Omega'\|\leqq n(\|\Omega\|^{n-1}+\|\Omega'\|^{n-1})\,\|\Omega-\Omega'\|\,.$$

Summing over n yields (III.11) for $\alpha_t^q - \alpha_t^{q'}$.

The convergence of the sum (III.7) shows that α_t^q is well defined and bounded, when defined as this sum. We identify α_t^q as (III.6) by showing that α_t^q satisfies the differential equation

$$-i\frac{d}{dt}\alpha_t^q(a) = D_q\alpha_t^q(a) \tag{III.14}$$

with the initial data satisfying

$$\lim_{t\to 0}\|\alpha_t^q(a) - a\| = 0. \tag{III.15}$$

The uniqueness of the continuous solution to (III.14) with given initial data shows that α_t^q is the desired automorphism (III.6).

Let us first observe that the group property of $\alpha_t^q(a)$ follows from the rearrangement of the power series (III.7), namely

$$\alpha_t^q\alpha_s^q = \alpha_{t+s}^q.$$

Secondly, we observe that continuity of α_t^q in t follows from the group property and continuity at $t=0$. Since by assumption

$$\lim_{t\to 0}\|\alpha_t(a) - a\| = 0,$$

we conclude that as $t\to 0$,

$$\begin{aligned}\|\alpha_t^q(a) - a\| &\leqq \|\alpha_t^q(a) - \alpha_t(a)\| + \|\alpha_t(a) - a\| \\ &\leqq O(t)\,\|\Omega\|\,\|a\| + o(1) \leqq o(1),\end{aligned}$$

where we use (III.11) to dominate the first term.

Thus our analysis of α_t^q is complete if we can establish (III.14) for a in the domain of D_q, which equals the domain of D. Using the group property, we need only establish the equation for $t=0$. Furthermore, the integration over s in (III.14) ensures that, if a is in the domain $\mathscr{D}(D)$ [or $\mathscr{D}(d)$] then each term in (III.14) is also in $\mathscr{D}(D)$ (or $\mathscr{D}(d)$). Thus we can use the estimate (III.13) to justify term by term differentiation of the series (III.7) to obtain

$$-i\frac{d}{dt}\alpha_t^q(a)|_{t=0} = (D + \mathrm{ad}(\Omega))(a) = D_q(a).$$

Alternatively, we see that $\alpha_t^q : \mathscr{D}(d)\to\mathscr{D}(d)$. This completes the analysis of α_t^q. The study of γ_t^q is similar, except that we obtain the equation

$$-i\frac{d}{dt}\gamma_t^q(a) = (D+\Omega)\gamma_t^q(a). \tag{III.16}$$

In the course of proving the theorem, we have also established the following

Proposition III.2. *With the assumption $q\in\mathscr{A}_-\cap\mathscr{D}(d)$, the groups α_t^q and γ_t^q map $\mathscr{D}(d)$ into $\mathscr{D}(d)$. If furthermore $a\in\mathscr{D}(D)$, then $\alpha_t^q(a)$ and $\gamma_t^q(a)$ are differentiable and satisfy*

$$-i\frac{d}{dt}\alpha_t^q(a) = D_q\alpha_t^q(a) = \alpha_t^q(D_q a) \tag{III.17}$$

and

$$-i\frac{d}{dt}\gamma_t^q(a)=(D+\Omega)\gamma_t^q(a). \qquad \text{(III.18)}$$

We remark that if there is a self adjoint H such that

$$\alpha_t(a)=e^{itH}ae^{-itH},$$

then the perturbed automorphism has the form

$$\alpha_t^q(a)=e^{it(H+\Omega)}ae^{-it(H+\Omega)}, \qquad \text{(III.19)}$$

and

$$\gamma_t^q(a)=e^{it(H+\Omega)}ae^{-itH}. \qquad \text{(III.20)}$$

These transformations α_t^q and γ_t^q do not necessarily continue analytically to complex t on the algebra $\mathscr{A}_\alpha$, unless also $\Omega\in\mathscr{A}_\alpha$.

Theorem III.3. (i) *Let* $q\in\mathscr{A}_-\cap\mathscr{D}(d)$ *and let* ω *be an* sKMS *functional for* $(\mathscr{A},\Gamma,\alpha_t,d)$. *Let* $a_j\in\mathscr{A}$, $j=1,2,3$. *Then the function*

$$\omega(a_1\gamma_z^q(a_2)\alpha_z(a_3)) \qquad \text{(III.21)}$$

which is defined for real z, *has an analytic continuation in* z *into the strip*

$$0\leqq\operatorname{Im}z\leqq1$$

to a function which we also denote by (III.21).

(ii) *In addition,*

$$|\omega(a_1\gamma_z^q(a_2)\alpha_z(a_3))|\leqq|\omega|(\mathbf{1})e^{M_1}\prod_{j=1}^{3}\|a_j\|, \qquad \text{(III.22)}$$

where

$$M_1=2c|\operatorname{Re}z|+|z|\,\|\Omega\|\,e^{c|\operatorname{Re}z|}. \qquad \text{(III.23)}$$

(iii) *If* q,q' *are two such* q*'s, then for each* z *in the strip*

$$|\omega(a_1\gamma_z^q(a_2)\alpha_z(a_3))-\omega(a_1\gamma_z^{q'}(a_2)\alpha_z(a_3))|\leqq O(\|\Omega-\Omega'\|)\prod_{j=1}^{3}\|a_j\|, \qquad \text{(III.24)}$$

as $\|\Omega-\Omega'\|\to0$.

(iv) *For* $z=i$, *and* $a,b\in\mathscr{A}$, *we have*

$$\omega(ab\gamma_i^q(\mathbf{1}))=\omega(b^\Gamma\gamma_i^q(\mathbf{1})\alpha_i(a)). \qquad \text{(III.25)}$$

Proof. We use the expansion (III.8) for α_z^q to write for real z

$$f(z)=\sum_{n=0}^{\infty}(iz)^n\int_{\sigma_n}\omega(a_1\alpha_{zs_1}(\Omega)\dots\alpha_{zs_n}(\Omega)\alpha_z(a_2)\alpha_z(a_3))ds_1\dots ds_n. \qquad \text{(III.26)}$$

We now infer from the fundamental Theorem II.2 that each term on the right of (III.26) can be analytically continued in z into the strip $0\leqq\operatorname{Im}z\leqq1$. This requires $0\leqq s_1\leqq\dots\leqq s_n\leqq1$, which is the case on σ_n.

We now prove that the series (III.26) is absolutely convergent and bounded by (III.22). It follows that the limiting function $f(z)$ is also analytic. From (II.11), we conclude that the n^{th} term in the sum (III.26) is bounded in absolute value by

$$|\omega|(\mathbf{1})\left(\prod_{j=1}^{3}\|a_j\|\right)\|\Omega\|^n|z|^n e^{2c|\mathrm{Re}z|}\int_{\sigma_n}\exp\left(c|\mathrm{Re}z|\sum_{j=1}^{n}s_j\right)ds_1\ldots ds_n$$

which by (III.12) is bounded by

$$|\omega|(\mathbf{1})\left(\prod_{j=1}^{3}\|a_j\|\right)e^{2c|\mathrm{Re}z|}\frac{1}{n!}(|z|\,\|\Omega\|\,e^{c|\mathrm{Re}z|})^n .$$

Summing this bound over n yields (III.22). The proof of (III.24) is similar.

The proof of (III.25) follows from applying the sKMS property for ω to the analytic continuation of (III.26) to $z=i$, in the case $a_2=\mathbf{1}$. We can first assume $\Omega\in\mathscr{A}_\alpha$, and then pass to a limiting $\Omega\in\mathscr{A}$ using (III.24).

IV. Stability of the sKMS Property

Let ω denote an sKMS functional on $(\mathscr{A},\Gamma,\alpha_t,d)$. Let $q\in\mathscr{A}_-\cap\mathscr{D}(d)$ and let $\Omega=\Omega_q=dq+q^2$. As a consequence of Theorem III.2, the functional

$$\omega^q(a)=\omega(a\gamma_i^q(\mathbf{1})) \tag{IV.1}$$

can be obtained by analytic continuation of $\omega(a\gamma_t^q(\mathbf{1}))$ for $t\in\mathbb{R}$. This has the sKMS property associated with $d_q=d+\delta_q$.

Theorem IV.1. *If ω is an* sKMS *functional for $(\mathscr{A},\Gamma,\alpha_t,d)$, and $q\in\mathscr{A}_-\cap\mathscr{D}(d)$, then ω^q is an* sKMS *functional for $(\mathscr{A},\Gamma,\alpha_t^q,d_q)$.*

Lemma IV.2. *For t real, $a,b\in\mathscr{A}$, q as above, and ω an* sKMS *functional for $(\mathscr{A},\Gamma,\alpha_t,d)$, we have*

$$\omega(a(d\gamma_t^q(\mathbf{1})+q\gamma_t^q(\mathbf{1})-\gamma_t^q(\mathbf{1})\alpha_t(q)))=0, \tag{IV.2}$$

and

$$\omega(b(\alpha_t^q(a)\gamma_t^q(\mathbf{1})-\gamma_t^q(\mathbf{1})\alpha_t(a)))=0. \tag{IV.3}$$

Proof. Note that $\mathbf{1}\in\mathscr{D}(d)$, so Proposition III.3 ensures that $d\gamma_t^q(\mathbf{1})$ is defined. It is then sufficient to establish (IV.2–3) for $q\in\mathscr{A}_\alpha$, since the continuity estimate (III.11) then allows us to pass by continuity to $q\in\mathscr{A}_-\cap\mathscr{D}(d)$.

The first identity follows by studying the derivatives of the operator valued function

$$e(t)=d\gamma_t^q(\mathbf{1})+q\gamma_t^q(\mathbf{1})-\gamma_t^q(\mathbf{1})\alpha_t(q). \tag{IV.4}$$

Clearly $e(0)=0$, and we now see that $e^{(n)}(0)=0$. Using (III.16) for γ_t^q, we obtain

$$-i\frac{de(t)}{dt}=(d+q)(D+\Omega)\gamma_t^q(\mathbf{1})-\{(D+\Omega)\gamma_t^q(\mathbf{1})\}\alpha_t(q)-\gamma_t^q(\mathbf{1})\alpha_t(Dq).$$

We use

$$D+\Omega=(d+q)^2,$$

so $D+\Omega$ commutes with $d+q$. Therefore

$$-i\frac{de(t)}{dt}=(D+\Omega)e(t).$$

It follows that

$$e^{(n)}(0)=\left.\frac{d^n e(t)}{dt^n}\right|_{t=0}=i^n(D+\Omega)^n e(t)\Big|_{t=0}=0.$$

Each term in (IV.2) is real analytic in t, as a consequence of Theorem III.3.(i). Thus the vanishing of all derivations at $t=0$ ensures the vanishing of (IV.2).

In order to establish (IV.3) we study

$$E(t)=\alpha_t^q(a)\gamma_t^q(\mathbf{1})-\gamma_t^q(\mathbf{1})\alpha_t(a).$$

Again $E(0)=0$. We claim that

$$-i\frac{dE(t)}{dt}=(D+\Omega)E(t), \tag{IV.5}$$

which establishes (IV.3) by repeating the above analysis. But

$$\begin{aligned}-i\frac{dE(t)}{dt}&=\{(D+\mathrm{ad}(\Omega))\alpha_t^q(a)\}\gamma_t^q(\mathbf{1})+\alpha_t^q(a)(D+\Omega)\gamma_t^q(\mathbf{1})\\&\quad-\{(D+\Omega)\gamma_t^q(\mathbf{1})\}\alpha_t(a)-\gamma_t^q(\mathbf{1})D\alpha_t(a)\\&=(D+\Omega)E(t),\end{aligned}$$

to complete the proof.

Proof of Theorem IV.1. We first show that

$$\omega^q(d_q a)=0, \qquad a\in\mathscr{D}(d).$$

Using integration by parts (II.7), and the fact that γ_t^q is even, we obtain

$$\omega(a^\Gamma d\gamma_t^q(\mathbf{1}))=-\omega(da\gamma_t^q(\mathbf{1})).$$

Hence by (IV.2),

$$\omega((da-a^\Gamma q)\gamma_t^q(\mathbf{1}))=-\omega(a^\Gamma\gamma_t^q(1)\alpha_t(q))=\omega(a\gamma_t^q(1)\alpha_t(q)).$$

Using Theorem III.3, this analytically continues in t to the point $t=i$. Using definition (IV.1), and property (III.25), we obtain

$$\omega^q(da-a^\Gamma q)=\omega(a\gamma_i^q(\mathbf{1})\alpha_i(q))=\omega(q^\Gamma a\gamma_i^q(\mathbf{1}))=-\omega^q(qa).$$

In other words

$$\omega^q(da+qa-a^\Gamma q)=\omega^q(d_q a)=0,$$

as claimed.

Next we use (IV.3) to give

$$\omega(b^\Gamma\alpha_t^q(a)\gamma_t^q(\mathbf{1}))=\omega(b^\Gamma\gamma_t^q(\mathbf{1})\alpha_t(a)).$$

Again Theorem III.3 assures an analytic continuation to $t=i$, while using (III.25) we obtain

$$\omega^q(b^\Gamma \alpha_i^q(a)) = \omega^q(ab).$$

Hence we have verified the sKMS property as claimed.

V. Homotopy Invariance of the Chern Character

Let ω be a sKMS functional. Recall [JLO 1–2, K] that the Chern character $\tau = \{\tau_n\}_{n \geqq 0}$ on the quantum algebra $(\mathscr{A}, \Gamma, \alpha_t, d)$ defined by ω is

$$\tau_n(a_0, a_1, \ldots, a_n) = i^{\varepsilon_n} \int_{\sigma_n} \omega(a_0 \alpha_{is_1}(da_1^\Gamma) \alpha_{is_2}(da_2) \alpha_{is_3}(da_3^\Gamma) \ldots \alpha_{is_n}(da_n^{\Gamma^n})) ds_1 \ldots ds_n, \tag{V.1}$$

where $\varepsilon_n = n \bmod 2$. Our present formulation does not require that α_t is a $*$-automorphism, but the algebraic constructions of [JLO 1–2, K] still apply. The analytic estimates are a consequence of Theorem II.2. We introduce Connes' entire cyclic complex with coboundary operator ∂, see [C3, JLO 1–2] for the definitions.

We consider now a deformation of d defined by

$$d_{q(\lambda)} = d + \delta_{q(\lambda)}, \tag{V.2}$$

where $[0,1] \ni \lambda \to q(\lambda) \in \mathscr{A}_- \cap \mathscr{D}(d)$ is continuously differentiable. Let $\omega^{q(\lambda)}$ be the corresponding sKMS functional defined in Sect. IV, and let τ^λ denote the Chern character associated with $\omega^{q(\lambda)}$.

Theorem V.1. *With the above definitions, τ^λ and τ are in the same cohomology class. In fact*

$$\frac{d}{d\lambda}\tau^\lambda = \partial G^\lambda, \tag{V.3}$$

where $G^\lambda \in \mathscr{C}(\mathscr{A})$ is given by

$$G_{n-1}^\lambda(a_0, a_1, \ldots, a_{n-1}) = -i^{\varepsilon_n} \sum_{j=0}^{n-1} (-1)^j \int_{\sigma_n} \omega^{q(\lambda)}(a_0 \alpha_{is_1}^{q(\lambda)}(d_{q(\lambda)} a_1^\Gamma) \ldots \times \alpha_{is_j}^{q(\lambda)}(d_{q(\lambda)} a_j^{\Gamma^j}) \alpha_{is_{j+1}}^{q(\lambda)}(\dot{q}(\lambda)) \alpha_{is_{j+2}}^{q(\lambda)}(d_{q(\lambda)} a_{j+1}^{\Gamma^{j+2}}) \ldots \alpha_{is_n}^{q(\lambda)}(d_{q(\lambda)} a_{n-1}^{\Gamma^n})) d^n s. \tag{V.4}$$

Proof. The sequence $G^\lambda = \{G_n^\lambda\}$ satisfies the entire growth condition, as follows from the estimates of Sects. II–IV. In fact,

$$|G_n^\lambda(a_0, a_1, \ldots, a_n)| \leqq \frac{1}{n!} C^n |\omega^{q(\lambda)}|(\mathbf{1}) \, \|\dot{q}(\lambda)\| \, \|a_0\| \prod_{j=1}^n \|d_{q(\lambda)} a_j\|,$$

with C a constant independent of λ. Since

$$\|d_{q(\lambda)} a\| \leqq \|da\| + 2\|q(\lambda)\| \, \|a\| \leqq C(\|a\| + \|da\|) = C\|a\|_*,$$

uniformly in $\lambda \in [0, 1]$, it follows that

$$|G_n^{\lambda}(q_0, q_1, \ldots, q_n)| \leqq \frac{1}{n!} C_1 C_2^n \prod_{j=0}^{n} \|a_j\|_*, \tag{V.5}$$

with $C_1, C_2 < \infty$.

The algebraic identity (V.3) is a consequence of the following two identities:

$$\frac{d}{d\lambda} \alpha_{it}^{q(\lambda)}(a) = \int_0^t [\alpha_{it}^{q(\lambda)}(a), \alpha_{is}^{q(\lambda)}(d_{q(\lambda)} \dot{q}(\lambda))] \, ds \tag{V.6}$$

and

$$\frac{d}{d\lambda} \gamma_i^{q(\lambda)}(\mathbf{1}) = -\int_0^1 \alpha_{is}^{q(\lambda)}(d_{q(\lambda)} \dot{q}(\lambda)) \, ds \gamma_i^{q(\lambda)}(\mathbf{1}), \tag{V.7}$$

and the identities of [JLO1, GS, K, EFJL].

Definition V.2. The index of the super-derivation d relative to ω is defined by

$$i_{\omega}(d) = \omega(\mathbf{1}). \tag{V.8}$$

Corollary V.3. *The index of a super-derivation is stable under bounded deformations of the form* (V.2),

$$i_{\omega}(d) = i_{\omega^q}(d_q). \tag{V.9}$$

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Note added in proof. In a recent preprint entitled "Algebra cochains and cyclic cohomology", D. Quillen has related the cocycle constructed in [JLO1] to his notion of superconnection.