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A Direct Method for Deriving a Multi-Soliton Solution for the Problem of Interaction of Waves on the x, y Plane

V. K. Mel'nikov

Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Head Post Office P.O. Box 79, SU-101000 Moscow, USSR

Abstract. Explicit expressions are found for a multi-soliton solution of the system of equations describing the interaction of waves on the x, y plane. The proof of all necessary statements follows from the theory of matrices and is not based on the inverse scattering method. The obtained results are closely related to some problems of mathematical physics.

In the present paper we obtained explicit expressions for a multi-soliton solution of the system of equations

$$3\frac{\partial^2 u}{\partial y^2}-\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(3u^2+\frac{\partial^2 u}{\partial x^2}+8\kappa|\varphi|^2\right)\right]=0, \quad i\frac{\partial\varphi}{\partial y}=u\varphi+\frac{\partial^2\varphi}{\partial x^2}, \tag{1}$$

describing (in a certain approximation) the interaction of a long wave with a short-wave packet propagating on the x, y plane at an angle to each other [1, 2]. Here u is the long wave amplitude, φ is the complex short-wave envelope and the parameter κ satisfies the condition $\kappa^2=1$. Though this solution was derived first by using the ideas underlying the inverse scattering method, our proofs here are based only on some very simple facts related to matrices of a very special form and have no relation to the afore-mentioned method. This is achieved in the following way.

1. Solution of an Auxiliary System of Equations

Let B be the square matrix of order $r_0=r_1+2r_2$, $r_1>0$, $r_2>0$, with the elements $B_{r,s}$, $r,s=1,\dots,r_0$. Assume that nonzero elements of the matrix B have the form

$$B_{r,s}=\begin{cases}\dfrac{f_r\exp[(\omega_r-\sigma_s)x-4(\omega_r^3-\sigma_r^3)y]}{\omega_r-\sigma_s}, \\ \quad \text{if} \quad r=1,\dots,r_1,r_1+r_2+1,\dots,r_0 \quad \text{and} \quad s=1,\dots,r_1+r_2, \\ -\dfrac{f_r\exp[-4(\omega_r^3-\sigma_r^3)y]}{\omega_r^3-\sigma_s^3}, \quad \text{if} \quad r_1<r\leqq r_1+r_2<s\leqq r_0.\end{cases} \tag{1.1}$$

The rest elements of the matrix B are assumed to be zero, i.e.

$$B_{r,s}=0, \quad \text{if} \begin{cases} 1) \ 1 \leqq r \leqq r_1, r_1+r_2<s \leqq r_0, \\ 2) \ r_1<r \leqq r_1+r_2, \ 1 \leqq s \leqq r_1+r_2, \\ 3) \ r_1+r_2<r, \ s \leqq r_0. \end{cases} \tag{1.2}$$

It is also assumed that the quantities $f_1, \ldots, f_{r_0}$, $\omega_1, \ldots, \omega_{r_0}$, $\sigma_1, \ldots, \sigma_{r_0}$ are independent of the x, y coordinates. Moreover, assume that the quantities $f_1, \ldots, f_{r_0}$ depend on the time t so as to fulfill the equalities

$$\begin{aligned} &\frac{\partial f_r}{\partial t}+i(\omega_r^2-\sigma_r^2)f_r=0 \quad \text{at} \quad r=1, \ldots, r_1, \\ &\frac{\partial f_r}{\partial t}-i\sigma_r^2 f_r=0 \quad \text{at} \quad r=r_1+1, \ldots, r_1+r_2, \\ &\frac{\partial f_r}{\partial t}+i\omega_r^2 f_r=0 \quad \text{at} \quad r=r_1+r_2+1, \ldots, r_0. \end{aligned} \tag{1.3}$$

However, the quantities $\omega_1, \ldots, \omega_{r_0}$, $\sigma_1, \ldots, \sigma_{r_0}$ are thought to be independent of t.

Now we take the column vectors λ and ℓ with the components λ_r and ℓ_r, respectively, of the form

$$\lambda_r=\begin{cases} f_r \exp[\omega_r x-4(\omega_r^3-\sigma_r^3)y], & \text{if} \quad r=1, \ldots, r_1, r_1+r_2+1, \ldots, r_0, \\ f_r \exp[-4(\omega_r^3-\sigma_r^3)y], & \text{if} \quad r_1<r \leqq r_1+r_2, \end{cases} \tag{1.4}$$

$$\ell_r=\begin{cases} \exp(-\sigma_r x), & \text{if} \quad 1 \leqq r \leqq r_1+r_2, \\ 1, & \text{if} \quad r_1+r_2<r \leqq r_0. \end{cases} \tag{1.5}$$

Then, we use the diagonal matrices I, I_0, I_1, and I_2 of order r_0 of the form

$$\begin{aligned} I&=\operatorname{diag}(1, \ldots, 1, 1, \ldots, 1, 0, \ldots, 0), \\ I_0&=\operatorname{diag}(1, \ldots, 1, 0, \ldots, 0, 1, \ldots, 1), \\ I_1&=\operatorname{diag}(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0), \\ I_2&=\operatorname{diag}(0, \ldots, 0, 0, \ldots, 0, 1, \ldots, 1), \end{aligned} \tag{1.6}$$

where the first groups of zeros and unities are of the length r_1; and the second and third, of the length r_2.

Assume

$$D=\det(\mathbb{1}+B), \tag{1.7}$$

$$\Phi=\det\begin{vmatrix} 0 & \tilde{\ell} I_2 \\ I_0\lambda & \mathbb{1}+B \end{vmatrix}, \qquad \Psi=\det\begin{vmatrix} 0 & \tilde{\ell} I \\ I_1\lambda & \mathbb{1}+B \end{vmatrix}, \tag{1.8}$$

where $\mathbb{1}$ is the unit matrix of order r_0 and the tilde denotes transposition, i.e., in particular, a passage from a column vector to a row vector. Assume now that in some vicinity of the point $x=x_0, y=y_0, t=t_0$, the inequality $D \neq 0$ holds. Define the functions u, φ, and ψ by the equalities

$$u=2\frac{\partial^2}{\partial x^2}\ln D, \qquad \varphi=\frac{\Phi}{D}, \qquad \psi=\frac{\Psi}{D}. \tag{1.9}$$

Then, the following theorem is valid.

Theorem 1. *The functions u, φ, and ψ defined by* (1.1)–(1.9) *satisfy in the vicinity of the above-mentioned point $x=x_0$, $y=y_0$, $t=t_0$ the system of equations*

$$
\begin{gathered}
3\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}+\frac{\partial}{\partial x}\left(3u^2+\frac{\partial^2 u}{\partial x^2}+8\varphi\psi\right)\right]=0\,,\\
i\frac{\partial\varphi}{\partial t}=u\varphi+\frac{\partial^2\varphi}{\partial x^2}\,,\qquad i\frac{\partial\psi}{\partial t}+u\psi+\frac{\partial^2\psi}{\partial x^2}=0\,.
\end{gathered}
\tag{1.10}
$$

The proof of this theorem is based on the following elementary lemma.

Lemma. *Let A be the square matrix of order $m+n+1$ with $m>0$ and $n>0$. Let then $A_{\mu,\nu}$ be the square matrix of order $m+n$ resulting from the matrix A after cancelling the elements of the μ^{th} row and ν^{th} column, and $\alpha_{\mu,\nu}=\det A_{\mu,\nu}$, $\mu,\nu=1,\dots,m+n+1$. Let finally A_0 be the minor of the n^{th} order in the right bottom angle of the matrix A, and the matrix $\mathscr{A}_0$ has the form*

$$
\mathscr{A}_0=\begin{vmatrix}\alpha_{1,1}\ \dots\ \alpha_{1,m+1}\\ \dots\dots\dots\dots\\ \alpha_{m+1,1}\ \dots\ \alpha_{m+1,m+1}\end{vmatrix}. \tag{1.11}
$$

Then, the following equality is valid:

$$
(\det A)^m\det A_0=\det\mathscr{A}_0\,. \tag{1.12}
$$

Proof. First, consider the case when $\det A\neq 0$. We take the matrix $\hat{A}$ of the form

$$
\hat{A}=\begin{vmatrix}\mathbb{1}_{m+1} & A_1\\ 0 & A_0\end{vmatrix}, \tag{1.13}
$$

where $\mathbb{1}_{m+1}$ is the unit matrix of order $m+1$ and A_1 is the minor of the matrix A formed by the elements at the intersection of rows with numbers $\mu=1,\dots,m+1$ and columns with numbers $\nu=m+2,\dots,m+n+1$. Then, the following equality holds:

$$
A^{-1}\hat{A}=\begin{vmatrix}\hat{A}_0 & 0\\ \hat{A}_1 & \mathbb{1}_n\end{vmatrix}, \tag{1.14}
$$

where $\mathbb{1}_n$ is the unit matrix of order n, $\hat{A}_0$ is the minor of the $(m+1)^{\text{th}}$ order that is in the left upper angle of the matrix A^{-1} and $\hat{A}_1$ is the minor of the matrix A^{-1} formed by the elements at the intersection of the rows with numbers $\mu=m+2,\dots,m+n+1$ and columns with numbers $\nu=1,\dots,m+1$. Owing to (1.11) and (1.13) it follows from (1.14) that relation (1.12) is valid.

In the case when $\det A=0$, we substitute the matrix A by the matrix $A'=A+\varepsilon\mathbb{1}_{m+n+1}$, where $\mathbb{1}_{m+n+1}$ is the unit matrix of order $m+n+1$. For the matrix A' the lemma is valid for all sufficiently small $\varepsilon\neq 0$. Passing to the limit $\varepsilon\to 0$, we get that in this case the equality $\det\mathscr{A}_0=0$ is valid, i.e. relation (1.12) is valid also for $\det A=0$.
The lemma is proved.

Proof of the Theorem. By substituting directly expressions (1.9) into (1.10), one may be convinced that system (1.10) will be satisfied provided that the quantities D,

Φ, and Ψ satisfy the relations

$$\left(3\frac{\partial^2 D}{\partial t^2}-\frac{\partial^2 D}{\partial x\partial y}-\frac{\partial^4 D}{\partial x^4}\right)D-3\left[\left(\frac{\partial D}{\partial t}\right)^2+\left(\frac{\partial^2 D}{\partial x^2}\right)^2\right]$$

$$+\left(\frac{\partial D}{\partial y}+4\frac{\partial^3 D}{\partial x^3}\right)\frac{\partial D}{\partial x}-4\Phi\Psi=0, \tag{1.15}$$

$$\left(i\frac{\partial \Phi}{\partial t}-\frac{\partial^2 \Phi}{\partial x^2}\right)D=\left(i\frac{\partial D}{\partial t}+\frac{\partial^2 D}{\partial x^2}\right)\Phi-2\frac{\partial D}{\partial x}\frac{\partial \Phi}{\partial x}, \tag{1.16}$$

$$\left(i\frac{\partial \Psi}{\partial t}+\frac{\partial^2 \Psi}{\partial x^2}\right)D=\left(i\frac{\partial D}{\partial t}-\frac{\partial^2 D}{\partial x^2}\right)\Psi+2\frac{\partial D}{\partial x}\frac{\partial \Psi}{\partial x}. \tag{1.17}$$

Let us prove them. According to (1.1), (1.2), and (1.4)–(1.6) the following equalities are valid:

$$\frac{\partial B}{\partial x}=\omega I_0 B-BI\sigma=I_0\lambda\tilde{\ell}I, \qquad \frac{\partial \lambda}{\partial x}=\omega I_0\lambda, \qquad \frac{\partial \ell}{\partial x}=-\sigma I\ell, \tag{1.18}$$

where the product $\lambda\tilde{\ell}$ of the column vector λ times the row vector $\tilde{\ell}$ is assumed as a matrix product, and consequently, is the square matrix of order r_0 with the elements $\lambda_r\ell_s$, $r,s=1,\dots,r_0$, and the diagonal matrices ω and σ have the form

$$\omega=\mathrm{diag}(\omega_1,\dots,\omega_{r_0}), \qquad \sigma=\mathrm{diag}(\sigma_1,\dots,\sigma_{r_0}). \tag{1.19}$$

For arbitrary integers $m\geqq 0$ and $n\geqq 0$ we define the square matrices $F_{m,n}$, G_m, and H_n of order r_0+1 of the form

$$F_{m,n}=\begin{vmatrix} 0 & \tilde{\ell}I\sigma^n \\ \omega^m I_0\lambda & \mathbb{1}+B \end{vmatrix}, \tag{1.20}$$

$$G_m=\begin{vmatrix} 0 & \tilde{\ell}I_2 \\ \omega^m I_0\lambda & \mathbb{1}+B \end{vmatrix}, \qquad H_n=\begin{vmatrix} 0 & \tilde{\ell}I\sigma^n \\ I_1\lambda & \mathbb{1}+B \end{vmatrix}. \tag{1.21}$$

Let further K be the square matrix of order r_0+1 of the form

$$K=\begin{vmatrix} 0 & \tilde{\ell}I_2 \\ I_1\lambda & \mathbb{1}+B \end{vmatrix}. \tag{1.22}$$

Finally, we choose the square matrices U, U_0, V, and W of order r_0+2 of the form

$$U=\begin{vmatrix} 0 & 0 & \tilde{\ell}I\sigma \\ 0 & 0 & \tilde{\ell}I \\ \omega I_0\lambda & I_0\lambda & \mathbb{1}+B \end{vmatrix}, \qquad U_0=\begin{vmatrix} 0 & 0 & \tilde{\ell}I_2 \\ 0 & 0 & \tilde{\ell}I \\ I_1\lambda & I_0\lambda & \mathbb{1}+B \end{vmatrix}, \tag{1.23}$$

$$V=\begin{vmatrix} 0 & 0 & \tilde{\ell}I_2 \\ 0 & 0 & \tilde{\ell}I \\ \omega I_0\lambda & I_0\lambda & \mathbb{1}+B \end{vmatrix}, \qquad W=\begin{vmatrix} 0 & 0 & \tilde{\ell}I\sigma \\ 0 & 0 & \tilde{\ell}I \\ I_1\lambda & I_0\lambda & \mathbb{1}+B \end{vmatrix}. \tag{1.24}$$

From (1.7), (1.8), and (1.18)–(1.24) we have

$$\frac{\partial D}{\partial x}=-\det F_{0,0}, \quad \frac{\partial^2 D}{\partial x^2}=-\det F_{1,0}+\det F_{0,1}, \tag{1.25}$$

$$\frac{\partial^3 F}{\partial x^3}=-\det F_{2,0}+2\det F_{1,1}-\det F_{0,2}, \tag{1.26}$$

$$\frac{\partial^4 D}{\partial x^4}=-\det F_{3,0}+3\det F_{2,1}-3\det F_{1,2}+\det F_{0,3}-2\det U, \tag{1.27}$$

$$\frac{\partial \Phi}{\partial x}=\det G_1, \quad \frac{\partial^2 \Phi}{\partial x^2}=\det G_2-\det V, \tag{1.28}$$

$$\frac{\partial \Psi}{\partial x}=-\det H_1, \quad \frac{\partial^2 \Psi}{\partial x^2}=\det H_2+\det W. \tag{1.29}$$

Now we take the matrix T of the form

$$T=\exp(i\sigma^2 It), \tag{1.30}$$

and let

$$\hat{B}=T^{-1}BT. \tag{1.31}$$

Using (1.1)–(1.3), (1.6), (1.19), (1.30), and (1.31) we find that the nonzero elements $\hat{B}_{r,s}$ of the matrix $\hat{B}$ have the form

$$\hat{B}_{r,s}=\begin{cases}\dfrac{g_r\exp[(\omega_r-\sigma_s)x-i(\omega_r^2-\sigma_s^2)t-4(\omega_r^3-\sigma_r^3)y]}{\omega_r-\sigma_s}, \\ \quad \text{if} \quad r=1,\dots,r_1,\ r_1+r_2+1,\dots,r_0, \text{ and } s=1,\dots,r_1+r_2, \\ -\dfrac{g_r\exp[-4(\omega_r^3-\sigma_r^3)y]}{\omega_r^3-\sigma_s^3}, \quad \text{if} \quad r_1<r\leqq r_1+r_2<s\leqq r_0,\end{cases} \tag{1.32}$$

where the quantities $g_1,\dots,g_{r_0}$ are related with $f_1,\dots,f_{r_0}$ by

$$g_r=\begin{cases} f_r\exp[i(\omega_r^2-\sigma_r^2)t], & \text{if} \quad 1\leqq r\leqq r_1, \\ f_r\exp(-i\sigma_r^2 t), & \text{if} \quad r_1<r\leqq r_1+r_2, \\ f_r\exp(i\omega_r^2 t), & \text{if} \quad r_1+r_2<r\leqq r_0,\end{cases} \tag{1.33}$$

and consequently, are independent of t. The rest elements of the matrix B are obviously equal to zero. By virtue of (1.4)–(1.6) and (1.30)–(1.33) we get that

$$\begin{aligned}&\frac{\partial \hat{B}}{\partial t}=-i\omega^2 I_0\hat{B}+i\hat{B}I\sigma^2=-iT^{-1}(\omega I_0\lambda\tilde{\ell}I+I_0\lambda\tilde{\ell}I\sigma)T,\\ &\frac{\partial}{\partial t}(T^{-1}I_0\lambda)=-iT^{-1}\omega^2 I_0\lambda,\\ &\frac{\partial}{\partial t}(TI\ell)=iT\sigma^2 I\ell,\\ &\frac{\partial}{\partial t}(T^{-1}I_1\lambda)=\frac{\partial}{\partial t}(TI_2\ell)=0.\end{aligned} \tag{1.34}$$

Then, according to (1.7), (1.8), (1.30), and (1.31) the following equalities hold:

$$D=\det(\mathbb{1}+\hat{B}),$$

$$\Phi=\det\begin{vmatrix}0 & \tilde{\ell}I_2T\\ T^{-1}I_0\lambda & \mathbb{1}+\hat{B}\end{vmatrix},\qquad \Psi=\det\begin{vmatrix}0 & \tilde{\ell}IT\\ T^{-1}I_1\lambda & \mathbb{1}+\hat{B}\end{vmatrix},$$

which result, on the basis of (1.34), in the relations

$$i\frac{\partial D}{\partial t}=-\det\begin{vmatrix}0 & \tilde{\ell}IT\\ T^{-1}\omega I_0\lambda & \mathbb{1}+\hat{B}\end{vmatrix}-\det\begin{vmatrix}0 & \tilde{\ell}I\sigma T\\ T^{-1}I_0\lambda & \mathbb{1}+\hat{B}\end{vmatrix},$$

$$i\frac{\partial \Phi}{\partial t}=\det\begin{vmatrix}0 & \tilde{\ell}I_2T\\ T^{-1}\omega^2 I_0\lambda & \mathbb{1}+\hat{B}\end{vmatrix}+\det\begin{vmatrix}0 & 0 & \tilde{\ell}I_2T\\ 0 & 0 & \tilde{\ell}IT\\ T^{-1}\omega I_0\lambda & T^{-1}I_0\lambda & \mathbb{1}+\hat{B}\end{vmatrix},$$

$$i\frac{\partial \Psi}{\partial t}=-\det\begin{vmatrix}0 & \tilde{\ell}I\sigma^2T\\ T^{-1}I_1\lambda & \mathbb{1}+\hat{B}\end{vmatrix}+\det\begin{vmatrix}0 & 0 & \tilde{\ell}I\sigma T\\ 0 & 0 & \tilde{\ell}IT\\ T^{-1}I_1\lambda & T^{-1}I_0\lambda & \mathbb{1}+\hat{B}\end{vmatrix},$$

$$\begin{aligned}\frac{\partial^2 D}{\partial t^2}=&\det\begin{vmatrix}0 & \tilde{\ell}IT\\ T^{-1}\omega^3 I_0\lambda & \mathbb{1}+\hat{B}\end{vmatrix}-\det\begin{vmatrix}0 & \ell I\sigma^2T\\ T^{-1}\omega I_0\lambda & \mathbb{1}+\hat{B}\end{vmatrix}\\ &+\det\begin{vmatrix}0 & \tilde{\ell}I\sigma T\\ T^{-1}\omega^2 I_0\lambda & \mathbb{1}+\hat{B}\end{vmatrix}-\det\begin{vmatrix}0 & \tilde{\ell}I\sigma^3T\\ T^{-1}I_0\lambda & \mathbb{1}+\hat{B}\end{vmatrix}\\ &+2\det\begin{vmatrix}0 & 0 & \tilde{\ell}I\sigma T\\ 0 & 0 & \tilde{\ell}IT\\ T^{-1}\omega I_0\lambda & T^{-1}I_0\lambda & \mathbb{1}+\hat{B}\end{vmatrix},\end{aligned}$$

i.e. in conformity with (1.20)–(1.24) we have

$$i\frac{\partial D}{\partial t}=-\det F_{1,0}-\det F_{0,1},\tag{1.35}$$

$$i\frac{\partial \Phi}{\partial t}=\det G_2+\det V,\qquad i\frac{\partial \Psi}{\partial t}=-\det H_2+\det W,\tag{1.36}$$

$$\frac{\partial^2 D}{\partial t^2}=\det F_{3,0}+\det F_{2,1}-\det F_{1,2}-\det F_{0,3}+2\det U.\tag{1.37}$$

Hence, on the basis of (1.25), (1.28), and (1.29) we have

$$i\frac{\partial D}{\partial t}+\frac{\partial^2 D}{\partial x^2}=-2\det F_{1,0},\qquad -i\frac{\partial D}{\partial t}+\frac{\partial^2 D}{\partial x^2}=2\det F_{0,1},\tag{1.38}$$

$$i\frac{\partial \Phi}{\partial t}-\frac{\partial^2 \Phi}{\partial x^2}=2\det V,\qquad i\frac{\partial \Psi}{\partial t}+\frac{\partial^2 \Psi}{\partial x^2}=2\det W.\tag{1.39}$$

Now let us use the lemma proved before. Assume that $A=V$. Putting $m=1$, $n=r_0$, with (1.20), (1.21), and (1.24) in mind, we get that $A_0=\mathbb{1}+B$, $A_{1,1}=F_{0,0}$, $A_{1,2}=F_{1,0}$, $A_{2,1}=G_0$, and $A_{2,2}=G_1$. By virtue of (1.7), (1.8), (1.21), (1.25), (1.28), (1.38), and (1.39), equality (1.12) in this case has the form (1.16). Analogously, setting $A=W$, at $m=1$, $n=r_0$ we find that $A_0=\mathbb{1}+B$, $A_{1,1}=F_{0,0}$, $A_{1,2}=H_0$, $A_{2,1}=F_{0,1}$, $A_{2,2}=H_1$. According to (1.7), (1.8), (1.21), (1.25), (1.29), (1.38), and (1.39), equality (1.12) acquires the form (1.17).

Now we use the matrix Y of the form

$$Y=\exp(4\sigma^3 y) \tag{1.40}$$

and put

$$\breve{B}=Y^{-1}BY. \tag{1.41}$$

According to (1.1), (1.19), (1.40), and (1.41) the nonzero elements $\breve{B}_{r,s}$ of the matrix $\breve{B}$ have the form

$$\breve{B}_{r,s}=\begin{cases} \dfrac{f_r\exp[(\omega_r-\sigma_s)x-4(\omega_r^3-\sigma_s^3)y]}{\omega_r-\sigma_s}, \\ \quad \text{if} \quad r=1,\dots,r_1,r_1+r_2+1,\dots,r_0 \text{ and } s=1,\dots,r_1+r_2, \\ -\dfrac{f_r\exp[-4(\omega_r^3-\sigma_s^3)y]}{\omega_r^3-\sigma_s^3}, \quad \text{if} \quad r_1<r\leqq r_1+r_2<s\leqq r_0. \end{cases} \tag{1.42}$$

The rest elements of the matrix $\breve{B}$ are obviously equal to zero. With the help of (1.4)–(1.6) and (1.40)–(1.42) we get that

$$\begin{aligned} \frac{\partial\breve{B}}{\partial y} &= -4\omega^3 I_0\breve{B}+4\breve{B}I\sigma^3-4\omega^3 I_1\breve{B}I_2+4I_1\breve{B}I_2\sigma^3 \\ &= -4Y^{-1}(\omega^2 I_0\lambda\tilde{\ell}I+\omega I_0\lambda\tilde{\ell}I\sigma+I_0\lambda\tilde{\ell}I\sigma^2-I_1\lambda\tilde{\ell}I_2)Y. \end{aligned} \tag{1.43}$$

Then, by virtue of (1.7) and (1.41) we have $D=\det(\mathbb{1}+\breve{B})$. Hence, on the basis of (1.43) we have

$$\begin{aligned} \frac{\partial D}{\partial y} &= 4\det\begin{vmatrix} 0 & \tilde{\ell}IY \\ Y^{-1}\omega^2 I_0\lambda & \mathbb{1}+\breve{B}\end{vmatrix} + 4\det\begin{vmatrix} 0 & \tilde{\ell}I\sigma Y \\ Y^{-1}\omega I_0\lambda & \mathbb{1}+\breve{B}\end{vmatrix} \\ &\quad + 4\det\begin{vmatrix} 0 & \tilde{\ell}I\sigma^2 Y \\ Y^{-1}I_0\lambda & \mathbb{1}+\breve{B}\end{vmatrix} - 4\det\begin{vmatrix} 0 & \tilde{\ell}I_2 Y \\ Y^{-1}I_1\lambda & \mathbb{1}+\breve{B}\end{vmatrix}, \end{aligned}$$

i.e. according to (1.20) and (1.22) we get

$$\frac{\partial D}{\partial y}=4\det F_{2,0}+4\det F_{1,1}+4\det F_{0,2}-4\det K. \tag{1.44}$$

Then, using (1.26) and (1.44) we derive

$$\frac{\partial D}{\partial y}+\frac{\partial^3 D}{\partial x^3}=3\det F_{2,0}+6\det F_{1,1}+3\det F_{0,2}-4\det K.$$

Hence, it follows that

$$\frac{\partial^2 D}{\partial x \partial y} + \frac{\partial^4 D}{\partial x^4} = 3 \det F_{3,0} + 3 \det F_{2,1} - 3 \det F_{1,2}$$
$$- 3 \det F_{0,3} - 6 \det U + 4 \det U_0 . \tag{1.45}$$

Thus, according to (1.37) and (1.45) we obtain

$$3 \frac{\partial^2 D}{\partial t^2} - \frac{\partial^2 D}{\partial x \partial y} - \frac{\partial^4 D}{\partial x^4} = 12 \det U - 4 \det U_0 . \tag{1.46}$$

Moreover, with the help of (1.25) and (1.35) we are convinced that

$$\left(\frac{\partial D}{\partial t} \right)^2 + \left(\frac{\partial^2 D}{\partial x^2} \right)^2 = -4 \det (F_{1,0} F_{0,1}) , \tag{1.47}$$

and by virtue of (1.26) and (1.44) we obtain the equality

$$\frac{\partial D}{\partial y} + 4 \frac{\partial^3 D}{\partial x^3} = 12 \det F_{1,1} - 4 \det K . \tag{1.48}$$

We use the lemma proved before again. Put $A = U$. Taking $m = 1$, $n = r_0$, with (1.20) and (1.23) in mind, we derive $A_0 = \mathbb{1} + B$, $A_{1,1} = F_{0,0}$, $A_{1,2} = F_{1,0}$, $A_{2,1} = F_{0,1}$, $A_{2,2} = F_{1,1}$. Thus, on the basis of equality (1.12) the following relation holds:

$$D \det U = \det (F_{0,0} F_{1,1}) - \det (F_{1,0} F_{0,1}) . \tag{1.49}$$

Analogously, putting $A = U_0$ at $m = 1$, $n = r_0$ we derive $A_0 = \mathbb{1} + B$, $A_{1,1} = F_{0,0}$, $A_{1,2} = H_0$, $A_{2,1} = G_0$, and $A_{2,2} = K$; in accordance with equality (1.12) we derive the relation

$$D \det U_0 = \det (F_{0,0} K) - \det (G_0 H_0) . \tag{1.50}$$

Now we multiply equality (1.49) by 12 and from the result obtained subtract equality (1.50) multiplied by 4. As a result, we get

$$(12 \det U - 4 \det U_0) D + 12 \det (F_{1,0} F_{0,1})$$
$$= (12 \det F_{1,1} - 4 \det K) \det F_{0,0} + 4 \det (G_0 H_0) .$$

With the help of (1.8), (1.21), (1.25), and (1.46)–(1.48) we prove that this equality results in relation (1.15).

The theorem is proved.

It is to be noted that if the quantities $\omega_1, \ldots, \omega_{r_0}$, $\sigma_1, \ldots, \sigma_{r_0}$ are chosen to obey the condition

$$\omega_r^3 = \sigma_r^3 , \qquad r = 1, \ldots, r_0 ,$$

the solution of system (1.10) derived is independent, according to (1.1) and (1.4), of y and consequently, satisfies the system of equations

$$3 \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2}{\partial x^2} \left(3u^2 + \frac{\partial^2 u}{\partial x^2} + 8 \varphi \psi \right) = 0 ,$$

$$i \frac{\partial \varphi}{\partial t} = u \varphi + \frac{\partial^2 \varphi}{\partial x^2} , \qquad i \frac{\partial \psi}{\partial t} + u \psi + \frac{\partial^2 \psi}{\partial x^2} = 0 .$$

2. Invariant Manifold of System (1.10)

It follows from Theorem 1 that if the functions D, Φ, and Ψ defined by (1.1)–(1.8) satisfy the relations[1]

$$D=\bar{D}, \quad \Psi=\kappa\bar{\Phi}, \quad \kappa^2=1, \tag{2.1}$$

then the functions u, φ, and ψ defined by (1.9) belong to an invariant manifold $u=\bar{u}$, $\psi=\kappa\bar{\varphi}$ of system (1.10), and consequently, the functions

$$u=2\frac{\partial^2}{\partial x^2}\ln D, \quad \varphi=\frac{\Phi}{D} \tag{2.2}$$

are the solutions of the system of equations

$$\begin{gathered} 3\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}+\frac{\partial}{\partial x}\left(3u^2+\frac{\partial^2 u}{\partial x^2}+8\kappa|\varphi|^2\right)\right]=0, \\ i\frac{\partial \varphi}{\partial t}=u\varphi+\frac{\partial^2 \varphi}{\partial x^2}. \end{gathered} \tag{2.3}$$

The following theorem contains sufficient conditions for fulfilling relation (2.1).

Theorem 2. *If the quantities* $f_1, \ldots, f_{r_0}$, $\omega_1, \ldots, \omega_{r_0}$, $\sigma_1, \ldots, \sigma_{r_0}$ *entering into the definition of the matrix* B *and vectors* λ *and* ℓ *satisfy the conditions*

$$\begin{aligned} &1) \quad f_r \neq 0 \quad at \quad r=1, \ldots, r_0, \\ &2) \quad f_r=\bar{f}_r, \quad \sigma_r=-\bar{\omega}_r \quad at \quad r=1, \ldots, r_1, \\ &3) \quad f_{r_1+r_2+r}=\kappa\bar{f}_{r_1+r}, \quad \sigma_{r_1+r}=-\bar{\omega}_{r_1+r_2+r}, \\ &\qquad \sigma_{r_1+r_2+r}=-\bar{\omega}_{r_1+r} \quad \text{at} \quad r=1, \ldots, r_2, \end{aligned} \tag{2.4}$$

then the functions D, Φ, and Ψ obtained with the help of (1.1)–(1.8) satisfy relations (2.1).

Proof. Represent the matrix B in the following block form:

$$B=\begin{vmatrix} \alpha & \beta & 0 \\ 0 & 0 & -\gamma \\ a & b & 0 \end{vmatrix}, \tag{2.5}$$

where α is the minor at the intersection of the first r_1 rows and first r_1 columns; β is the minor at the intersection of the rows with numbers $r=1, \ldots, r_1$ and the columns with numbers $s=r_1+1, \ldots, r_1+r_2$; $-\gamma$ is the minor at the intersection of the rows with numbers $r=r_1+1, \ldots, r_1+r_2$ and the columns with numbers $s=r_1+r_2+1, \ldots, r_0$; a is the minor at the intersection of the rows with numbers $r=r_1+r_2+1, \ldots, r_0$ and the columns with numbers $s=1, \ldots, r_1$; and finally b is the minor at the intersection of the rows with numbers $r=r_1+r_2+1, \ldots, r_0$ and the columns

[1] Hereafter a bar above any quantity denotes complex conjugation and an asterisk denotes a Hermitian conjugation of matrices (and vectors)

with numbers $s=r_1+1,\dots,r_1+r_2$. Let

$$f=\mathrm{diag}\{f_1\exp[-4(\omega_1^3+\bar\omega_1^3)y],\dots,f_{r_1}\exp[-4(\omega_{r_1}^3+\bar\omega_{r_1}^3)y],\qquad g=\mathrm{diag}(g_1,\dots,g_{r_2}),\quad h=\mathrm{diag}(h_1,\dots,h_{r_2}),\tag{2.6}$$

where, according to the definition, at $r=1,\dots,r_2$ we have

$$g_r=f_{r_1+r}\exp[-4(\omega_{r_1+r}^3+\bar\omega_{r_1+r_2+r}^3)y],\qquad h_r=f_{r_1+r_2+r}\exp[-4(\bar\omega_{r_1+r}^3+\omega_{r_1+r_2+r}^3)y].\tag{2.7}$$

By virtue of (2.4) we derive that

$$f^*=f,\quad h=\kappa g^*.\tag{2.8}$$

Then, according to (1.1) and (2.4)–(2.8) we obtain

$$\alpha f=f\alpha^*,\quad af=h\beta^*,\quad bh^*=hb^*,\quad \gamma g^*=g\gamma^*.\tag{2.9}$$

Now we put

$$S=\begin{vmatrix} f & 0 & 0\\ 0 & 0 & \kappa g\\ 0 & h & 0\end{vmatrix},\qquad S_0=\det S.\tag{2.10}$$

According to (2.5)–(2.10) the following equalities hold:

$$S=S^*,\quad SB^*=BS^*=BS.$$

Hence, it follows that

$$S(\mathbb{1}+B^*)=(\mathbb{1}+B)S,\tag{2.11}$$

i.e. using inequalities $f_r\neq 0$ at $r=1,\dots,r_0$ we derive the equality

$$\det(\mathbb{1}+B)=\det(\mathbb{1}+B^*).$$

Therefore, the first relation in (2.1) is proved.

Let us prove the second one in (2.1). By virtue of (1.8) and (1.10) we have

$$S_0\bar\Phi=\det\begin{vmatrix} 0 & \lambda^* I_0\\ SI_2\ell & S(\mathbb{1}+B^*)\end{vmatrix},\qquad \Psi S_0=\det\begin{vmatrix} 0 & \tilde\ell IS\\ I_1\lambda & (\mathbb{1}+B)S\end{vmatrix}.\tag{2.12}$$

According to (1.4)–(1.6), (2.4), (2.6), (2.7), and (2.10) the following equalities are valid:

$$\tilde\ell IS=\lambda^* I_0,\quad I_1\lambda=\kappa SI_2\ell.\tag{2.13}$$

According to (2.11) and (2.13) the second relation in (2.1) follows from equalities (2.12).

The theorem is proved.

It follows from this theorem that if the quantities $\omega_1,\dots,\omega_{r_0}$ are chosen under the conditions

$$\begin{aligned}&1)\qquad \omega_r^2-|\omega_r|^2+\bar\omega_r^2=0\quad\text{at}\quad r=1,\dots,r_1,\\ &2)\qquad \omega_{r_1+r}^3+\bar\omega_{r_1+r_2+r}^3=0\quad\text{at}\quad r=1,\dots,r_2,\end{aligned}\tag{2.14}$$

the solution of system (2.3) thus obtained is independent of y, and consequently, satisfies the system of equations

$$3\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2}{\partial x^2}\left(3u^2+\frac{\partial^2 u}{\partial x^2}+8\kappa|\varphi|^2\right)=0, \quad i\frac{\partial\varphi}{\partial t}=u\varphi+\frac{\partial^2\varphi}{\partial x^2},$$

playing an important role in some branches of mathematical physics.

Now let us find out what requirements are to be imposed additionally on the quantities $f_1, \ldots, f_{r_0}, \omega_1, \ldots, \omega_{r_0}$ in order that the determinant D would not vanish at any real values of x, y, and t. The answer is in the following theorem.

Theorem 3. *If the quantities $f_1, \ldots, f_{r_0}, \omega_1, \ldots, \omega_{r_0}$ entering into the matrix B satisfy the conditions*

1) $$\operatorname{sign} f_1=\ldots=\operatorname{sign} f_{r_1}=\operatorname{sign}(\operatorname{Re}\omega_1)=\ldots=\operatorname{sign}(\operatorname{Re}\omega_{r_1}), \tag{2.15}$$

2) $$\operatorname{sign}(\operatorname{Re}\omega_{r_1+r_2+1})=\ldots=\operatorname{sign}(\operatorname{Re}\omega_{r_0}), \tag{2.16}$$

3) $$\kappa\operatorname{Re}[(\omega^3_{r_1+r}+\bar{\omega}^3_{r_1+r})\omega_{r_1+r_2+r}]>0 \quad \text{at} \quad r=1,\ldots,r_2, \tag{2.17}$$

the determinant $D=\det(\mathbb{1}+B)$ differs from zero at any real x, y, and t.

Proof. Consider a homogeneous system of linear algebraic equations

$$X+\alpha X+\beta Y=0, \quad Y-\gamma Z=0, \quad aX+bY+Z=0, \tag{2.18}$$

where the matrices α, β, γ, a, and b have earlier been defined with the help of representation (2.5) of the matrix B, and X and Y, Z are the column vectors with the r_1 and r_2 components, respectively. We show that if the conditions (2.15)–(2.17) are fulfilled, system (2.18) has only a trivial solution. With this purpose we make in (2.18) a substitution

$$X=f\hat{X}, \quad Y=h^*\hat{Y}, \quad Z=\hat{Z},$$

where the matrices f and h are defined by (2.6) and (2.7). As a result, we get the system

$$(f+\hat{\alpha})\hat{X}+\hat{\beta}\hat{Y}=0, \quad \hat{Y}-\hat{\gamma}\hat{Z}=0, \quad \hat{a}\hat{X}+\hat{b}\hat{Y}+\hat{Z}=0, \tag{2.19}$$

where

$$\hat{\alpha}=\alpha f, \quad \hat{\beta}=\beta h^*, \quad \hat{\gamma}=(h^*)^{-1}\gamma, \quad \hat{a}=af, \quad \hat{b}=bh^*. \tag{2.20}$$

According to (2.8), (2.9), and (2.20) we have

$$\hat{\alpha}^*=\hat{\alpha}, \quad \hat{\beta}^*=\hat{a}, \quad \hat{\gamma}^*=\hat{\gamma}, \quad \hat{b}^*=\hat{b}, \tag{2.21}$$

i.e. the matrices $f+\hat{\alpha}$, $\hat{b}$, and $\hat{\gamma}$ are Hermitian. Then, from system (2.19) there follows the equality

$$-\hat{X}^*(f+\hat{\alpha})\hat{X}+\hat{Y}^*\hat{b}\hat{Y}+\hat{Z}^*\hat{\gamma}\hat{Z}=0. \tag{2.22}$$

A simple analysis shows that if condition (2.17) is fulfilled and the equality

$$\begin{aligned}\operatorname{sign} f_1=\ldots=\operatorname{sign} f_{r_1}&=\operatorname{sign}(\operatorname{Re}\omega_1)=\ldots=\operatorname{sign}(\operatorname{Re}\omega_{r_1})\\ &=-\operatorname{sign}(\operatorname{Re}\omega_{r_1+r_2+1})=\ldots=-\operatorname{sign}(\operatorname{Re}\omega_{r_0})\end{aligned} \tag{2.23}$$

is valid, the matrices $-(f+\hat{\alpha})$, $\hat{b}$, and $\hat{\gamma}$ will simultaneously be either non-negative or non-positive. Moreover, by virtue of the first row of formula (2.23) we have $\det(f+\hat{\alpha})\neq 0$. Hence it follows that equality (2.22) holds only at $\hat{X}=0$. This means that for any solution of system (2.19) the following relations are valid:

$$\hat{Y}-\hat{\gamma}\hat{Z}=0, \quad \hat{b}\hat{Y}+\hat{Z}=0. \tag{2.24}$$

From the afore-said we get that all eigenvalues of the matrices $\hat{b}\hat{\gamma}$ and $\hat{\gamma}\hat{b}$ are non-negative. Consequently, the determinant of system (2.24) differs from zero, i.e. $\hat{Y}=\hat{Z}=0$. Thus, provided that conditions (2.17) and (2.23) are fulfilled, system (2.18) has only a trivial solution.

Consider now the matrices A_0 and A_1 of the form

$$A_0=\begin{vmatrix} f+\hat{\alpha} & \hat{\beta} \\ \hat{a} & \hat{b} \end{vmatrix}, \quad A_1=\hat{\gamma}.$$

According to (2.21) the matrices A_0 and A_1 are Hermitian. A simple analysis shows that if condition (2.17) is fulfilled and the equality

$$\operatorname{sign} f_1=\ldots=\operatorname{sign} f_{r_1}=\operatorname{sign}(\operatorname{Re}\omega_1)=\ldots=\operatorname{sign}(\operatorname{Re}\omega_{r_1})$$
$$=\operatorname{sign}(\operatorname{Re}\omega_{r_1+r_2+1})=\ldots=\operatorname{sign}(\operatorname{Re}\omega_{r_0})$$

holds, the matrices A_0 and A_1 will simultaneously be either non-negative or non-positive. Now we choose the matrix Q of the form

$$Q=\begin{vmatrix} \mathbb{1}_{r_2} & 0 \\ -q & \mathbb{1}_{r_2} \end{vmatrix},$$

where $\mathbb{1}_{r_2}$ is the unit matrix of order r_2 and $q=\hat{a}(f+\hat{\alpha})^{-1}$. On the basis of (2.21) we have $q^*=(f+\hat{\alpha})^{-1}\hat{\beta}$. One can easily see that the Hermitian matrix $\hat{A}_0=QA_0Q^*$ has the form

$$\hat{A}_0=\begin{vmatrix} f+\hat{\alpha} & 0 \\ 0 & \hat{b}-\hat{a}(f+\hat{\alpha})^{-1}\hat{\beta} \end{vmatrix}.$$

The matrices A_0 and $\hat{A}_0$ will simultaneously be either non-negative or non-positive. It follows from the afore-said that the Hermitian matrices

$$f+\hat{\alpha}, \quad A_1=\hat{\gamma}, \quad A_2=\hat{b}-\hat{a}(f+\hat{\alpha})^{-1}\hat{\beta} \tag{2.26}$$

will simultaneously be either non-negative or non-positive. Using the first equation of system (2.19) we express vector X through vector Y, i.e. we put

$$\hat{X}=-(f+\hat{\alpha})^{-1}\hat{\beta}\hat{Y}. \tag{2.27}$$

This is possible, since by virtue of the first row of formula (2.25) the equality $\det(f+\hat{\alpha})\neq 0$ holds. After substituting expression (2.27) into the third equation of system (2.19), we get

$$\hat{Y}-A_1\hat{Z}=0, \quad A_2\hat{Y}+\hat{Z}=0, \tag{2.28}$$

where the matrices A_1 and A_2 are defined by (2.26). In view of the afore-said all eigenvalues of the matrices A_1A_2 and A_2A_1 are non-negative. Hence, the determinant of system (2.28) differs from zero, i.e. $\hat{Y}=\hat{Z}=0$. Then, by (2.27) we get $\hat{X}=0$. Thus, provided that conditions (2.17) and (2.25) are fulfilled, system (2.18) has only a trivial solution.

Since conditions (2.15) and (2.16) result in the validity of either (2.23) or (2.25) conditions, it follows that conditions (2.15)–(2.17) guarantee the absence in (2.18) of a nontrivial solution, which is equivalent as is known, to the difference of the determinant D from zero.

The theorem is proved.

It is to be noted that conditions (2.14) do not contradict conditions (2.15)–(2.17).

System (2.3) is derived from system (1) by changing t by y and y by t. This means that performing the same change in solution (2.2) we should obviously derive a solution of system (1).

Let us make some remarks concerning this solution. In a typical case the solution obtained describes the interaction of r_1+r_2 solitary waves of two types. Waves of the first type have the form

$$u=\frac{2\mu_1^2}{\cosh^2[\mu_1(x+2v_1y-\tau_1t)]}, \qquad \varphi=0,$$

where the real parameters μ_1, v_1, and τ_1 satisfy the condition $\tau_1=4(\mu_1^2-3v_1^2)$, and are the well-known solutions [3] of the Kadomtsev-Petviashvili equation [4]. Waves of the second type have the form

$$u=\frac{2\mu_2^2}{\cosh^2[\mu_2(x+2v_2y-\tau_2t)]},$$

$$\varphi=c_0\frac{\exp[iv_2(x+2v_2y)+i\omega t]}{\cosh[\mu_2(x+2v_2y-\tau_2t)]}\exp[-i(\mu_2^2+v_2^2)y],$$

where the real parameters μ_2, v_2, τ_2, ω and the complex quantity c_0 satisfy the only condition

$$[\tau_2-4(\mu_2^2-3v_2^2)]\mu_2^2=4\kappa|c_0|^2,$$

and consequently, a wave of this type can exist under the condition

$$[\tau_2-4(\mu_2^2-3v_2^2)]\kappa>0.$$

In a typical case the interaction of all the waves is elastic, i.e. the result of interaction manifests itself in the relevant phase shifts of all interacting waves.

The situation changes radically if some additional conditions are imposed on the quantities $\omega_1,\ldots,\omega_{r_0}$. In this case in the solution obtained there appear waves having essentially different asymptotics as $t\to-\infty$ and $t\to\infty$. The simplest example of this phenomenon has been found in our paper [5]. However, that example does not exhaust all the possibilities of this phenomenon. A detailed analysis of all possible variants will be published elsewhere.

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