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Renormalization Theory in Four Dimensional Scalar Fields (II)

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Abstract. We interpret the results of the preceding paper (I) in terms of partial resummations of the perturbative series for the effective interaction. As an application we sketch how our resummation method leads to a simple summation rule leading to a convergent expansion for the Schwinger functions of the planar Φ_4^4-theory.

1. Introduction

In the preceding paper [Commun. Math. Phys. **100**, 545–590 (1985), referred to here as (I)] we have shown how to find "$n!$-bounds" on the perturbative coefficients of the effective potential and of the Schwinger functions. The method is based on a renormalization group approach to renormalization theory. In this paper we show that there is a natural way of collecting together various perturbative contributions through a general analysis of summation rules for divergent series. For each resummation we introduce some recursive relation between formal power series so that the resummation is by definition a non-formal solution of the same recursive relation. The similarity of this procedure with the non-perturbative methods based on the "beta function" is manifest. As applications we discuss the theory of the leading contribution to the effective potentials at momentum p as $p \to \infty$ and the convergence of the planar Φ_4^4-theory sketching a simple proof of the theorems of 't Hooft and Rivasseau [13].

Another aim of this paper is to provide some technical details not explicitly presented in (I); they have all been collected in Appendix A. In this paper the formulae preceded by a I [e.g. (I, 2.10)] refer to the paper (I).

Many of the ideas appearing in this paper overlap with those of references [1–14].

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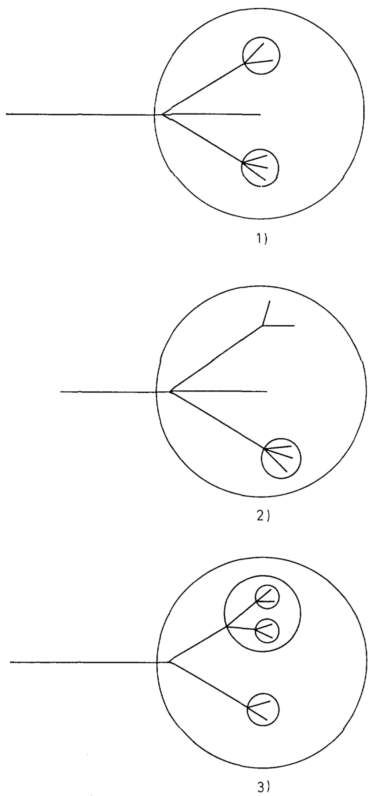
1.1. Resummations

The series in g, (I, 2.16), which have been implicity discussed in (I) are very likely divergent. In this section we discuss a summation rule which transforms them into better ones; this idea stems out of the appendix to [9] written by Rivasseau and applied by him to study the planar Φ_4^4-theory. This summation rule can be stated very simply in our formalism: given a dressed tree γ we say that it is "completely dressed" if every bifurcation of γ is enclosed in a frame (otherwise we'll call it partially dressed)



1) Completely dressed

2) partially dressed

3) superdivergent

Fig. 1.1

A dressed tree will be called "superdivergent" if it contains only dressed vertices and its bifurcations give always rise to only two new lines. The reason of the name "superdivergent" should be clear from Theorem 2 and the fact that these trees are those with the greatest f for a given n (see Sect. 7 of I). We can give two essentially equivalent, as far as the final result is concerned, summation rules which, in some sense are at the extremes of a variety of summation rules.

Consider a dressed tree γ whose frame structure is "weakly divergent" which means that no frame has solidly packed frames inside it, i.e. the subtree enclosed in a given frame is never completely dressed.

Call Ψ_∞ the operation which associates to a dressed tree its "skeleton" obtained by chopping off the dressed terminal lines those frames which are enclosing completely dressed trees, together with their content of lines and internal frames:

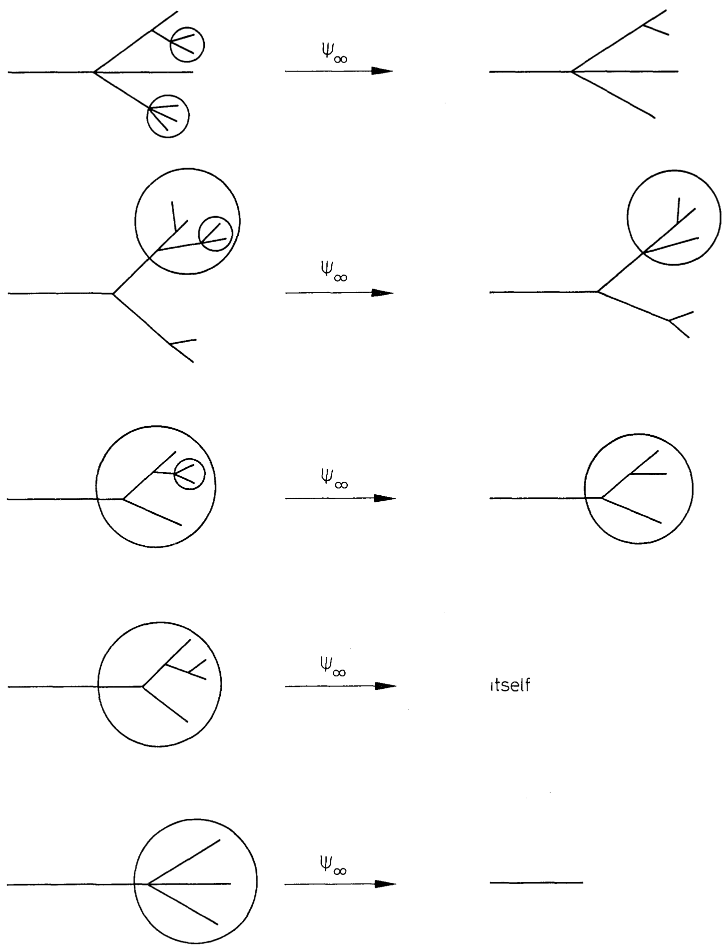


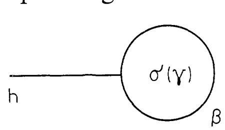
Fig. 1.2

Therefore $\Psi_\infty(\gamma)$ is a "weakly divergent tree".

We can similarly define a "weakly 2-divergent" tree as a tree which does not have subtrees which are superdivergent, and the operation Ψ_2 over the generic dressed tree γ as the operation of chopping from γ all the superdivergent ends. Therefore $\Psi_2(\gamma)$ is a "weakly 2-divergent" tree. Clearly

$$\Psi_\infty^2 = \Psi_\infty, \qquad \Psi_2^2 = \Psi_2. \tag{1.1}$$

Given now a weakly divergent tree we associate to it a tree obtained attaching to its final lines the factor corresponding to an extra frame with a completely divergent tree inside,



$$\equiv \lambda_h^{(\beta)}(\sigma') \tag{1.2}$$

Therefore given a weak divergent tree ($\Psi_\infty(\gamma) = \gamma$), one gets from it a new tree whose final lines have factors $\lambda_h^{(\beta)}(\sigma)$, the frequency being the frequency of the next bifurcation; this frequency will be fixed by γ if the corresponding bifurcation is not inside any other frame, otherwise it will be summed. We do that for the weak 2-divergent trees ($\Psi_2(\gamma)=\gamma$), the frame we attach to any end has inside a superdivergent tree.

The $\lambda_k^{(\beta)}(\sigma)$'s were called in (I) $r_\beta^{(N)}(\sigma;k)$ [see (I, 6.44) and the remark before (I, 7.15)]; here they are denoted by λ as they are playing the role of "bare" constants and in the case of weak 2-divergent trees their σ correspond to superdivergent trees. β will be as usual 0, 2, 2′, 4. Suppose now to drop any restriction on N and n; therefore at the frequency k the interaction looks formally

$$V_R^{(k)}(\varphi)=\sum_{k(\gamma)=k}^{\gamma}\frac{1}{n(\gamma)}V(\gamma)=\sum_{\substack{k(\hat\gamma)=k\\ \hat\gamma\in\left\{\substack{\text{weak 2-div.}\\ \text{trees}}\right\}}}^{\hat\gamma}\left(\sum_{\Psi_2(\gamma)=\hat\gamma}^{\gamma}\frac{1}{n(\gamma)}V(\gamma)\right). \tag{1.3}$$

The second sum in (1.3) corresponds just to the operation of attaching at all the ends of $\hat\gamma$ all the possible superdivergent trees obtaining finally

$$V_R^{(k)}(\varphi)=\sum_{\substack{k(\hat\gamma)=k\\ \hat\gamma\in\left\{\substack{\text{weak 2-div.}\\ \text{trees}}\right\}}}^{\hat\gamma}\frac{1}{n(\hat\gamma)}V_R(\hat\gamma), \tag{1.4}$$

where $V_R(\hat\gamma)$ differs from $V(\hat\gamma)$ only because at the end of every final line of $\hat\gamma$ there is a factor

$$\lambda^{(\beta)}(h)\equiv g^{(\beta)}+\left(\sum_\sigma \lambda_h^{(\beta)}(\sigma)\right), \tag{1.5}$$

which we describe graphically as

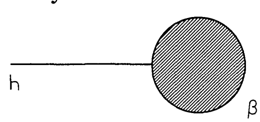


Fig. 1.3

Remarks. a) The combinatorial factor $n(\hat\gamma)$ is the usual one [see (I, 4.1)], remembering that two final lines in $\hat\gamma$ have to be considered different if their ends have different β's.

b) In (1.5) $\sum_\sigma$ runs over the shapes of all the possible superdivergent trees; $\lambda^{(\beta)}(h)$ is a formal power series in g.

From the estimates of (I) we cannot infer that (1.5) converges. Nevertheless we can observe that $\lambda^{(\beta)}(k)$ verifies a recursion relation due to the fact that a superdivergent tree has the following structure:

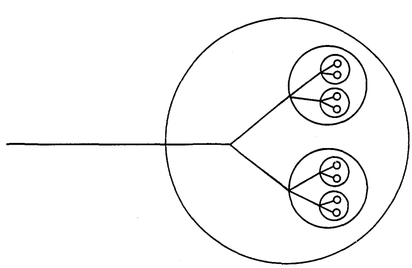
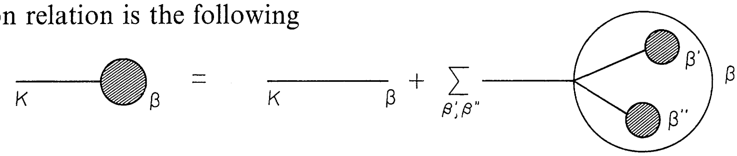


Fig. 1.4

and inside each frame this structure is repeated an arbitrary number of times. The recursion relation is the following



$$\lambda^{(\beta)}(k)=g^{(\beta)}+\sum_{\beta',\beta''}\sum_{0}^{k}{}_h\int d\zeta K^{(h)}_{\beta;\beta',\beta''}(\zeta)\,\lambda^{(\beta')}(h)\,\lambda^{(\beta'')}(h)\,, \tag{1.6}$$

where $g^{(4)}=g$, $g^{(2)}=m$, $g^{(2')}=a$, $g^{(0)}=0$ and we are assuming the slightly more general interaction

$$V^{(N)}(\varphi)=-g\int:\varphi_\xi^4:d\xi-m\int:\varphi_\xi^2:d\xi-a\int:(\partial\varphi_\xi)^2:d\xi\,. \tag{1.7}$$

Equation (1.6), written explicitly, omitting the $\beta=0$ case which is simpler and can be treated separately, becomes

$$\begin{aligned}
\lambda^{(4)}(k)=g&+\frac{1}{2}\sum_{0}^{k}{}_h\left[\binom{4}{2}^2 2!\int d(\xi-\eta)\,(C^{(h)2}_{\xi\eta}-C^{(h-1)2}_{\xi\eta})\right]\lambda^{(4)}(h)^2\\
&+\sum_{0}^{k}{}_h\left[\binom{2}{1}\binom{4}{2}\int d(\xi-\eta)\,(C^{(h)}_{\xi\eta}-C^{(h-1)}_{\xi\eta})\right]\lambda^{(4)}(h)\,\lambda^{(2)}(h)\\
\lambda^{(2)}(k)=m&+\frac{1}{2}\sum_{0}^{k}{}_h\left[\binom{4}{3}^2 3!\int d(\xi-\eta)\,(C^{(h)3}_{\xi\eta}-C^{(h-1)3}_{\xi\eta})\right]\lambda^{(4)}(h)^2\\
&+\sum_{0}^{k}{}_h\left[\binom{4}{2}\binom{2}{2}2!\int d(\xi-\eta)\,(C^{(h)2}_{\xi\eta}-C^{(h-1)2}_{\xi\eta})\right]\lambda^{(4)}(h)\,\lambda^{(2)}(h)\\
&+\sum_{0}^{k}{}_h\left[\binom{4}{2}2!\int d(\xi-\eta)\,((\underline{\partial}C^{(h)}_{\xi\eta})^2-(\underline{\partial}C^{(h-1)}_{\xi\eta})^2)\right]\lambda^{(4)}(h)\,\lambda^{(2')}(h)\\
&+\frac{1}{2}\sum_{0}^{k}{}_h\left[\binom{2}{1}^2\int d(\xi-\eta)\,(C^{(h)}_{\xi\eta}-C^{(h-1)}_{\xi\eta})\right]\lambda^{(2)}(h)^2\\
\lambda^{(2')}(k)=a&-\frac{1}{2}\sum_{0}^{k}{}_h\left[\binom{4}{3}^2 3!\frac{1}{8}\int d(\xi-\eta)\,(\underline{\xi}-\underline{\eta})^2\,(C^{(h)3}_{\xi\eta}-C^{(h-1)3}_{\xi\eta})\right]\lambda^{(4)}(h)^2\\
&-\frac{1}{2}\sum_{0}^{k}{}_h\left[\binom{2}{1}^2\frac{1}{8}\int d(\xi-\eta)\,(\underline{\xi}-\underline{\eta})^2\,(C^{(h)}_{\xi\eta}-C^{(h-1)}_{\xi\eta})\right]\lambda^{(2)}(h)^2\\
&+\sum_{0}^{k}{}_h\left[\binom{2}{1}^2\frac{1}{8}\int d(\xi-\eta)\,(\underline{\xi}-\underline{\eta})\cdot\underline{\partial}(C^{(h)}_{\xi\eta}-C^{(h-1)}_{\xi\eta})\right]\lambda^{(2')}(h)\,\lambda^{(2)}(h)\,.
\end{aligned} \tag{1.8}$$

It is clear that the iterative solution of (1.6) for $\lambda^{(\beta)}(k)$ $(\beta:2,2',4)$ expressed as a power series in g (if $m=a=0$) or $\underline{g}=(g,m,a)$ in the more general case (which can be worked out exactly in the same way) and substituted in $V_R(\hat{\gamma})$ [Eq. (1.4)] gives the formal power series (1.3) for $V_R^{(k)}(\varphi)$ as before. However it might be that (1.6) admits true non-formal solutions admitting an asymptotic expansion in $\underline{g}=(g,m,a)$, near $\underline{g}=0$, which necessarily must agree with the power series giving $\underline{\lambda}(k)=(\lambda^{(4)}(k),\lambda^{(2)}(k),\lambda^{(2')}(k))$ as functions of $\underline{g}$. Then one can state the following "summation rule":

Summation Rule. The sum $\left(\sum_{\Psi_2(\gamma)=\hat{\gamma}}^{\gamma} \frac{1}{n(\gamma)} V(\gamma)\right)$ is by definition the expression $\frac{1}{n(\hat{\gamma})} V_R(\hat{\gamma})$ computed with the same rule used for $V(\gamma)$ but with the "coupling constants" associated to the final lines given by the true solutions $\lambda^{(\beta)}(k)$ of Eqs. (1.6), ..., (1.8). This will be a great improvement if the solutions have a well defined ultraviolet limit as $k \to \infty$.

We now investigate the resummations of the superdivergent trees in some detail.

1.2. The Resummation of Superdivergent Trees

Let's write the asymptotic expressions of Eqs. (1.8): we define

$$\lambda^{(4)}(k) \equiv \lambda(k),\ \gamma^{-2k}\lambda^{(2)}(k) = \mu(k),\ \lambda^{(2')}(k) = \alpha(k), \tag{1.9}$$

then for very large k, from (1.8) we have

$$\begin{aligned} \lambda(k) - \lambda(k-1) &= G^{(4,4)}\lambda^2(k) + G^{(4,2)}\lambda(k)\mu(k), \\ \mu(k) - \mu(k-1) &= -(\gamma^2-1)\,\mu(k) + M^{(4,4)}\lambda^2(k) + M^{(2,2)}\mu^2(k) \\ &\quad + M^{(4,2)}\lambda(k)\,\mu(k) + M^{(4,2')}\lambda(k)\,\alpha(k), \\ \alpha(k) - \alpha(k-1) &= -A^{(4,4)}\lambda^2(k) - A^{(2,2)}\mu^2(k) + A^{(2',2)}\alpha(k)\,\mu(k), \end{aligned} \tag{1.10}$$

where

$$G^{(i,j)} \equiv \lim_{k\to\infty} G^{(i,j)}(k), \qquad M^{(i,j)} \equiv \gamma^2 \lim_{k\to\infty} \gamma^{-2k} M^{(i,j)}(k), \qquad A^{(i,j)} \equiv \lim_{k\to\infty} A^{(i,j)}(k), \tag{1.11}$$

and $G^{(i,j)}(h)$, $M^{(i,j)}(h)$, $A^{(i,j)}(h)$ are the coefficients of $\lambda^{(i)}(h)\,\lambda^{(j)}(h)$ in the three equations (1.8) respectively. The connection between Eqs. (1.10) and (1.8) together with the explicit expressions for $G^{(i,j)}$, $A^{(i,j)}$, $M^{(i,j)}$ is discussed in Appendix B.

We study (1.10) as if k were a continuous parameter and the finite difference equations a system of differential equations. The possibility of doing so is also discussed in Appendix B. Therefore we have

$$\begin{aligned} \frac{d\lambda}{dt} &= \bar{G}^{(4,4)}\lambda^2 + \bar{G}^{(4,2)}\lambda\mu, \\ \frac{d\mu}{dt} &= -\mu + \bar{M}^{(4,4)}\lambda^2 + \bar{M}^{(2,2)}\mu^2 + \bar{M}^{(4,2)}\lambda\mu + \bar{M}^{(4,2')}\lambda\alpha, \\ \frac{d\alpha}{dt} &= -\bar{A}^{(4,4)}\lambda^2 - \bar{A}^{(2,2)}\mu^2 + \bar{A}^{(2',2)}\alpha\mu, \end{aligned} \tag{1.12}$$

where $t = k\cdot(\gamma^2-1)$, $\bar{G}^{(i,j)} = \lim_{\gamma\to 1}(\gamma^2-1)^{-1}G^{(i,j)}$ and same definitions for $\bar{M}^{(i,j)}$ and $\bar{A}^{(i,j)}$.

We determine the attractive manifold in the (λ, μ, α) space. We define

$$\mu = \mu(\lambda, \alpha) = \tfrac{1}{2}(L\lambda^2 + 2I\lambda\alpha + A\alpha^2) + (\text{higher orders}). \tag{1.13}$$

Neglecting the orders higher than 2 we have

$$\begin{aligned}\frac{d\mu}{dt} &\equiv \dot{\mu} = L\lambda\dot{\lambda} + I(\dot{\lambda}\alpha + \lambda\dot{\alpha}) + A\alpha\dot{\alpha} \\ &= L\bar{G}^{(4,4)}\lambda^3 + I(\bar{G}^{(4,4)}\lambda^2\alpha - \bar{A}^{(4,4)}\lambda^3) + A(-\bar{A}^{(4,4)}\alpha\lambda^2) \\ &= -\tfrac{1}{2}(L\lambda^2 + 2I\lambda\alpha + A\alpha^2) + \bar{M}^{(4,4)}\lambda^2 + \bar{M}^{(4,2')}\lambda\alpha \\ &\quad + (\text{higher order terms}),\end{aligned} \tag{1.14}$$

which by consistency implies

$$A = 0, \quad \tfrac{1}{2}L = \bar{M}^{(4,4)}, \quad I = \bar{M}^{(4,2')} \tag{1.15}$$

and

$$\mu(\lambda, \alpha) = (\bar{M}^{(4,4)}\lambda^2 + \bar{M}^{(4,2')}\lambda\alpha) + (\text{higher orders}). \tag{1.16}$$

Equation (1.12) becomes, neglecting terms of order higher than three

$$\begin{aligned}\frac{d\lambda}{dt} &= \bar{G}^{(4,4)}\lambda^2 + \bar{G}^{(4,2)}(\bar{M}^{(4,4)}\lambda^3 + \bar{M}^{(4,2')}\lambda^2\alpha), \\ \frac{d\alpha}{dt} &= -\bar{A}^{(4,4)}\lambda^2 + \bar{A}^{(2',2)}(\bar{M}^{(4,4)}\lambda^2\alpha + \bar{M}^{(4,2')}\lambda\alpha^2),\end{aligned} \tag{1.17}$$

with (see Appendix B)

$$\bar{G}^{(4,4)} = \frac{4}{c_1}\bar{A}^{(4,4)} > 0, \quad \bar{G}^{(4,2)} = 4\bar{A}^{(2',2)} > 0, \tag{1.18}$$

and c_1 is a positive constant which can be explicitly computed.

Looking now for a solution $(\lambda(k), \alpha(k)) \to 0$ as $k \to \infty$ it is easy to realize that (1.17) can be approximated by

$$\begin{aligned}\frac{d\lambda}{dt} &= \bar{G}^{(4,4)}\lambda^2, \\ \frac{d\alpha}{dt} &= -\kappa\bar{G}^{(4,4)}\lambda^2, \quad \kappa = \frac{c_1}{4},\end{aligned} \tag{1.19}$$

which implies $\alpha = -\kappa\lambda$ and

$$\lambda(k) = \frac{\lambda(0)}{1 - \lambda(0)G^{(4,4)}k}. \tag{1.20}$$

Looking at Eq. (1.8) it follows that, with obvious definitions,

$$\begin{aligned}\lambda(0) &= g + f_1(\lambda, \mu, \alpha), \\ \mu(0) &= m + f_2(\lambda, \mu, \alpha), \\ \alpha(0) &= a + f_3(\lambda, \mu, \alpha).\end{aligned} \tag{1.21}$$

Therefore it must be possible to choose g, m, a in such a way that $(\lambda(k), \mu(k), \alpha(k)) \to 0$ as $k \to \infty$ and if g is small enough we can write

$$\lambda(k) \simeq \frac{g}{1 - (\bar{G}^{(4,4)}\log\gamma^2)gk}, \tag{1.22}$$

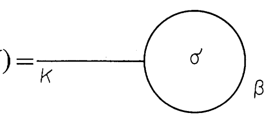
where

$$\bar{G}^{(4,4)} = \lim_{\gamma\to 1} \frac{G^{(4,4)}}{\log\gamma^2} > 0 . \tag{1.23}$$

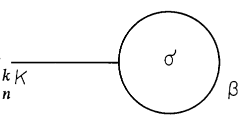
If $g<0$, $\lambda(k)$ is defined everywhere in k, while if $g>0$, it has a pole for $k = \frac{1}{(\bar{G}^{(4,4)}\log\gamma^2)g}$.

1.3. More about Resummations

We want to connect now the $\lambda^{(\beta)}(k)$ that we have discussed in the last section, with the "form factors" $\bar{r}_\beta^{(N)}(w, \tilde{\mathscr{G}}, k)$ introduced in Sect. 7 of (I), [see Eq. (I, 7.18)] and with the more standard notion of the bare coupling constant. Let us collect together some definitions introduced in Sect. 1.1 and some new ones:

$\lambda_k^{(\beta)}(\sigma, N) =$ 

(1.2)

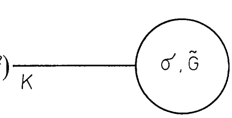
$\lambda_n^{(\beta)}(k; N) = \sum_{\substack{k(\sigma)=k \\ v(\sigma)=n}} \sigma$ 

(1.24)

$$\lambda^{(\beta)}(k; N) = \sum_{n=1}^{\infty} \lambda_n^{(\beta)}(k; N) , \tag{1.5}$$

Remark. The first equation is just Definition (1.2) where we have written explicitly the dependence on the cutoff N. The same thing can be said for the third line which is just the definition (1.5) and the meaning of $\lambda_n^{(\beta)}(k; N)$ is completely clear. Nevertheless the fact that we have written the explicit N-dependence is important as, for instance, the general recursive relation as for instance, the analogue of (1.35) in the non-planar case, interpreted as a true finite difference equation is different for N fixed from that obtained in the $N\to\infty$ limit because not every tree with a high enough number of final lines n can contribute to the kernel if N is kept fixed. The resummations we consider here are, anyway, those obtained in the $N\to\infty$ limit.

Comparing now the definition (1.24) with the definition of $\bar{r}_\beta^{(N)}(w, \tilde{\mathscr{G}}; k)$, we have the following relation

$\gamma^{-2k\varepsilon(\beta)} \cdot \lambda_n^{(\beta)}(k; N) = \sum_{f=1}^{n-1} \sum_{\substack{n(w)=n \\ f(w)=f}} w \sum_{\substack{\mathscr{G} \\ w\,\text{fixed}}} \left\{ \sum_{\substack{k(\sigma)=k \\ v(\sigma)=n \\ f(\sigma)=f \\ w(\sigma)=w}} \sigma\, \chi(\sigma, \mathscr{G})\right.$ 

$\left.\right\}$

$$= \sum_{f=1}^{n-1} \sum_{\substack{n(w)=n \\ f(w)=f}} w \sum_{\substack{\mathscr{G} \\ w\,\text{fixed}}} \bar{r}_\beta^{(N)}(w, \tilde{\mathscr{G}}; k) ,$$

$$\text{where} \quad \begin{cases} \varepsilon(\beta) = 1 & \text{if} \quad \beta = 2 \\ \varepsilon(\beta) = 0 & \text{otherwise} \end{cases} \tag{1.25}$$

[see Eq. (I, 7.21)].

From (1.25) and the estimates proved in Sect. 7 of (I), Theorem 2, we have

$$\begin{aligned}
\gamma^{-2k\varepsilon(\beta)}|\lambda_n^{(\beta)}(k;N)| &\leqq \sum_{1}^{n-1}{}_f \sum_{\substack{n(w)=n\\ f(w)=f}}{}_w \sum_{\substack{\mathscr{G}\\ w\,\text{fixed}}} c_7^{n+(f-1)} f! \sum_0^f{}_j \frac{(bk)^j}{j!} \\
&\leqq \sum_1^{n-1}{}_f\, c_{12}^{n+f} c_7^{n+f-1} c_9^n (n-f)!\, f! \sum_0^f{}_j \frac{(bk)^j}{j!} \\
&\leqq c_{13}^n n! \sum_0^{n-1}{}_j \frac{(bk)^j}{j!}\,,
\end{aligned} \tag{1.26}$$

where we used Theorem 2 and Lemmas 4 and 5 of Sect. 7 of (I). Therefore we have the estimate

$$\gamma^{-2k\varepsilon(\beta)}|\lambda_n^{(\beta)}(k;N)| \leqq c_{13}^n n! \sum_0^{n-1}{}_j \frac{(bk)^j}{j!}\,.$$

The connection between $\lambda^{(\beta)}(k,N)$ and the "bare" coupling constant on one side and the "running" coupling constant on the other is the following

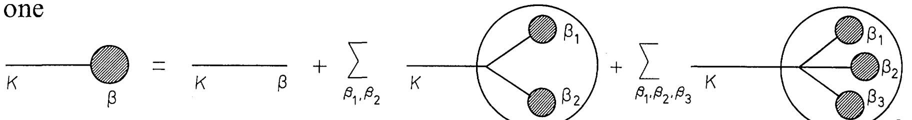
$$\begin{aligned}
&\text{“Bare coupling constants”} = \lambda^{(\beta)}(k=N;N); \qquad \beta = 0,2,2',4\,, \\
&\text{“Running coupling constants”}^{1} = \lambda^{(\beta)}(k) = \lim_{N\to\infty} \lambda^{(\beta)}(k;N); \qquad \beta=0,2,2',4\,.
\end{aligned} \tag{1.27}$$

The running coupling constants are the solutions of the finite difference Eq. (1.6), or better, of the more general ones that we are going to discuss shortly. The bare coupling constants are defined as formal power series by the following equation:

$$\begin{aligned}
V_R^{(N)} = V^{(N)} + \sum_2^\infty{}_l \Delta_l^{(N)} V = -\lambda^{(4)}(N,N) \int_\Lambda d\xi : \varphi_\xi^{[\leqq N]4} : \\
-\lambda^{(2)}(N,N) \int_\Lambda d\xi : \varphi_\xi^{[\leqq N]2} : -\lambda^{(2')}(N,N) \int_\Lambda d\xi : (\partial \varphi_\xi^{[\leqq N]})^2 : -\lambda^{(0)}(N,N)
\end{aligned} \tag{1.28}$$

[see Eqs. (I, 6.1) and (I, 6.2), where there the coefficients $\lambda_n^{(\beta)}(N,N)$ were called $r_n(N,\beta)$].

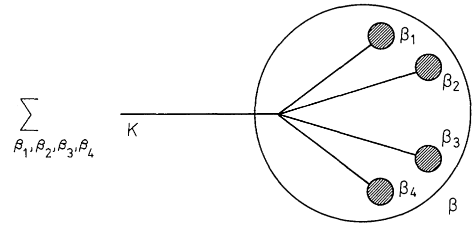
We are now in a better position to discuss more general resummations. Remembering how we performed a resummation over the superdivergent trees it is clear how one can imagine more general ones involving larger classes of completely dressed trees. For instance a possible recursion relation is the following one



(1.29)

which can be still generalized adding extra completely dressed trees, in the obvious way, the next step being of course to add

1 We don't claim, the definition of "running coupling constant" used here is same as that used by physicists, but they should be, at least, connected



(1.30)

As in (1.29) the last term is of order λ^3, if we can find for the equation (1.29) a solution $\bar{\lambda}(k)=\{\gamma^{-2\varepsilon(\beta)k}\lambda^{(\beta)}(k)\}$ going to zero as $k\to\infty$, then it should coincide in the limit $k\to\infty$ with the solution, found in the previous section, of Eq. (1.6).

We can also think of a much more drastic resummation where one resums all the trees with frames, partially and completely dressed, to be left with only undressed trees with R at all the bifurcations and decorated tops. A decorated top will correspond to the complete "running coupling constant" $\lambda^{(\beta)}(k)$, with k the frequency of the first vertex where its line merges,

$\lambda^{(\beta)}(k)=$ 

(1.31)

Assuming again that also in this case one could prove that $\gamma^{-2\varepsilon(\beta)k}\lambda^{(\beta)}(k)\to 0$ as $k\to\infty$ one could try to proceed in this way: let $V^{(k)}_{n(k),R}(\varphi^{[\leqq k]})$ be the contribution to $V^{(k)}_R(\varphi^{[\leqq k]})$ of the undressed trees with n final lines, $n\leqq n(k)$, and with the coefficients $\lambda^{(\beta)}(k)$ instead of $\underline{g}^{(\beta)}$ appended to the final lines, then [see Eqs. (1.3), (1.4)]

$$V^{(k)}_{n(k),R}(\varphi^{[\leqq k]})=\sum_{1}^{n(k)}{}_n\left\{\sum_{\substack{\hat{\gamma}\\ k(\hat{\gamma})=k\\ v(\hat{\gamma})=n\\ \hat{\gamma}:\text{unframed}}}\frac{1}{n(\hat{\gamma})}V_R(\hat{\gamma})\right\}. \tag{1.32}$$

$V^{(k)}_{n(k),R}$ is well defined and if one expands in formal power series of $\underline{g}$ (1.32) the standard perturbative expression is obtained again. The improvement would be of course that, due to the fact $\gamma^{-2\varepsilon(\beta)k}\lambda^{(\beta)}(k)\to 0$ as $k\to\infty$ (to prove), which is what is commonly called "asymptotic freedom," one can hope to give a well defined meaning to the limit $k\to\infty$ of a measure

$$\exp V^{(k)}_{n(k),R}(\varphi^{[\leqq k]})\,P(d\varphi^{[\leqq k]}), \tag{1.33}$$

provided $n(k)$ is chosen going to infinity as $k\to\infty$ in the appropriate way. This program is far from being completely settled but in our opinion deserves further investigation.

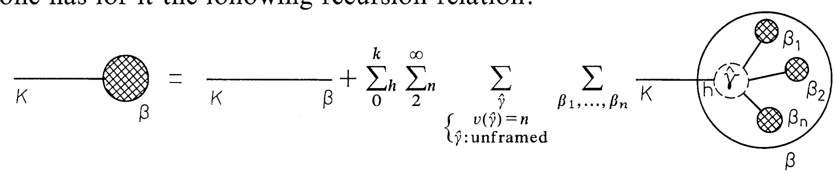
We show now that also without solving the general resummation equations what we have proved is enough to make some definite statements about planar theory.

1.4. Some Heuristic Results on Planar Theory

The planar Φ^4_4-theory is defined as the perturbative Φ^4_4-theory where only the planar Feynman diagrams are taken into account. The obvious simplification of this theory as compared with the complete one is in the fact that the number of F-diagrams is much smaller

$$\#\,[F\text{-diagrams (top.) in } \Phi_4^4 \text{ of order } n] \leqq c_0^n n!\,,$$
$$[F\text{-diagrams (top.) in } \Phi_4^4\text{-planar, of order } n] \leqq c_0^n\,, \tag{1.34}$$

where c_0 has been explicitly estimated in ('t Hooft, [13]). This suggests, remembering the nature of the problems connected to the Borel summability of the Φ_4^4-perturbative expansions that these results should be easier to obtain in this case as well as stronger ones. In fact in this case one can study the most general summation rule in the following way: Remembering the definition (1.31) of $\lambda^{(\beta)}(k)$, one has for it the following recursion relation:



(1.35)

which, defining

$$\bar\lambda^{(\beta)}(k)=\gamma^{-2k\varepsilon(\beta)}\lambda^{(\beta)}(k), \qquad \bar g^{(\beta)}=\gamma^{-2k\varepsilon(\beta)}g^{(\beta)}, \qquad \begin{array}{ll}\varepsilon(\beta)=1 & \text{if } \beta=2,\\ \varepsilon(\beta)=0 & \text{otherwise}\end{array} \tag{1.36}$$

[see (I, 7.12)] can be written analytically

$$\bar\lambda^{(\beta)}(k)=\bar g^{(\beta)}+\sum_{h=0}^{k}\sum_{n=2}^{\infty}\sum_{\beta_1,\dots,\beta_n}\sum_{k_1,\dots,k_n\geqq n}\gamma^{-2k\varepsilon(\beta)}\mathscr{K}^{(n)}_{\beta;\beta_1,\dots,\beta_n}(h;k_1,\dots,k_n) \tag{1.37}$$
$$\cdot\,\bar\lambda^{(\beta_1)}(k_1)\dots\bar\lambda^{(\beta_n)}(k_n)\,,$$

where

$$\gamma^{-2\varepsilon(\beta)k}\mathscr{K}^{(n)}_{\beta;\beta_1,\dots,\beta_n}(h;k_1,\dots,k_n)=\sum_{\substack{\hat\gamma\\ v(\hat\gamma)=n\\ k(\hat\gamma)=-1\\ \hat\gamma:\text{unframed}}}^{*} K^{(n)}_{\beta;\beta_1,\dots,\beta_n}(\hat\gamma)\,. \tag{1.38}$$

Here $\hat\gamma$ is, as in the proof of Theorem 2 of (I), a tree without frames with an R to all its bifurcations except its lowest one, whose frequency we call h, and with root $k(\hat\gamma)=-1$. $\sum_{\hat\gamma}^{*}$ means that the sum is over all the $\hat\gamma$ with the constraint that the frequencies of the bifurcations where the final lines merge and the frequency h are kept fixed. Observe that in (1.37) the sum over h, the frequency of the unrenormalized bifurcation, goes from 0 to k instead of from 0 to N.

It is clear that the estimate of the kernel in (1.38) mimics exactly the estimates of Theorems 1 and 2 of (I) with the further modification that each final line brings a factor $\lambda^{(\beta_i)}(k_i)$ instead of g, β_i being the index running over $2, 2', 4$ appended to each final line and k_i the frequency of the bifurcation where the final line merges. Therefore with the same notations of Sect. 7 of (I),

$$K^{(n)}_{\beta;\beta_1,\dots,\beta_n}(\hat\gamma)=\sum_{\tilde{\mathscr{G}}}K^{(n)}_{\beta;\beta_1,\dots,\beta_n}(\hat\gamma,\tilde{\mathscr{G}})\,, \tag{1.39}$$

and $\exists c_\varepsilon>0$, such that for $\varepsilon>0$

$$|K^{(n)}_{\beta;\beta_1,\dots,\beta_n}(\hat\gamma,\tilde{\mathscr{G}})|\leqq c_\varepsilon^n\prod_{\{B\}_{\hat\gamma}}\gamma^{-(1-\varepsilon)(h(B)-h(B'))}\,. \tag{1.40}$$

As in the planar case $\sum_{\substack{\mathscr{G},\beta_{\mathscr{G}}\\ v(\mathscr{G})=n}} 1 \leqq \bar{c}^n$, where $\bar{c}$ is a constant proportional to c_0

$$|K^{(n)}_{\beta;\beta_1,\dots,\beta_n}(\hat{\gamma})| \leqq (\bar{c})^n \prod_{\{B\}_{\hat{\gamma}}} \gamma^{-\frac{1}{2}(h(B)-h(B'))}, \tag{1.41}$$

where $\{B\}_{\hat{\gamma}}$ is the set of all the bifurcations of $\hat{\gamma}$. Therefore from (1.38), (1.41) and Lemma 2 of (I)

$$\sum_{\beta_1,\dots,\beta_n} \sum_{k_1,\dots,k_n} \gamma^{-2k\varepsilon(\beta)} |\mathscr{K}^{(n)}_{\beta;\beta_1,\dots,\beta_n}(h;k_1,\dots,k_n)| \leqq (\mathrm{const})^n, \qquad \forall h \leqq k. \tag{1.42}$$

Proceeding now as in the case of the resummation of the superdivergent trees we can write

$$\begin{aligned}\bar{\lambda}^{(\beta)}(k) - \bar{\lambda}^{(\beta)}(k-1) = -(\gamma^2-1)\,\delta_{2,\beta}\bar{\lambda}^{(\beta)}(k) + \sum_{2}^{\infty}{}_n \sum_{\beta_1,\dots,\beta_n} \sum_{k_1,\dots,k_n} \gamma^{-2k\varepsilon(\beta)} \cdot \gamma^{2\varepsilon(\beta)}\\ \cdot \mathscr{K}^{(n)}_{\beta;\beta_1,\dots,\beta_n}(k;k_1,\dots,k_n)\,\bar{\lambda}^{(\beta_1)}(k_1)\dots\bar{\lambda}^{(\beta_n)}(k_n).\end{aligned} \tag{1.43}$$

Due to (1.42) the expression in the right-hand side of (1.43) is convergent, provided $|\bar{\lambda}^{(\beta)}(k)| \leqq \delta$ with δ small enough.

We call $\mathscr{B}^{(\beta)}(\bar{\lambda}^{(\cdot)}(\cdot);k)$ the functional defined by the series in (1.43), then

$$\bar{\lambda}^{(\beta)}(k) - \bar{\lambda}^{(\beta)}(k-1) = -(\gamma^2-1)\,\delta_{2,\beta}\bar{\lambda}^{(\beta)}(k) + \mathscr{B}^{(\beta)}(\bar{\lambda}^{(\cdot)}(\cdot);k). \tag{1.44}$$

If $\bar{\lambda}^{(\beta)}(k) \to 0$ as $k \to \infty$ (1.44) tends to coincide with Eq. (1.10) that one gets resumming only the superdivergent trees.

Let us call $\bar{\lambda}^{(\beta)}(k;g)$ the solution of (1.44) and assume it has the same properties that we expect for the solution of (1.10), namely that:

a) It is analytic in g, for complex g, $\mathrm{Re}\,g<0$ and $|g|$ sufficiently small.

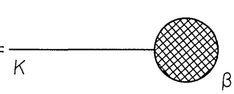
b) It satisfies on the boundary of the analyticity region the following estimates

$$\left|\frac{\partial^{q+1}\bar{\lambda}^{(\beta)}(k;g)}{\partial^{q+1}}\right| \leqq (\mathrm{const})^{q+1}(q+1)!\,k^q. \tag{1.45}$$

$V_R^{(k)}(\varphi)$ has to be defined as [see Eq. (1.4)]

$$V_R^{(k)}(\varphi) = \sum_{\substack{k(\hat{\gamma})=k\\ \hat{\gamma}:\,\mathrm{unframed}}} \hat{\gamma}\, \frac{1}{n(\hat{\gamma})} V_R(\hat{\gamma}), \tag{1.46}$$

where $V_R(\hat{\gamma})$ differs from $V(\hat{\gamma})$ only because at the end of every final line of $\hat{\gamma}$ there is a factor

$\lambda^{(\beta)}(k) =$ 

(1.31)

Apart from that, $V_R^{(k)}(\varphi)$ has exactly the same expression as (I, 7.45) with the only other *crucial* difference being that the estimates analogous to (I, 7.48), don't have the factor $n!$ in the planar case we are considering.[2] The generic truncated Schwinger function $\hat{S}^T(f;p)$ built from this $V_R^{(k)}(\varphi)$ is therefore analytic for g

2 This implies the convergence of the planar perturbation theory as a power series in the $\bar{\lambda}^{(\beta)}(k)$'s. Note that to have convergence one only needs that $\bar{\lambda}^{(\beta)}(k,g)$ is small for g small, which should be an immediate consequence of (1.44) and of the properties of the $\mathscr{B}^{(\beta)}(\cdot)$ functional

complex, $\operatorname{Re} g<0$ and $|g|$ sufficiently small and finally satisfies the appropriate bounds on the boundary of this region. Reexpressing the "running" coupling constants $\bar{\lambda}^{(\beta)}(k, g)$ as a power series of g one gets again the formal power series of $\hat{S}^T(f; p)$. Moreover the following estimate holds:

$$\left|\hat{S}^T(f;p)-\sum_0^n{}_k\, g^k S_k^T(f;p)\right| \leqq (\text{const})^{n+1} g^{n+1}(n+1)! \tag{1.47}$$

in the same analiticity region which follows from the assumptions a) and b) on $\bar{\lambda}^{(\beta)}(k; g)$, from (1.42) and from the estimate (1.26); (1.47) proves the Borel summability of the renormalized planar expansion which is asymptotic to the Schwinger function, [13].

Appendix (A)

We collect in this appendix some technical proofs of the results discussed in (I). All the formulas quoted here refer to (I).

Lemma 1 (By Giovanni Felder, Zürich). *Let $\tilde{\mathscr{G}}$ be an unlabelled Feynman graph with n vertices, and let γ be a tree with n endpoints. Then the number $N(\tilde{\mathscr{G}}, \gamma, \{n_\beta\}_\gamma)$ of labellings of $\tilde{\mathscr{G}}$ compatible with γ and such that for all vertices B the subgraph of $\tilde{\mathscr{G}}$ corresponding to B has n_B external lines, is bounded above by $c_\varepsilon^n n(\sigma)\exp\varepsilon\sum_B n_B$, for all $\varepsilon>0$, and some constant c_ε, if σ is the shape of the tree γ.*

Proof. Consider $\tilde{\mathscr{G}}$, γ, $\{n_B\}_\gamma$ fixed. Let γ_B be the subtree of γ with root $h(B)$, and $N_B(J)$ the number of ways of choosing and labelling a subgraph of $\tilde{\mathscr{G}}$ compatible with γ_B and having an external line connected to the vertex J of $\tilde{\mathscr{G}}$. Let furthermore $B_1, \dots, B_{S_B}$ be the vertices following B in γ. Since the subgraphs $\tilde{\mathscr{G}}_{B_1}, \dots, \tilde{\mathscr{G}}_{B_{S_B}}$ corresponding to $B_1, \dots, B_{S_B}$ have to be connected together, there exists at least one tree diagram T_B with vertices $B_1, \dots, B_{S_B}$ whose lines correspond to propagators connecting $\tilde{\mathscr{G}}_{B_1}, \dots, \tilde{\mathscr{G}}_{B_{S_B}}$. Let d_{B_i} be the number of lines of T_B emerging from B_i. We have the estimate:

$$N_B(J) \leqq \left\{\prod_1^{S_B}{}_i\left(\max_{J'\in\tilde{\mathscr{G}}} N_{B_i}(J')\right)\right\} \sum_{\substack{d_{B_i} d_{B_{S_B}}\geqq 1\\ \Sigma_i (d_{B_i}-1)=S_B-2}} \left(\prod_1^{S_B}{}_i\,(n_{B_i})^{d_{B_i}-1}\right) \frac{S_B(S_B-2)!}{\prod_1^{S_B}{}_i\,(d_{B_i}-1)!}, \tag{F.1}$$

where the last ratio is the Cayley formula for the number of rooted trees T_B with fixed coordination numbers (see Moon, J.W.: Enumerating labelled trees. In: Graph Theory and Theoretical Physics. Harary, F. ed. London, New York: Academic Press, 1967), and $\prod_1^{S_B}{}_i\,(n_{B_i})^{d_{B_i}-1}$ is a bound on the number of ways of choosing external lines of $\tilde{\mathscr{G}}_{B_i}$ corresponding to the lines of T_B. The sum over d_{B_i} can be performed explicitly:

$$N_B(J) \leqq \left\{\prod_1^{S_B}{}_i\left(\max_{J'} N_{B_i}(J')\right)\right\} S_B\left(\textstyle\sum_i n_{B_i}\right)^{S_B-2}, \tag{F.2}$$

and using $x^k \leqq k!\varepsilon^{-k}\exp\varepsilon k$, $\sum_B (S_B - 1) = n-1$ we get:

$$N(\gamma, \tilde{\mathscr{G}}, \{n_B\}_\gamma) \leqq \bar{c}_\varepsilon^n \Big(\prod_{B\in\gamma} S_B!\Big) \exp\varepsilon \sum_B n_B \,. \tag{F.3}$$

But $\Big(\prod_{B\in\gamma} S_B!\Big)\Big/ n(\sigma) = \prod_{B\in\gamma}\Big(S_B!\Big/\prod_i t_{i,B}!\Big)$ (where $t_{i,B}$ are the multiplicities of the different tree shapes of the trees that start from B) is just the number of ways of drawing the shape σ by choosing at each vertex how to order the trees starting from it: this number is bounded by the number of ways of drawing all the trees with n endpoints, which, by the same argument used to count the trees is bounded by c^n for some constant c. □

Proof of Theorem 1. To prove Theorem 1 we proceed in a recursive way. Start considering the innermost unmarked boxes, each of them has a well defined frequency, a fixed number of vertices inside (true vertices or marked boxes shrunk to points) and a certain number of lines going out from it. First one has to find an estimate for the coefficient of the Wick monomial corresponding to the external lines going out from the generic innermost unmarked box, then one has to consider the next generation boxes (all the boxes here are unmarked). In the next step the previous boxes behave essentially as if they were shrunk to points (due to the fact that they are on a smaller scale as their associate frequency is higher) and therefore as generalized vertices from which an arbitrary number lines of different type $(\varphi, \partial\varphi, D, S, D^1)$ can go out. One has to prove again that the coefficient of this generic box satisfies the estimate of the theorem, then the proof is completed as the argument can be iterated as many times as needed until we get the final box enclosing the whole $\mathscr{G}$.

We start considering the innermost box in Fig. 1a

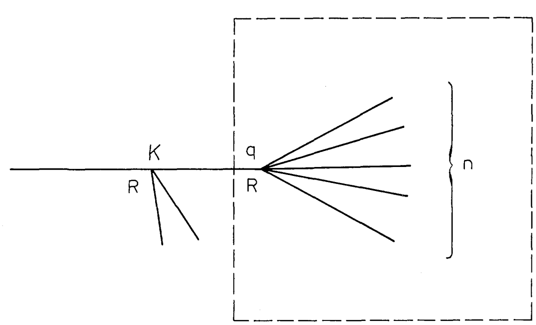


Fig. 1a

The box B corresponds to the subtree of γ_a in the dotted frame. The contribution from this subtree can be written in the following way

$$I_B(\varphi)_{(\Delta_1\times\ldots\times\Delta_l)} = \int_{\Delta_1\times\ldots\times\Delta_l} d\eta_1 \ldots d\eta_l \int_{(\Delta)^{n-l}} dx_1 \ldots dx_{n-l} \cdot F_B^{(q)}(\eta_1, \ldots, \eta_l; x_1, \ldots, x_{n-l}) : P^{[\leqq k]}_{\mathscr{G},B,\beta_{\mathscr{G}}}(\varphi):, \tag{1a}$$

where Δ has linear size γ^{-k} and l is the number of coordinates from which the external legs of the box B start. $P^{[\leqq k]}_{\mathscr{G},B,\beta_{\mathscr{G}}}(\varphi)$ is defined as in Eq. (7.3): $P^{[\leqq k]}_{\mathscr{G}_B,\beta_{\mathscr{G}_B}}$, with $\mathscr{G}_B$

the subgraph of $\mathscr{G}$ contained in B. The estimate we need for the contribution of the box B is an estimate for

$$\mathscr{I}_{\Delta_1 \times \ldots \times \Delta_l}(B) \equiv \int_{\Delta_1 \times \ldots \times \Delta_l} d\eta_1 \ldots d\eta_l \int_{(\Delta)^{n-l}} dx_1 \ldots dx_{n-l} |F_B^{(q)}(\eta_1, \ldots, \eta_l; x_1, \ldots, x_{n-l})| \tag{2a}$$

or $\mathscr{I}(B) \equiv \mathscr{I}_{(\Delta \times \ldots \times \Delta)}(B)$.

B has a well defined frequency q, a fixed number of vertices inside: n, some of them, possibly none, can be marked boxes shrunk to points, with n different coordinates $x_1, x_2, \ldots, x_n$, and each one with an index β running in the set $\{0, 2, 2'\}$, where here 0 means that from this vertex four lines of type φ come out, 2 means that two lines "φ" come out and $2'$ that two lines of "type $\partial\varphi$" come out. Let's call n_0, n_2, n_2' the number of vertices with $\beta: 0, 2, 2'$ respectively and $l_0, l_2, l_{2'}$ the number of half internal lines going out from the vertices of type $0, 2, 2'$ respectively. The following relations hold

$$4n_0 = l_0 + N_{B_0}, \quad 2n_2 = l_2 + N_{B_2}, \quad 2n_{2'} = l_{2'} + N_{B_{2'}}, \tag{3a}$$

where N_{B_i} are the external lines going out from the vertices of type "i". Therefore $N_{B_{0,2}} = N_{B_0} + N_{B_2}$ is the total number of lines "φ" going out from B, before the $\mathscr{R}$-operation has been applied, and $N_{B_2^1}$ is the total number of lines "$\partial\varphi$" going out also before the $\mathscr{R}$-operation.

Remembering the definition of the truncated expectation [see (3.8)] and that $\mathscr{E}_{[q]}^T$ is a truncated expectation with respect to $\varphi^{[q]}$, it follows that its contribution associated to $(\mathscr{G}, B)$ is of the following type:

$$C(n, \mathscr{G} \cap B) \times \left[\sum_{\tau} \prod_{\lambda' \notin \tau} C_{\lambda'}^{(q-1)} \prod_{\lambda \in \tau} (C_\lambda^{(q)} - C_\lambda^{(q-1)}) \right], \tag{4a}$$

where τ represent a subset of the set of lines of $B \cap \mathscr{G}$ which has the property that $\bigcup_{\lambda \in \tau} \lambda$ is a connected set (see [15]). $C(n, \mathscr{G} \cap B)$ is a function bounded by (const)^n, which takes care of the possible permutations of the half lines. The nature of the covariances $C_{\lambda'}^{(q-1)}$ or $(C_\lambda^{(q)} - C_\lambda^{(q-1)})$ depends on the nature of the half lines which join to form λ or λ'. Equation (4a) and the fact that $\bigcup_{\lambda \in \tau} \lambda$ is a connected set implies that from the truncated expectation, associated to B, we can extract a factor

$$e^{-\kappa\gamma^q(|x_{i_1} - x_{i_2}| + |x_{i_2} - x_{i_3}| + \ldots + |x_{i_{n-1}} - x_{i_n}|)}, \quad \kappa > 0, \tag{5a}$$

where $(i_1, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$. This factor can always be bounded above by

$$e^{-\kappa\gamma^q d(x_1, x_2, \ldots, x_n)}, \tag{6a}$$

where $d(x_1, \ldots, x_n)$ is the length of the shortest polygonal path connecting all the points $x_1, \ldots, x_n$.

In the estimate of each covariance we have that:

1/2 line of type φ brings a factor γ^q,

1/2 line of type $\partial\varphi$ brings a factor γ^{2q}.

Moreover, remembering (7.12), any vertex of type 2 has an extra factor γ^{2q}. From all that it follows that the product of covariances associated to B brings a factor we can upper bound with

$$c^n e^{-\kappa\gamma^q d(x_1,\ldots,x_n)}\gamma^{q(l_0+l_2+2l_{2'})}\gamma^{2qn_2}=c^n e^{-\kappa\gamma^q d(x_1,\ldots,x_n)}\gamma^{q(4n-N_B-N_{B_{2'}})}. \tag{7a}$$

The $\mathscr{R}$-operation is effective when the number of external lines of B is less than or equal to 4 (to be more precise when the "k-dimension" of the Wick monomial is less than or equal to 4) and in this case it substitutes some φ's or $\partial\varphi$'s with some other fields which have the appropriate zeroes. As the Wick monomial $P^{[\leqq k]}_{\mathscr{G},B,\beta_{\mathscr{G}}}(\varphi)$ has been divided by its zeroes [see (I, 7.2)] in $F^{(q)}_B$ we have a factor of this kind

$$\prod_{\{i,j\}}(\gamma^k|\eta_i-\eta_j|)\prod_{\{l,s\}}(\gamma^k|\eta_l-\eta_s|)\prod_{\{r,t\}}(\gamma^k|\eta_r-\eta_t|)^2, \tag{8a}$$

where the first product runs over all the zeroes produced by the fields of type D going out from B in number of $n_B(D)$, the second over all the zeroes produced by the fields of type D^1 [let their number be $n_B(D^1)$] and the third refers to the $n_B(S)$ fields of type S. It follows immediately that

$$n_B(D)+n_B(D^1)+2n_B(S)<4, \tag{9a}$$

and, after the $\mathscr{R}$-operation has been applied:

$$\begin{aligned} N_B\equiv N_{B_{0,2}}+N_{B_{2'}}&=n_B(\varphi)+n_B(D)+n_B(S)+n_B(\partial\varphi)+n_B(D^1),\\ N_{B_{2'}}&=n_B(\partial\varphi)+n_B(D^1) \end{aligned} \tag{10a}$$

hold.

The factor (8a) can be written in the following way

$$[(8a)]=(\gamma^{-(q-k)[n_B(D)+n_B(D^1)+2n_B(S)]})\cdot\left(\prod_{\{i,j\}}(\gamma^q|\eta_i-\eta_j|)\prod_{\{l,s\}}(\gamma^q|\eta_l-\eta_s|)\prod_{\{r,t\}}(\gamma^q|\eta_r-\eta_t|)^2\right), \tag{11a}$$

and $\mathscr{I}_{\Delta_1\times\ldots\times\Delta_l}(B)$ can be estimated by

$$\begin{aligned}\mathscr{I}_{\Delta_1\times\ldots\times\Delta_l}(B)\leqq&\Big\{c^n e^{-\frac{\kappa}{2}\gamma^k d(\Delta_1,\ldots,\Delta_l)}\gamma^{q(4n-N_B-N_{B_{2'}})}\gamma^{-4q(n-1)}\gamma^{-4k}\gamma^{k(N_B+N_{B_{2'}})}\\ &\cdot\gamma^{-(q-k)(n_B(D)+n_B(D^1)+2n_B(S))}\Big\}\\ &\cdot\left(\int_{\Delta_1\times\ldots\times\Delta_l}\frac{d\eta_1\ldots d\eta_l\,dx_1\ldots dx_{n-l}}{|\Delta^{(k)}|\,|\Delta^{(q)}|^{n-1}}e^{-\frac{\kappa}{2}\gamma^q d(\eta_1,\ldots,\eta_l,x_1,\ldots,x_{n-l})}\right.\\ &\left.\cdot\left(\prod_{\{i,j\}}(\gamma^q|\eta_i-\eta_j|)\prod_{\{l,s\}}(\gamma^q|\eta_l-\eta_s|)\prod_{\{r,t\}}(\gamma^q|\eta_r-\eta_t|)^2\right)\right),\end{aligned} \tag{12a}$$

where

a) $\gamma^{-4k}\gamma^{-4q(n-1)}$ is the volume factor due to $|\Delta^{(k)}|\,|\Delta^{(q)}|^{n-1}$, ($\Delta^{(q)}$ has linear size γ^{-q}, $\Delta^{(k)}$, γ^{-k}).

b) The Wick monomial has been divided to make it of "k-dimension" $=0$ by the factor $\gamma^{k(N_B+N_{B_{2'}})}$.

c) $d(\Delta_1, \ldots, \Delta_l)$ is the length of the smallest polygonal joining $\{\Delta_1, \ldots, \Delta_l\}$.
The ( ) of (12a) can be estimated by

$$\frac{1}{|\Delta^{(k)}|\,|\Delta^{(q)}|^{n-1}} \int_{\Delta_1 \times \ldots \times \Delta_l} d\eta_1 \ldots d\eta_l \, dx_1 \ldots dx_{n-l} \, e^{-\frac{\kappa}{2}\gamma^q d(\eta_1, \ldots, x_{n-l})}(1+(\gamma^q d(\eta_1, \ldots, x_{n-l}))$$
$$\cdot (1+(\gamma^q d(\eta_1, \ldots, x_{n-l}))^4) \leqq (\text{const}). \tag{13a}$$

Therefore, using (10a),

$$\mathscr{I}_{(\Delta_1 \times \ldots \times \Delta_l)}(B) \leqq (\tilde{c})^n e^{-\kappa\gamma^k d(\Delta_1, \ldots, \Delta_l)} \gamma^{-(q-k)[n_B(\varphi)+2n_B(\partial\varphi)+2n_B(D)+3n_B(D^1)+3n_B(S)-4]}, \tag{14a}$$

which is the final estimate for the innermost boxes.

We summarize here the origin of the various factors in the { } of (12a),

$$[\text{Covariances}] \sim c^n e^{-\kappa\gamma^k d(\Delta_1, \ldots, \Delta_l)} \gamma^{q(4n - N_B - N_{B_{2'}})},$$
$$[\text{Volume factors and normal.}] \sim c\gamma^{-4k} \gamma^{-4q(n-1)} \gamma^{k(N_B + N_{B_{2'}})}, \tag{15a}$$
$$[\mathscr{R}\text{-operation}] \sim c\gamma^{-(q-k)(n_B(D)+n_B(D^1)+2n_B(S))}.$$

Consider now a next-generation box $\bar{B}$ containing innermost boxes, with k its frequency and h that of the next one surrounding $\bar{B}$. Inside it there will be a certain number of innermost boxes $B_1, B_2, \ldots, B_r$, of frequency $q_1, \ldots, q_r$ and also some vertices: $\bar{n}$ (true vertices or marked boxes shrunk to points, as in the case of the innermost ones)

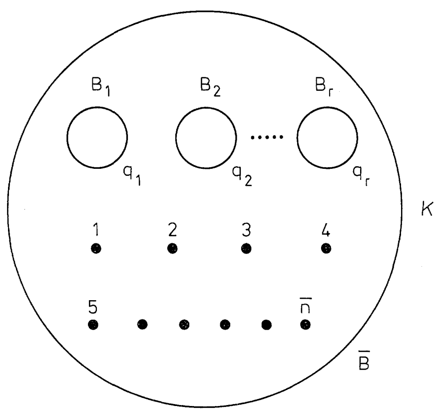


Fig. 2a

The contribution of a innermost box B will be written in the following way:

$$I_B(\varphi) = \bar{\mathscr{I}}(B) \int \frac{d\eta_1 \ldots d\eta_l \, dx_1 \ldots dx_{n-l}}{|\Delta^{(k)}|\,|\Delta^{(q)}|^{n-1}} \frac{F_B^{(q)}(\eta_1, \ldots, \eta_l; x_1, \ldots, x_{n-l})}{\bar{\mathscr{I}}(B)} : P^{[\leqq k]}_{\mathscr{G}, B, \beta_{\mathscr{G}}}(\varphi):, \tag{16a}$$

where

$$\bar{\mathscr{I}}(B) \equiv c^n \gamma^{-(q-k)[n_B(\varphi)+\ldots+3n_B(S)-4]}. \tag{17a}$$

Let us denote the contribution of the box $\bar{B}$ in the following way:

$$I_{\bar{B}}(\varphi) = \int d\bar{x} F_{\bar{B}}^{(k)}(\bar{x}) : P^{[\leqq h]}_{\mathscr{G}, \bar{B}, \beta_{\mathscr{G}}}(\varphi):, \tag{18a}$$

where $\bar{x}$ is the set of all the coordinates inside $\bar{B}$, including also those inside the preceding boxes. We look for an estimate of

$$\mathscr{I}_{\Delta_1 \times \ldots \times \Delta_\varrho}(\bar{B}) \equiv \int_{\Delta_1 \times \ldots \times \Delta_\varrho} d\bar{x} |F_{\bar{B}}^{(k)}(\bar{x})|, \tag{19a}$$

where $\Delta_1, \ldots, \Delta_\varrho$ are tesserae of linear size γ^{-h}, where the coordinates of the external lines of $\bar{B}$ live. As for the innermost boxes the estimate of (19a) is made in two parts: in the first, which is the more significant one, we have to consider all the factors $\gamma^{-k\alpha}$ which are produced by the covariances, the external lines, the volume factors and the $\mathscr{R}$-operation and prove that they collect all together in the right way; the second part consists just in proving that the remaining integration can be bounded by $(\text{const})^{n+\sum_1^r n_i}$.

a) *First Part.* When the Wick monomials of $I_{B_1}(\varphi), \ldots, I_{B_l}(\varphi)$ enter in the truncated expectation at the frequency k, as $:P^{[\leq k]}_{\mathscr{G}, B_i, \beta_{\mathscr{G}}}(\varphi):$ has "k-dimension" $=0$, it is clear that the half lines coming from them in the covariance do *not* give any γ^k factor. Vice versa the fields of the $P^{[\leq k]}_{\mathscr{G}, B_i, \beta_{\mathscr{G}}}(\varphi)$ monomials which are not contracted and remain as external lines of $\bar{B}$, now at the frequency h, as they have to keep their appropriate factors in the denominator, but now with γ^h instead of γ^k, produce a factor

$$\gamma^{-(k-h)[n_{B_i,\bar{B}}(\varphi) + 2n_{B_i,\bar{B}}(\partial\varphi) + 2n_{B_i,\bar{B}}(D) + 3n_{B_i,\bar{B}}(D^1) + 3n_{B_i,\bar{B}}(S)]}, \tag{20a}$$

where $n_{B_i,\bar{B}}(V)$ is the number of external lines of type V, of B_i which are also external lines of $\bar{B}$, V being $\varphi, \partial\varphi, D, \ldots$. The vertices in $\bar{B}$ are treated exactly as for the innermost boxes. Let $\bar{n}$ be the number of vertices in $\bar{B}$. Let r be the number of the innermost boxes. Let $\tilde{N}_{\bar{B}}$ be the number of external lines of $\bar{B}$ starting from the $\bar{n}$ vertices. Therefore the covariances of the truncated expectation $\mathscr{E}^T_{[k]}$ produce a factor bounded by

$$c^{\sum_1^r n_i + r} e^{-\kappa\gamma^k \tilde{d}(x_1, \ldots, x_{\bar{n}}, \bar{\eta}_1, \ldots, \bar{\eta}_{\bar{r}})} \gamma^{k(4\bar{n} - \tilde{N}_{\bar{B}} - \tilde{N}_{\bar{B}_{2'}})}, \qquad \bar{r} \geqq r, \tag{21a}$$

where the $\bar{\eta}_1, \ldots, \bar{\eta}_{\bar{r}}$ are coordinates of vertices in the innermost boxes $B_1, \ldots, B_r$, whose choice depends on $(\mathscr{G}, \bar{B}, \beta_{\mathscr{G}})$, $\tilde{d}(x_1, \ldots, x_{\bar{n}}, \bar{\eta}_1, \ldots, \bar{\eta}_{\bar{r}})$ is defined

$$\tilde{d}(x_1, \ldots, x_{\bar{n}}; \bar{\eta}_1, \ldots, \bar{\eta}_{\bar{r}}) = \sum_s |\bar{\eta}_s - \bar{\eta}_{s+1}| + \sum_j |x_{l_j} - \eta_j| + \sum_i |x_{l_i} - x_{l_{i+1}}|, \tag{22a}$$

where the two coordinates in each $|\ |$ belong always to different B_i's, and the exponential factor $e^{-\kappa\gamma^k \tilde{d}(x_1, \ldots, \bar{\eta}_{\bar{r}})}$ is an upper bound to the exponential decay factor produced by the covariances associated to the contribution $(\mathscr{G}, \bar{B}, \beta_{\mathscr{G}})$. $\mathscr{I}_{\Delta_1 \times \ldots \times \Delta_r}(\bar{B})$ [see Eq. (19a)] satisfies, before introducing the $\mathscr{R}$-operation,

$$\begin{aligned}
\mathscr{I}_{\Delta_1 \times \ldots \times \Delta_s}(\bar{B}) \leqq & \Big\{ (\bar{c})^{\sum_1^r i n_i + \bar{n}} (\bar{\mathscr{I}}(B_1) \ldots \bar{\mathscr{I}}(B_r)) e^{-\frac{\kappa}{2}\gamma^h d(\Delta_1, \ldots, \Delta_r)} \\
& \cdot \gamma^{k(4\bar{n} - \tilde{N}_{\bar{B}} - \tilde{N}_{\bar{B}_{2'}})} \gamma^{-4h} \gamma^{-4k(\bar{n}-1)} \gamma^{h(\tilde{N}_{\bar{B}} + \tilde{N}_{\bar{B}_{2'}})} \gamma^{-(k-h)\sum_1^r \left[\sum_V d(V) n_{B_i,\bar{B}}(V) - 4\right]} \Big\} \\
& \cdot \Bigg(\int \frac{dx_1 \ldots dx_{\bar{n}}}{|\Delta^{(h)}| \, |\Delta^{(k)}|^{\bar{n}-1}} \prod_{1}^{r} {}_i \int \frac{d\eta_1^{(i)} \ldots d\eta_{l_i}^{(i)} \, dx_1^{(i)} \ldots dx_{n_i - l_i}^{(i)}}{|\Delta^{(k)}| \, |\Delta^{(q_i)}|^{n_i - 1}} e^{-\frac{\kappa}{2}\tilde{d}(x_1, \ldots, x_{\bar{n}}, \bar{\eta}_1, \ldots, \bar{\eta}_{\bar{r}})} \\
& \cdot F_{B_i}^{(q_i)}(\underline{\eta}^{(i)}, \underline{x}^{(i)}) / \bar{\mathscr{I}}(B_i) \Bigg).
\end{aligned} \tag{23a}$$

The integration part of (23a) will be estimated later on.

Examine now the { } part of (21a): As

$$\tilde{N}_{\bar{B}}=\tilde{n}_{\bar{B}}(\varphi)+\tilde{n}_{\bar{B}}(\partial\varphi), \qquad \tilde{N}_{\bar{B}_{2'}}=\tilde{n}_{\bar{B}}(\partial\varphi), \tag{24a}$$

where $\tilde{n}_{\bar{B}}(V)$ are the external lines of type V, from $\bar{B}$, starting from the vertices $1, \ldots, \bar{n}$ of $\bar{B}$ itself, the total number of external lines of $\bar{B}: N_{\bar{B}}$ satisfies

$$N_{\bar{B}}=\sum_V\left(\tilde{n}_{\bar{B}}(V)+\sum_1^r{}_i\, n_{B_i,\bar{B}}(V)\right)\equiv\sum_V n_{\bar{B}}(V), \tag{25a}$$

moreover, as $\tilde{n}_{\bar{B}}(D)=\tilde{n}_{\bar{B}}(D^1)=\tilde{n}_{\bar{B}}(S)=0$,

$$\begin{aligned}&\tilde{N}_{\bar{B}}+\tilde{N}_{\bar{B}_{2'}}+\sum_1^r{}_i \sum_{\{\varphi,\partial\varphi,D,D^1,S\}}{}_V\, d(V)\, n_{B_i,\bar{B}}(V)\\&=[n_{\bar{B}}(\varphi)+2n_{\bar{B}}(\partial\varphi)+2n_{\bar{B}}(D)+3n_{\bar{B}}(D^1)+3n_{\bar{B}}(S)],\end{aligned} \tag{26a}$$

then

$$\begin{aligned}\{(23a)\}\leqq&(\mathscr{I}(B_1)\ldots\mathscr{I}(B_r))\,(\bar{c})^{\sum_1^r{}_i n_i+\bar{n}}\,e^{-\frac{\kappa}{2}\gamma^h d(\Delta_1,\ldots,\Delta_r)}\\&\cdot\gamma^{-(k-h)\left[\sum_{\{\varphi,\partial\varphi,D,D^1,S\}}{}_V\, d(V)n_{\bar{B}}(V)-4\right]},\end{aligned} \tag{27a}$$

which is also the estimate for $\mathscr{I}_{\Delta_1\times\ldots\times\Delta_r}(\bar{B})$ if (23a) is bounded by $(\text{const})^{\sum_1^r{}_i n_i+\bar{n}}$.

Before looking at the effects of the $\mathscr{R}$-operation, observe that one does not have to worry about a possible proliferation of factors $(\text{const})^{n_i}$ going to the next bifurcations of lower frequency; this is clear remembering that from each point at most four lines can go out, which can contribute at most to four bifurcations.

We consider now what is going to change if we take into account the effect of the $\mathscr{R}$-operation. When the Wick monomial in (18a) has "h-dimension" $\leqq 4$, some lines change into other ones; this, as we are going to see, does not change the estimate (27a) but excludes the possibility of having monomials with "h-dimension" $\leqq 4$ in (18a). To prove it, examine the effect of the $\mathscr{R}$-operation only on the { } part of (23a). Let's consider first the following example: $n_{\bar{B}}(\varphi)=2$ and $n_{\bar{B}}(V)=0\ \forall V\neq\varphi$ before the $\mathscr{R}$-operation. The application of $\mathscr{R}$, see Sect. 6 of (I), changes $n_{\bar{B}}(\varphi)$ into zero and $n_{\bar{B}}(S)$ into two if the corresponding index β is 1 (see the end of Sect. 7.4 of I); each S brings a factor $\gamma^{-2(k-h)}$ [due to its second order zero, see (11a)] and therefore the original

$$\gamma^{-(k-h)n_{\bar{B}}(\varphi)}=\gamma^{-(k-h)2} \quad \text{becomes} \quad \gamma^{-(k-h)n_{\bar{B}}(S)}\gamma^{-2(k-h)n_{\bar{B}}(S)}=\gamma^{-(k-h)3n_{\bar{B}}(S)}. \tag{28a}$$

The same happens for all the other monomials with "h-dimension $\leqq 4$ except that two new fields are produced: S^1 and T both of "h-dimension" $=4$. Therefore after the $\mathscr{R}$-operation is applied, again

$$\begin{aligned}\{(23a)\}\leqq&(\mathscr{I}(B_1)\ldots\mathscr{I}(B_r))\,(\bar{c})^{\sum_1^r{}_i n_i+\bar{n}}\,e^{-\frac{\kappa}{2}\gamma^h d(\Delta_1,\ldots,\Delta_r)}\\&\cdot\gamma^{-(k-h)[n_{\bar{B}}(\varphi)+2n_{\bar{B}}(\partial\varphi)+\ldots+3n_{\bar{B}}(S)+4n_{\bar{B}}(S^1)+4n_{\bar{B}}(T)-4]}.\end{aligned} \tag{29a}$$

In fact S^1 appears only in the case of D^1, and as it brings an extra $\gamma^{-(k-h)}$, it follows that after the $\mathscr{R}$-operation,

$$3n_{\bar{B}}^{\mathrm{old}}(D^1) = 3(n_{\bar{B}}^{\mathrm{new}}(D^1) + n_{\bar{B}}^{\mathrm{new}}(S^1)), \tag{30a}$$

and

$$\gamma^{-(k-h)(3n_{\bar{B}}^{0}(D^1))} \to \gamma^{-(k-h)(3n_{\bar{B}}^{\mathrm{n}}(D^1)+3n_{\bar{B}}^{\mathrm{n}}(S^1))} \gamma^{-(k-h)n_{\bar{B}}^{\mathrm{n}}(S^1)} = \gamma^{-(k-h)(3n_{\bar{B}}^{\mathrm{n}}(D^1)+4n_{\bar{B}}^{\mathrm{n}}(S^1))}. \tag{31a}$$

Similar argument works for the T-field.

To complete the first part of the proof of Theorem 1 we have to show that the $\mathscr{R}$-operation does not change, in any case, the factor $\gamma^{-(k-h)\cdot[\ ]}$ of Eq. (29a); the effect of the $\mathscr{R}$-operation is, of course, that some monomials do not appear anymore (those with "k-dimension" $\leqq 4$). This is proved in the following way: We indicate by $n_{\bar{B}}^{0}(V) \equiv n^0(V)$ the external lines (with respect to $\bar{B}$) of type V before the $\mathscr{R}$-operation is applied and $n(V)$ those after the $\mathscr{R}$-operation which have not changed their type; finally by $n_{V'}(V)$ we indicate the number of lines of type V which before the $\mathscr{R}$-operation were of type $V'(n_V(V) = n(V))$. Looking at the way $\mathscr{R}$ operates [see Eqs. (6.36)–(6.40)] it is clear that the following relations hold

$$\begin{aligned}
n^0(\varphi) &= n(\varphi) + n_\varphi(D) + n_\varphi(S), \\
n^0(\partial\varphi) &= n(\partial\varphi) + n_{\partial\varphi}(D^1), \\
n^0(D) &= n(D) + n_D(\partial\varphi) + n_D(S) + n_D(D^1) + n_D(T), \\
n^0(S) &= n(S) + n_S(T); \qquad n^0(S^1) = n(S^1), \\
n^0(D^1) &= n(D^1) + n_{D^1}(S^1); \qquad n^0(T) = n(T).
\end{aligned}$$

Let's also observe that each time a line is transformed into another one by the $\mathscr{R}$-operation, a factor $\gamma^{-(k-h)}$ to some power is produced, coming from the extra zeroes, the power being just the difference of the "k-dimension" of V and $V': d(V) - d(V')$. Therefore the factor $\gamma^{-(k-h)\cdot[\ ]}$ transforms in the following way:

$$\begin{aligned}
&\gamma^{-(k-h)[\Sigma_V d(V) n^0(V) - 4]} \\
&\xrightarrow{\mathscr{R}} \gamma^{-(k-h)[\Sigma_V d(V)n(V) + n_\varphi(D) + (2n_{\partial\varphi}(D^1) + 2n_D(D^1)) + (2n_D(S) + n_\varphi(S)) + (3n_S(T) + 2n_D(T)) + 3n_{D^1}(S^1)]} \\
&\cdot \gamma^{-(k-h)[n_\varphi(D) + n_{\partial\varphi}(D^1) + n_D(D^1) + n_D(S) + 2n_\varphi(S) + n_S(T) + 2n_D(T) + n_{D'}(S')]} \\
&= \gamma^{-(k-h)[\Sigma_V d(V) n^n(V) - 4]}.
\end{aligned}$$

We consider now how the integration in (23a) is modified. Observe that the extra factors $\gamma^{-(k-h)}$ are produced by the new zeroes which the Wick monomials have after the $\mathscr{R}$-operation [see Eq. (11a)]. These zeroes are of the following type

$$\prod_{\substack{(l,r)\\(i,j)\\l \neq r}} (\gamma^h |\eta_i^{(l)} - \eta_j^{(r)}|)^{\alpha(l,r;i,j)} \prod_{\substack{s,t\\s \neq t}} (\gamma^h |x_s - x_t|)^{\beta(s,t)} \prod_{\substack{l\\(i,p)}} (\gamma^h |\eta_i^{(l)} - x_p|)^{\gamma(l,i,p)}, \tag{32a}$$

where α, β, and γ are less than or equal to 2 and

$$\sum_{(\ )} \alpha(\ ) + \sum_{(\ )} \beta(\ ) + \sum_{(\ )} \gamma(\ ) < 4. \tag{33a}$$

Again we rewrite these products as

$$[(32a)]=\gamma^{-(k-h)[\Sigma_V a_V n_{\bar{\bar{B}}}^{\underline{n}/0}(V)]}\Big(\prod_{(\ )}(\gamma^k|\eta_i^{(l)}-\eta_j^{(r)}|)^{\alpha(\ )}\dots\Big), \tag{34a}$$

where the first factor in (32a) is what we have just considered in Eqs. (29a)–(31a) and

$$n_{\bar{\bar{B}}}^{\underline{n}/0}(V)\equiv n_{\bar{\bar{B}}}^{\underline{\text{new}}}(V)-n_{\bar{\bar{B}}}^{\underline{\text{old}}}(V). \tag{35a}$$

The second factor of (34a) modifies the integration part of (23a). The theorem is proved once the estimates of type (29a) are proved to hold for any subsequent box $\bar{\bar{B}}$, and therefore for the whole tree, and when the remaining global integral can be bounded by (const)^n.

The estimate (29a) for a generic next box $\bar{\bar{B}}$ can be proved exactly as before and produces the same estimate with the obvious changes of frequencies and indices. We are therefore left with the estimate of an integral which, after some reflection, one can realize is of the following type:

$$I(\gamma,\mathscr{G},\beta_{\mathscr{G}})=\Big(\prod_{\lambda\in\mathscr{A}}|\Delta^{(h_\lambda)}|\Big)^{-1}\int d\underline{X}\prod_{\lambda\in\mathscr{A}}e^{-\kappa\gamma^{h_\lambda}|\lambda|}[(\text{zeroes})], \tag{36a}$$

where

a) λ is an internal line of $\mathscr{G}$; $\mathscr{A}$ is the family of all the internal lines.[3]

b) h_λ is the frequency of the internal line λ (as previously defined). We observe that $\mathscr{A}$ is completely fixed once $(\gamma,\mathscr{G},\beta_{\mathscr{G}})$ is assigned.

c) [(zeroes)] represent the set of zeroes which are possibly, produced at each bifurcation by the $\mathscr{R}$-operation. We can write it

$$[(\text{zeroes})]=\prod_{\tilde{\lambda}}(\gamma^{h_{\tilde{\lambda}}}|\tilde{\lambda}|), \tag{37a}$$

where $|\tilde{\lambda}|=|\tilde{\xi}-\tilde{\eta}|$ and $\tilde{\lambda}$ is the line connecting $\tilde{\xi}$ and $\tilde{\eta}$ which may or may not be an internal line $\in\mathscr{A}$. $h_{\tilde{\lambda}}$ is the frequency of the first box containing both $\tilde{\xi}$ and $\tilde{\eta}$.

For $I(\gamma,\mathscr{G},\beta_{\mathscr{G}})$ we can prove the following estimate [see (13a)]:

$$|I(\gamma,\mathscr{G},\beta_{\mathscr{G}})|\leqq(\text{const})^n, \tag{38a}$$

where n is the number of vertices of $\mathscr{G}$ and (const) is an appropriate constant. First of all let's get rid of the factor [(zeroes)] in (37a). We prove the following estimate:

$$\prod_{\tilde{\lambda}}(\gamma^{h_{\tilde{\lambda}}}|\tilde{\lambda}|)\prod_{\lambda\in\mathscr{A}}e^{-\frac{\kappa}{2}\gamma^{h_\lambda}|\lambda|}\leqq(\text{const})^n \tag{39a}$$

for an appropriate constant. To prove (39a) let's observe that if $\tilde{\lambda}\equiv$ "$(\tilde{\xi},\tilde{\eta})$" is not an internal line, a family of internal lines exists connecting the boxes where $\tilde{\xi}$ and $\tilde{\eta}$ live, respectively, all with the same frequency $h(B)=h_{\tilde{\lambda}}$, where B is the smallest box containing $\tilde{\lambda}$ and $h(B)$ is its frequency. Graphically

3 $\mathscr{A}$ is a family of internal lines chosen in such a way that in each box of a well defined frequency the internal lines of that frequency are the minimum number to make a connected path between the inner boxes (therefore $|\mathscr{A}|=s-1$)

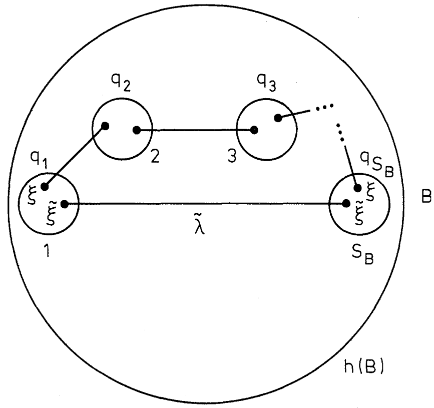


Fig. 3a

Let's call $d(B)$ the polygonal drawn in Fig. 3a to connect ξ to η (remember that this polygonal is modulo "inner boxes"); therefore the following estimate

$$|\tilde{\lambda}| \leqq d(B) + \sum_{1}^{s_B}{}_i\, d(B_i) + \sum_{1}^{s_B}{}_i \sum_{1}^{s_{B_i}}{}_j\, d(B_{ij}) + \sum_{1}^{s_B}{}_i \sum_{1}^{s_{B_i}}{}_j \sum_{1}^{s_{B_{ij}}}{}_k\, d(B_{ij_k}) + \dots \tag{40a}$$

holds, where $d(B_i)$ is the length of the polygonal in the box B_i modulo the inner boxes. Therefore the zero associated to $\tilde{\lambda}$ can be bounded by

$$\gamma^{h_{\tilde{\lambda}}}|\tilde{\lambda}| \leqq \Bigg[\gamma^{h(B)}\, d(B) + \sum_{1}^{s_B}{}_i\, \gamma^{(h(B)-q_i)}(\gamma^{q_i}\, d(B_i)) + \sum_{1}^{s_B}{}_i \sum_{1}^{s_{B_i}}{}_j\, \gamma^{(h(B)-q_{ij})}(\gamma^{q_{ij}}\, d(B_{ij})) + \dots\Bigg]. \tag{41a}$$

Remembering that the number of zeroes for any bifurcation (box) is bounded by 3, we can write

$$\prod_{\tilde{\lambda}\in\tilde{\mathscr{B}}} (\gamma^{h_{\tilde{\lambda}}}|\tilde{\lambda}|) \leqq \Bigg(\prod_{B}\Bigg[\gamma^{h(B)}\, d(B) + \sum_{1}^{s_B}{}_i\, \gamma^{(h(B)-q_i)}(\gamma^{q_i}\, d(B_i)) + \sum_{1}^{s_B}{}_i \sum_{1}^{s_{B_i}}{}_j\, \gamma^{(h(B)-q_{ij})} \cdot (\gamma^{q_{ij}}\, d(B_{ij})) + \dots\Bigg] \prod_{B_i}\Bigg[\gamma^{q_i}\, d(B_i) + \sum_{1}^{s_{B_i}}{}_j\, \gamma^{(q_i-q_{ij})}(\gamma^{q_{ij}}\, d(B_{ij})) + \dots\Bigg] \cdot \prod_{B_{ij}}\Bigg[\gamma^{q_{ij}}\, d(B_{ij}) + \sum_{1}^{s_{B_i}}{}_k\, \gamma^{(q_{ij}-q_{ijk})}(\gamma^{q_{ijk}}\, d(B_{ijk})) + \dots\Bigg]\Bigg)^3, \tag{42a}$$

where the products are over the boxes hierarchically ordered from the largest one (that corresponding to the frequency $k(\gamma)$] to the innermost ones. To bound this factor we use the exponential part

$$\prod_{\lambda\in\mathscr{A}} \exp -\frac{\kappa}{2}\gamma^{h_\lambda}|\lambda|.$$

We rearrange this expression in the following way, putting $\kappa/2=\delta$,

$$\prod_{\lambda\in\mathscr{A}} e^{-\delta\gamma^{h_\lambda}|\lambda|} \leqq \prod_{B\in\{B\}} e^{-\delta\gamma^{h(B)}d(B)} = e^{-\delta\sum_{B\in\{B\}}\gamma^{h(B)}d(B)}$$
$$= e^{-\delta\left\{\gamma^{h(B)}\,d(B) + \sum_{1}^{s_B}{}_i\,\gamma^{q_i}\,d(B_i) + \sum_{1}^{s_B}{}_i\sum_{1}^{s_{B_i}}{}_s\,\gamma^{q_{ij}}\,d(B_{ij}) + \sum_{1}^{s_B}{}_i\sum_{1}^{s_{B_i}}{}_j\sum_{1}^{s_{B_{ij}}}{}_k\,\gamma^{q_{ijk}}\,d(B_{ijk}) + \dots\right\}}, \tag{43a}$$

where B is the largest box, B_i are those external if we erase B and so on; q_i, q_{ij} ... are the corresponding frequencies. Now {(43a)} can be written in the following way:

$$\{(43a)\} = \left(\gamma^{h(B)} d(B) + \varepsilon \sum_{i=1}^{s_B} \gamma^{(h(B)-q_i)} \gamma^{q_i} d(B_i) + \varepsilon \sum_{i=1}^{s_B} \sum_{j=1}^{s_{B_i}} \gamma^{(h(B)-q_{ij})} \gamma^{q_{ij}} d(B_{ij}) + \dots\right)$$
$$+ \sum_{i=1}^{s_B} \left[(1 - \varepsilon\gamma^{(h(B)-q_i)}) \gamma^{q_i} d(B_i) + \varepsilon \sum_{j=1}^{s_{B_i}} \gamma^{(q_i - q_{ij})} \gamma^{q_{ij}} d(B_{ij}) + \dots \right]$$
$$+ \sum_{i=1}^{s_B} \sum_{j=1}^{s_{B_i}} \Bigg[(1 - \varepsilon(\gamma^{(h(B)-q_{ij})} + \gamma^{(q_i - q_{ij})})) \gamma^{q_{ij}} d(B_{ij})$$
$$+ \varepsilon \sum_{k=1}^{s_{B_{ij}}} \gamma^{(q_{ij} - q_{ijk})} \gamma^{q_{ijk}} d(B_{ijk}) + \dots \Bigg], \tag{44a}$$

where ε is chosen such that

$$\varepsilon \sum_{l=1}^{\infty} \gamma^{-l} = \varepsilon(\gamma - 1)^{-1} < 1,$$

and (45a)

$$1 - \varepsilon(\gamma - 1)^{-1} > \varepsilon \Leftrightarrow \varepsilon < \gamma^{-1}(\gamma - 1).$$

This is clearly possible, therefore

$$\prod_{\lambda \in \mathscr{A}} e^{-\delta\gamma^{h_\lambda}|\lambda|} \leqq \Bigg\{ \prod_B e^{-\delta\varepsilon\left[\gamma^{h(B)} d(B) + \sum_{i=1}^{s_B} \gamma^{(h(B)-q_i)} \gamma^{q_i} d(B_i) + \dots\right]}$$
$$\cdot \prod_{B_i} e^{-\delta\varepsilon\left[\gamma^{q_i} d(B_i) + \sum_{j=1}^{s_{B_i}} \gamma^{(q_i - q_{ij})} \gamma^{q_{ij}} d(B_{ij}) + \dots\right]}$$
$$\cdot \prod_{B_{ij}} e^{-\delta\varepsilon\left[\gamma^{q_{ij}} d(B_{ij}) + \sum_{k=1}^{s_{B_{ij}}} \gamma^{(q_{ij} - q_{ijk})} \gamma^{q_{ijk}} d(B_{ijk}) + \dots\right]} \Bigg\}, \tag{46a}$$

and

$$[(\text{left-hand side})\ (39a)] \leqq \left[\left(\prod_{B \in \{B\}} x e^{-\frac{\delta \cdot \varepsilon}{3} x} \right)^3 + \left(\prod_{B \in \{B\}} e^{-\delta\varepsilon x} \right) \right], \tag{47a}$$

and as the number of boxes B is $\leqq n-1$, where $n = v(\gamma)$ [(left-hand side) (39a)] $\leqq (\text{const})^n$ as claimed.

We are left with the estimate of

$$I_0(\gamma, \mathscr{G}, \beta_{\mathscr{G}}) = \prod_{\lambda \in \mathscr{A}} |\Delta^{(h_\lambda)}|^{-1} I_0(\mathscr{G}),$$

and (48a)

$$I_0(\mathscr{G}) = \int d\underline{X} \prod_{\lambda \in \mathscr{A}} e^{-\frac{\kappa}{2}\gamma^{h_\lambda}|\lambda|}.$$

To do that we proceed by induction. Let's consider the lowest frequency bifurcations (boxes) which we represent graphically in this way:

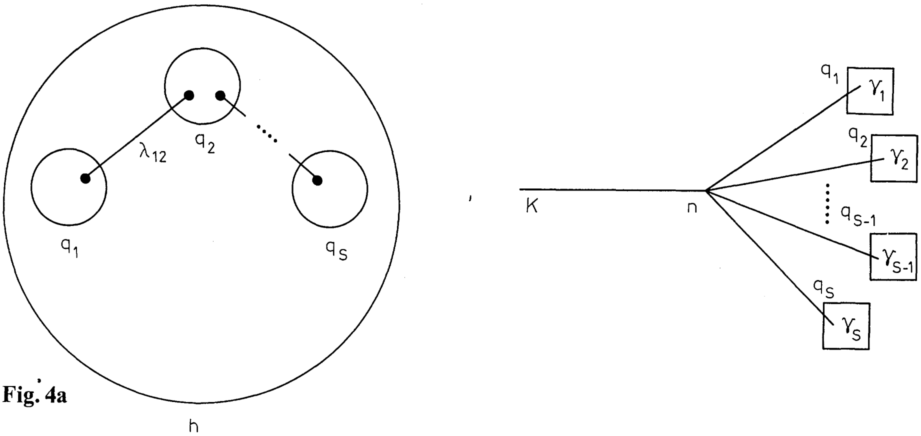


Fig. 4a

Between these boxes there will be lines connecting all of them. Let's erase some of those remaining, nevertheless, with the minimum number to form a connected polygonal (thinking of the boxes as points). We call η_1 the endpoint of $\lambda_{1,2}$ in the box "q_1"; all the variables in "q_1" are connected to those of the other boxes only through the line $\lambda_{1,2}$, therefore the dependence on η_1 in the integration associated to the variables in "q_1" can be eliminated by a change of variables obtaining

$$I_0(\mathscr{G})=I_0(\mathscr{G}_{q_1})\int d\underline{X}_{q_1}\prod_{\lambda\in\tilde{\mathscr{A}}\setminus\mathscr{A}_{q_1}} e^{-\frac{\kappa}{2}\gamma^{h_\lambda}|\lambda|}\int d\eta_1 e^{-\frac{\kappa}{2}\gamma^h|\lambda_{1,2}|}, \tag{49a}$$

where $\tilde{\mathscr{A}}$ is the set of all the internal lines except those we erased. $\mathscr{A}_{q_1}$ is the set of all the internal lines associated to the subgraph $\mathscr{G}_{q_1}\equiv\mathscr{G}\cap B_{q_1}$, with B_{q_1} the box of frequency q_1. Therefore

$$I_0(\mathscr{G})=(\text{const})\gamma^{-4h}I_0(\mathscr{G}_{q_1})\int d\underline{X}_{q_1}\prod_{\lambda\in\tilde{\mathscr{A}}\setminus\mathscr{A}_{q_1}} e^{-\frac{\kappa}{2}\gamma^{h_\lambda}|\lambda|}, \tag{50a}$$

where $\underline{X}_{q_1}$ are all the coordinates except those of the vertices in $\mathscr{G}_{q_1}$.

Now we can repeat the procedure for all the remaining boxes B_{q_i}, always integrating first on those connected with only one other one. We end up with

$$I_0(\mathscr{G})\leqq(\text{const})^s(\gamma^{-4h})^s I_0(\mathscr{G}_{q_1})\dots I_0(\mathscr{G}_{q_s}), \tag{51a}$$

and iterating we get

$$I_0(\mathscr{G})\leqq(\text{const})^n\prod_{\lambda\in\tilde{\mathscr{A}}}(\gamma^{-4h_\lambda}), \tag{52a}$$

which recalling (36a), (39a), and (48a) proves that

$$\begin{aligned} I(\gamma,\mathscr{G},\beta_{\mathscr{G}})&\leqq(\text{const})_1^n\, I_0(\gamma,\mathscr{G},\beta_{\mathscr{G}})=(\text{const})_1^n\Big(\prod_{\lambda\in\tilde{\mathscr{A}}}|\Delta^{(h_\lambda)}|^{-1}\Big)I_0(\mathscr{G})\\ &\leqq(\text{const})_1^n\,(\text{const})_2^n=(\overline{\text{const}})^n.\quad\square \end{aligned} \tag{53a}$$

Proof of Lemma 2. Given γ, without framed parts, with n final branches (the number of its bifurcation is $\leqq n-1$), let's call $\hat{\gamma}$ the tree γ to which we have erased the final lines. Then each line of $\hat{\gamma}$ brings a factor $\gamma^{-1/4(h-h')}$, where h and h' are the frequencies of its ends. See Fig. 3a.

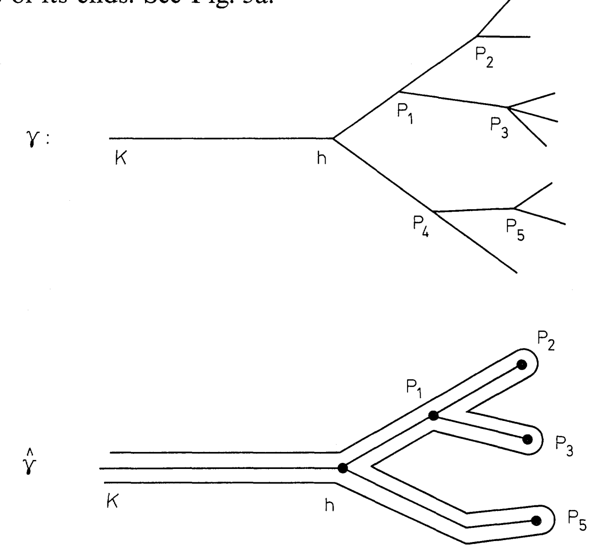


Fig. 3a

Now let's move along the branches starting from the root k, in the direction of the high frequencies, going back, on the opposite side, each time one arrives at the top of a final branch of $\hat{\gamma}$, until one is back again to k (see Fig. 3a). The path built in that way can be projected on the real axis: Fix on the real axis some points whose coordinates are just the frequencies of the tree (therefore # points $\leqq n-1$); the path can be described as a path which goes forward and backward on the real axis as described in Fig. 4a for the $\hat{\gamma}$ of Fig. 3a

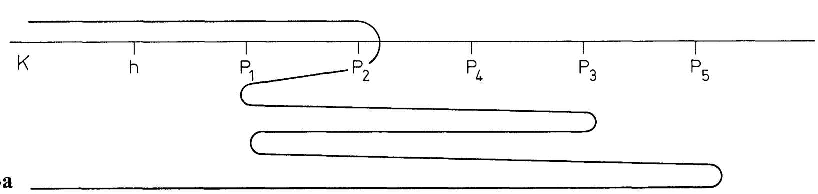


Fig. 4a

It is clear that given the path one can reconstruct $\hat{\gamma}$. Therefore to sum over all $\hat{\gamma}$ is equivalent to summing over all paths with less than n turning points whose coordinates are fixed arbitrarily on the real axis.

The weight for each tree is

$$\gamma^{-1/4\sum_{1}^{n-1}{}_i|x_i-x_{i-1}|} = \gamma^{-1/4L(\pi)}, \tag{54a}$$

where $L(\pi)$ is the length of the path π and $x_0=k$. Therefore

$$\sum_{\gamma}\prod_{\{B\}_\gamma}\gamma^{-\frac{1}{4}(q(B)-q(B'))} \leqq \#(n)\sum_{1}^{n-1}{}_s\Big(\sum_{x_1}\cdots\sum_{x_{s-1}}\gamma^{-\frac{1}{2}|x_1-x_0|}\cdots\gamma^{-\frac{1}{2}|x_{s-1}-x_{s-2}|}\Big)$$

$$\leqq \#(n)\sum_{1}^{n}{}_s\big(\textstyle\sum_x\gamma^{-\frac{1}{2}|x-x'|}\big)^s \leqq \#(n)\,n\cdot(M)^n \leqq (\bar{M})^n, \tag{55a}$$

where M and $\bar{M}$ are appropriate constants and $\#(n)$ is defined below. Let's call $\#(\hat{\gamma})$ a number which tells us in how many ways we can append n final branches to a tree $\hat{\gamma}$, and define

$$\#(n) = \max_{\gamma} \#(\hat{\gamma}) . \tag{56a}$$

It will be proved in the next lemma that $\#(n) \leqq (\text{const})^n$. Therefore, modulo Lemma 2a, we have proved that

$$\sum_{\gamma} \prod_{\{B\}_{\gamma}} \gamma^{-\frac{1}{4}(q(B)-q(B'))} \leqq c_2^n . \tag{57a}$$

Lemma 2a. *Given n objects and $k<n$ boxes, if we let $\mathcal{N}_{(k);n}$, denote the number of ways one can put m_1 objects in the first box, m_2 in the second, $\dots, m_k$ in the k^{th} one, with all possible $m_1, \dots, m_k$, such that $\sum_1^k {}_i m_i = n$, then $\mathcal{N}_{(k);n} \leqq (\text{const})^n$. Therefore*

$$\#(n) \leqq \sum_1^{n-1} {}_k (\text{const})^n \leqq \overline{(\text{const})}^n . \tag{58a}$$

Proof. As

$$0 \leqq m_k \leqq n - \sum_1^{k-1} {}_i m_i , \tag{59a}$$

$$\mathcal{N}_{(k);n} = \sum_0^n {}_{m_k} \mathcal{N}_{(k-1);n-m_k} . \tag{60a}$$

Therefore by iteration

$$\mathcal{N}_{(k);n} = \sum_0^n {}_{m_k} \sum_0^{n-m_k} {}_{m_{k-1}} \dots \sum_0^{n-\sum_3^k {}_l m_l} {}_{m_2} 1 , \tag{61a}$$

and defining $x_j = n - \sum_j^k {}_l m_l$,

$$\mathcal{N}_{(k);n} = \sum_0^n {}_{x_k} \sum_0^{x_k} {}_{x_{k-1}} \dots \sum_0^{x_3} {}_{x_2} 1 \leqq c \frac{n^k}{k!} \leqq (\text{const})^n . \quad \square \tag{62a}$$

Proof of Lemma 3. As $\sigma(\tilde{\gamma})$ is fixed, to sum over $\tilde{\gamma}$ means to sum over all the possible frequencies. A generic σ will be of the following type:

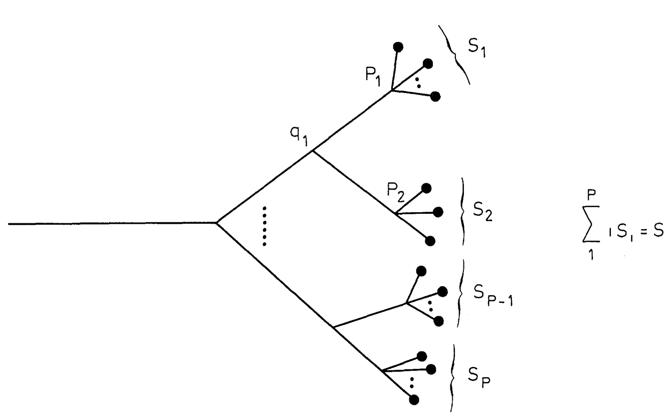


Fig. 5a

Therefore start summing over the frequencies of the first cluster.

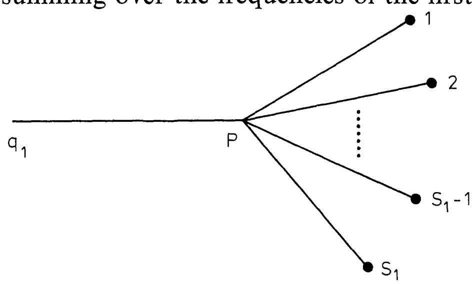


Fig. 6a

Call $I_{s_1}(q_1)$ the contribution of this cluster.

$$I_{s_1}(q_1)=\sum_{q_1+1}^{N}{}_p \prod_{1}^{s_1}{}_j\left(\bar{\mu}(w_j)f_j!\sum_{0}^{f_j}{}_i\frac{(bp)^i}{i!}\right)\gamma^{-\frac{1}{8}(p-q)}. \tag{63a}$$

Define $\theta=\frac{1}{8}\log\gamma$, and use the following estimate:

$$\sum_{q+1}^{\infty}{}_p\gamma^{-\frac{1}{8}(p-q)}p^i\leqq\gamma^{-1/8}\int_q^\infty\gamma^{-\frac{1}{8}(p-q)}(p-q+q)^i dp$$

$$=\gamma^{-1/8}\sum_{0}^{i}{}_r\binom{i}{r}\left(\frac{8}{\log\gamma}\right)^{i-r+1}\int_0^\infty e^{-t}t^{i-r}dt=\gamma^{-1/8}i!\theta^{-(i+1)}\sum_{0}^{i}{}_r\frac{(\theta q)^r}{r!}. \tag{64a}$$

Applying this estimate to (63a) we get

$$I_{s_1}(q_1)\leqq\gamma^{-1/8}\prod_{1}^{s_1}{}_j\bar{\mu}(w_j)f_j!\sum_{0}^{f_1}{}_{i_1}\dots\sum_{0}^{f_{s_1}}{}_{i_{s_1}}\frac{b^{(i_1+\dots+i_{s_1})}}{i_1!\dots i_{s_1}!}\theta^{-(i_1+\dots+i_{s_1})-1}$$

$$\cdot(i_1+\dots+i_{s_1})!\sum_{0}^{(i_1+\dots+i_{s_1})}{}_r\frac{(\theta q)^r}{r!}$$

$$=\gamma^{-1/8}\left(\prod_{1}^{s_1}{}_j\bar{\mu}(w_j)f_j!\right)\sum_{0}^{(f_1+\dots+f_{s_1})}{}_i\frac{(\theta q)^i}{i!}\frac{1}{\theta}$$

$$\cdot\left(\sum_{\substack{j_1\dots j_{s_1}\cdot\\(j_1+\dots+j_{s_1})\geqq i}}^{f_1,\dots,f_{s_1}}\left(\frac{b}{\theta}\right)^{j_1+\dots+j_{s_1}}\frac{(j_1+\dots+j_{s_1})!}{j_1!\dots j_{s_1}!}\right). \tag{65a}$$

Choose now $b<\theta$, then

$$\left(\sum_{\substack{j_1\dots j_{s_1}\\(j_1+\dots+j_{s_1})\geqq i}}^{f_1,\dots,f_{s_1}}\left(\frac{b}{\theta}\right)^{j_1+\dots+j_{s_1}}\frac{(j_1+\dots+j_{s_1})!}{j_1!\dots j_{s_1}!}\right)=\sum_{i}^{(f_1+\dots+f_{s_1})}{}_s\left(\frac{b}{\theta}\right)^s$$

$$\cdot\left[\sum_{\substack{j_1,\dots,j_{s_1}\\(j_1+\dots+j_{s_1})=s}}^{f_1,\dots,f_{s_1}}\frac{(j_1+\dots+j_{s_1})!}{j_1!\dots j_{s_1}!}\right]\leqq\left(\frac{b}{\theta}\right)^i\sum_{0}^{\infty}{}_s\left(\frac{b}{\theta}\right)^s\left[\sum_{\substack{j_1,\dots,j_{s_1}\\(j_1+\dots+j_{s_1})=s}}^{f_1,\dots,f_{s_1}}\frac{(j_1+\dots+j_{s_1})!}{j_1!\dots j_{s_1}!}\right]. \tag{66a}$$

Now the following lemma holds:

Lemma 3a. $\forall s\in[0,(f_1+\dots+f_{s_1})]$, $f_i\geqq i$ $\forall i\in[1,s_1]$, *the following inequality holds*

$$\sum_{\substack{j_1,\dots,j_{s_1}\\(j_1+\dots+j_{s_1})=s}}^{f_1,\dots,f_{s_1}}\frac{(j_1+\dots+j_{s_1})!}{j_1!\dots j_{s_1}!}\leqq\frac{(f_1+\dots+f_{s_1})!}{f_1!\dots f_{s_1}!}. \tag{67a}$$

We postpone the proof of this lemma.
Therefore

$$[(\ )]\leqq\left(\frac{b}{\theta}\right)^i\left(1-\frac{b}{\theta}\right)^{-1}\frac{(f_1+\ldots+f_{s_1})!}{f_1!\ldots f_{s_1}!},\tag{68a}$$

and

$$I_{s_1}(q_1)\leqq\gamma^{-1/8}(\theta-b)^{-1}\left(\prod_1^{s_1}{}_j\,\bar{\mu}(w_j)\right)(f_1+\ldots+f_{s_1})!\sum_0^{(f_1+\ldots+f_{s_1})}{}_i\frac{(bq_1)^i}{i!}.\tag{69a}$$

Therefore the contribution of the cluster of marked "ends" of Fig. 6a brings the same contribution as a simple line with a marked end,

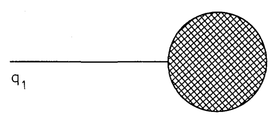


Fig. 7a

with

$$f=(f_1+\ldots+f_{s_1})\quad\text{and}\quad\bar{\mu}=\gamma^{-1/8}(\theta-b)^{-1}\left(\prod_1^{s_1}{}_j\,\bar{\mu}(w_j)\right).\tag{70a}$$

This argument can be repeated for all other clusters and then for clusters of clusters, getting at the very end

$$[(\text{left-hand side})\ (7.30)]\leqq c_3^s\left(\prod_1^s{}_i\,\bar{\mu}(w_i)\right)\left(\sum_1^s{}_i f_i\right)!\sum_0^{\sum_1^s{}_i f_i}{}_j\frac{(bk)^j}{j!},\tag{71a}$$

which is the statement of the lemma. □

Proof of Lemma 2a. We have to prove that

$$\frac{f_1!\ldots f_{s_1}!}{(f_1+\ldots+f_{s_1})!}\sum_{\substack{j_1,\ldots,j_{s_1}\\(j_1+\ldots+j_{s_1})=s}}^{f_1,\ldots,f_{s_1}}\frac{(j_1+\ldots+j_{s_1})!}{j_1!\ldots j_{s_1}!}\leqq 1.\tag{72a}$$

This inequality has to be proven recursively in the following way:

$$\begin{aligned}[(\text{left-hand side})\ (72\text{a})]=&\ \frac{f_1!\ldots f_{s_1-2}!(f_{s_1-1}+f_{s_1})!}{(f_1+f_2+\ldots+(f_{s_1-1}+f_{s_1}))!}\\ &\cdot\sum_{\substack{j_1,\ldots,j_{s_1-2},\tilde{j}\\(j_1+\ldots+j_{s_1-2}+\tilde{j})=s}}^{f_1,\ldots,f_{s_1-2},(f_{s_1-1}+f_{s_1})}\frac{(j_1+\ldots+j_{s_1-2}+\tilde{j})!}{j_1!j_2!\ldots j_{s_1-2}!\tilde{j}!}\\ &\cdot\left\{\sum_{\substack{j_{s_1-1},j_{s_1}\\(j_{s_1-1}+j_{s_1})=\tilde{j}}}^{f_{s_1-1},f_{s_1}}\frac{\tilde{j}!}{j_{s_1-1}!j_{s_1}!}\frac{f_{s_1-1}!f_{s_1}!}{(f_{s_1-1}+f_{s_1})!}\right\}.\end{aligned}\tag{73a}$$

Therefore once we prove that { } is less than or equal to 1, the result follows by recursion.

We have left to prove

$$\sum_{\substack{j_1, j_2 \\ (j_1+j_2)=j}}^{f_1, f_2} \binom{j}{j_1} \leqq \binom{f_1+f_2}{f_1}, \quad \forall f_1, f_2 > 0, 0 \leqq j \leqq f_1 + f_2. \tag{74a}$$

The proof follows from the relation

$$\binom{j}{j_1} + \binom{j}{j_1-1} = \binom{j+1}{j_1}. \quad \square \tag{75a}$$

Proof of Lemma 4. We have to estimate $\sum_{\substack{\mathscr{G} \\ w \text{fixed}}} 1$; remember that w fixes f, n, as the inner marked boxes are distributed inside the external ones and also how the vertices are distributed inside the different boxes.

Let us assume that w is such that there are s external boxes and m vertices outside the s-boxes; let's call, as usual,

$$\begin{aligned} f_i &= 1 + \#(\text{boxes inside the } i^{\text{th}} \text{ one}), \\ n_i &= \quad\ \#(\text{vertices inside the } i^{\text{th}} \text{ box}), \end{aligned} \tag{76a}$$

then

$$n = m + \sum_{1}^{s}{}_i\, n_i, \qquad f = \sum_{1}^{s}{}_i\, f_i. \tag{77a}$$

Let's call $\mathscr{N}^{(1)}_{n_i, f_i}(w)$ the number of subgraphs inside the i^{th} box with the structure of marked boxes relative to these subgraphs fixed by w. Therefore

$$\sum_{\substack{\mathscr{G} \\ w \text{fixed}}} 1 \leqq D^{m+s}(m+s)!\, \mathscr{N}^{(1)}_{n_1, f_1} \dots \mathscr{N}^{(1)}_{n_s, f_s}. \tag{78a}$$

Assume now that

$$\mathscr{N}^{(1)}_{n_i, f_i} \leqq c_{10}^{n_i} c_{11}^{f_i - 1}(n_i - f_i - 1)!. \tag{79a}$$

Then

$$\begin{aligned} \sum_{\substack{\mathscr{G} \\ w \text{fixed}}} 1 &\leqq D^{m+s} c_{10}^{n-m} c_{11}^{f-s} \prod_{1}^{s}{}_i\, (m+s)!\,(n_i - f_i - 1)! \\ &\leqq D^{m+s} c_{10}^{n-m} c_{11}^{f-s} (n-f)! \leqq c_9^n (n-f)!, \end{aligned} \tag{80a}$$

choosing c_9 appropriately.

The inequality (79a) is proven by induction: Let's assume that inside the i^{th} box there are s_i "external" boxes with $\bar{f}_1, \dots, \bar{f}_{s_i}$ boxes inside + the external one and that $\bar{n}_1, \dots, \bar{n}_{s_i}$ are the number of vertices inside these s_i boxes respectively; finally let's call m_i the number of vertices inside the i^{th} box and outside the s_i boxes. Therefore

$$f_i = 1 + \sum_{1}^{s_i}{}_j\, \bar{f}_j, \qquad n_i = m_i + \sum_{1}^{s_i}{}_j\, \bar{n}_j. \tag{81a}$$

Then

$$\begin{aligned}
\mathcal{N}^{(1)}_{n_i,f_i} &\leqq D^{m_i+s_i}(m_i+s_i)! \prod_1^{s_i}{}_j\, c_{10}^{\bar{n}_j} c_{11}^{\bar{f}_j-1}(\bar{n}_j-\bar{f}_j-1)! \\
&\leqq c_{10}^{n_i-m_i} c_{11}^{f_i-1-s_i} D^{m_i+s_i} \prod_1^{s_i}{}_j\,(m_i+s_i)!\,(\bar{n}_j-\bar{f}_j-1)! \\
&\leqq c_{10}^{n_i-m_i} c_{11}^{f_i-1-s_i} D^{m_i+s_i} \prod_1^{s_i}{}_j\,(m_i+s_i)!\,(\bar{n}_j-\bar{f}_j-1)! \\
&\leqq [c_{10}^{n_i} c_{11}^{f_i-1}(n_i-f_i-1)!] \\
&\quad \cdot \left\{ \frac{D^{m_i+s_i}(m_i+s_i)!}{c_{10}^{m_i} c_{11}^{s_i}(m_i+s_i-2)!} \left(\frac{(m_i+s_i-2)!\,((n_i-f_i-1)-(m_i+s_i-2))!}{(n_i-f_i-1)!} \right) \right\} \\
&\leqq [c_{10}^{n_i} c_{11}^{f_i-1}(n_i-f_i-1)!]\,,
\end{aligned} \tag{82a}$$

if

$$\left(\frac{D^{m_i+s_i}}{c_{10}^{m_i} c_{11}^{s_i}} (m_i+s_i)\,(m_i+s_i-1) \right) < 1\,, \tag{83a}$$

which is always possible by choosing appropriately c_{10} and c_{11}. The lemma is proven provided

$$\mathcal{N}^{(1)}_{n,1} \leqq c_{10}^{n}(n-2)!\,, \tag{84a}$$

which is true provided c_{10} is chosen appropriately.

Finally we have

Lemma 5.

$$\sum_{\substack{w \\ (n,f\,\text{fixed})}} 1 \leqq c_{12}^{n+f}\,. \tag{7.45}$$

Proof. Remember that fixing w means to fix

a) the total number of boxes,

b) how the boxes are included one into each other,

c) how many vertices are contained in each box.

Call symbolically $\{f\}$ an arrangement of spheres (boxes) and $\{n\}$ an arrangement of vertices. Then

$$\sum_w = \sum_{\{f\}} \left(\sum_{\substack{\{n\} \\ \{f\}\,\text{fixed}}} 1 \right). \tag{85a}$$

We study the second sum first. If the arrangement of spheres is fixed, the regions (differences between spheres) which belong to one sphere and not to any other inner one are also fixed and their number is f.

$\sum_{\substack{\{n\} \\ \{f\}\,\text{fixed}}} 1$ is the number of ways one can put identical vertices inside these regions in such a way that their total number is n. This is a number bounded by the result of Lemma 2a, therefore

$$\left(\sum_{\substack{\{n\} \\ \{f\}\,\text{fixed}}} 1 \right) \leqq (\text{const})^n\,. \tag{86a}$$

We are left with the estimate of $\sum_{\{f\}} 1$. An arrangement of f spheres can be described in this way: We number the spheres with an index $j: 0, \ldots, f$. $j=0$ refers to a fictitious sphere which encloses all the other ones. Let's order these spheres in the following way: let's order arbitrarily the more external ones (those contained only in the fictitious sphere s_0). Let's call these ones $s_1, s_2, \ldots, s_{n_0}$, then let's consider those inside s_1 and "external" and call them $s_{n_0+1}, s_{n_0+2}, \ldots, s_{n_0+n_1}$, then those inside s_2 and external $s_{n_0+n_1+1}, \ldots, s_{n_0+n_1+n_2}$ and so on, until this generation of spheres is completed and we start considering those inside s_{n_0+1} and "external" .... Therefore assigning the numbers $\{n_0, n_1, \ldots, n_{f-1}\}$ the arrangement of spheres is completely defined. Moreover $\sum_0^{f-1} {}_j n_j = f$. Nevertheless not any set $\{n_0, n_1, \ldots, n_{f-1}\}$ describes an arrangement of spheres unless some constraints between the n_i's are satisfied. Therefore

$$\sum_{\{f\}} 1 \leqq \sum_{\substack{\{n_0, n_1, \ldots, n_{f-1}\} \\ \sum_0^{f-1} {}_i n_i = f}} 1 \leqq (\text{const})^f, \tag{87a}$$

using again *Lemma 2a*; Collecting (56a) and (57a) together we get the final result

$$\sum_w 1 \leqq c_{12}^{n+f}, \tag{88a}$$

choosing c_{12} appropriately. □

Appendix B

We give an explicit expression of the coefficient $G^{(4,4)}$ [see Eq. (1.11)] and we examine the behaviour of $G^{(4,4)}(k)$ as $k \to \infty$.

$$G^{(4,4)} = \left[\frac{1}{2}\binom{4}{2}^2 2!\right] \lim_{k\to\infty} \int d\zeta (C_{0,\zeta}^{(k)2} - C_{0,\zeta}^{(k-1)2}), \tag{1b}$$

and [see Sect. 2 of (I)]

$$C_{\xi\eta}^{(k)} = \sum_0^k {}_q \tilde{C}_{\xi\eta}^{(q)}. \tag{2b}$$

From Eqs. (I, 2.7), (I, 2.8) we have

$$\tilde{C}_{0,\zeta}^{(k)} = \sum_0^k {}_t \hat{C}_{0,\zeta}^{(k,t)} = \gamma^{2k} \bar{C}^{(k)}(\gamma^k \zeta) \tag{3b}$$

where, looking at the equations of Sect. 2, of (I)

$$\begin{aligned} \bar{C}^{(k)}(x) = \gamma^2(\gamma^2-1)(\gamma^4-1) \sum_0^k {}_t \frac{1}{\gamma^{2t}} \Bigg\{ \sum_0^t {}_l \gamma^{-4l} \frac{1}{(2\pi)^4} \int d^{4k} e^{ik\cdot x} \\ \cdot \left(\frac{1}{k^8 + \tilde{A}k^6 + \tilde{B}k^4 + \tilde{C}k^2 + \gamma^8} - \frac{1}{k^8 + \tilde{A}k^6 + \tilde{B}k^4 + \tilde{C}k^2 + \gamma^{16}} \right) \Bigg\}. \end{aligned} \tag{4b}$$

As for

$$k \gg (C_{0,\zeta}^{(k)2} - C_{0,\zeta}^{(k-1)2}) = \tilde{C}_{0,\zeta}^{(k)}(C_{0,\zeta}^{(k)} + C_{0,\zeta}^{(k-1)}) \simeq 2\tilde{C}_{0,\zeta}^{(k)} C_{0,\zeta}^{(k)}, \tag{5b}$$

$$\begin{aligned} G^{(4,4)} &= 2\binom{4}{2}^2 \lim_{k\to\infty} \int d^4\zeta \, \gamma^{2k} \bar{C}^{(k)}(\gamma^k \zeta)\, C^{(k)}(\zeta) \\ &= 2\binom{4}{2}^2 \lim_{k\to\infty} \int d^4 z \frac{\bar{C}^{(k)}(z)\, C^{(k)}(\gamma^{-k} z)}{\gamma^{2k}} \\ &= 2\binom{4}{2}^2 \lim_{k\to\infty} \int \frac{d^4 z}{z^2} [\bar{C}^{(k)}(z)\, (\gamma^{-k}z)^2\, C^{(k)}(\gamma^{-k}z)] \\ &= 2\binom{4}{2}^2 \int \frac{d^4 z}{z^2} [\bar{C}(z) c_1], \end{aligned} \tag{6b}$$

where

$$\bar{C}(z) = \lim_{k\to\infty} \bar{C}^{(k)}(z) \tag{7b}$$

is defined for any z and bounded and

$$\begin{aligned} c_1 &= \lim_{\varrho\to 0} \varrho^2 C^{(\infty)}(\varrho) \equiv \lim_{k\to\infty} (\gamma^{-k}z)^2\, C^{(k)}(\gamma^{-k}z) \\ &= \lim_{\varrho\to 0} \varrho^2 \int d^4 k \frac{e^{ik\varrho}}{k^2+1} > 0 . \end{aligned} \tag{8b}$$

Therefore

$$G^{(4,4)} = 2\binom{4}{2}^2 c_1 \int \frac{d^4 z}{z^2} \bar{C}(z) . \tag{9b}$$

We have also the following result: for k enough large

$$|G^{(4,4)}(k) - G^{(4,4)}| = O(\gamma^{-2(1-\varepsilon)k}), \qquad \varepsilon > 0 . \tag{10b}$$

In fact

$$G^{(4,4)}(k) - G^{(4,4)} = 2\binom{4}{2}^2 \int \frac{d^4 z}{z^2} [\bar{C}^{(k)}(z)\, (\gamma^{-k}z)^2\, C^{(k)}(\gamma^{-k}z) - \bar{C}(z) c_1], \tag{11b}$$

and

$$[(11\text{b})] = (\bar{C}^{(k)}(z) - \bar{C}(z))\, (\gamma^{-k}z)^2\, C^{(k)}(\gamma^{-k}z) + \bar{C}(z)\, [(\gamma^{-k}z)^2\, C^{(k)}(\gamma^{-k}z) - c_1] . \tag{12b}$$

As $\bar{C}(z)$ and $(\gamma^{-k}z)^2\, C^{(k)}(\gamma^{-k}z)$ are bounded and $\bar{C}(z) \sim e^{-\delta|z|}$ as $|z| \to \infty$, $\delta > 0$,

$$|(11\text{b})| \leqq M\{|\bar{C}^{(k)}(z) - \bar{C}(z)| + e^{-\delta|z|} |(\gamma^{-k}z)^2\, C^{(k)}(\gamma^{-k}z) - c_1|\} . \tag{13b}$$

Then from (4b)

$$|\bar{C}^{(k)}(z) - \bar{C}(z)| \leqq N_1 \gamma^{-2k} e^{-\delta|z|} . \tag{14b}$$

Moreover

$$\begin{aligned} |(\gamma^{-k}z)^2\, C^{(k)}(\gamma^{-k}z) - c_1| \leqq\; & |(\gamma^{-k}z)^2\, (C^{(k)}(\gamma^{-k}z) - C^{(n)}(\gamma^{-k}z))| \\ & + |(\gamma^{-k}z)^2\, (C^{(n)}(\gamma^{-k}z) - C^{(\infty)}(\gamma^{-k}z))| \\ & + |(\gamma^{-k}z)^2\, C^{(\infty)}(\gamma^{-k}z) - c_1| , \end{aligned} \tag{15b}$$

and it is easy to realize that the first modulus can be bounded by $O(\gamma^{-2k})$, the second by $\varepsilon(n)$, which tends to zero as $n\to\infty$, and the third one by

$$z^2\left|\int d^4q\frac{e^{iqz}}{q^2+\gamma^{-2k}}-\lim_{n\to\infty}\int d^4q\frac{e^{iqz}}{q^2+\gamma^{-2n}}\right|. \tag{16b}$$

The result (10b) is achieved observing that

$$\begin{aligned}&\left|z^2\int d^4q\,e^{iqz}\frac{\gamma^{-2k}(\gamma^{-2(n-k)}-1)}{(q^2+\gamma^{-2n})(q^2+\gamma^{-2k})}\right|\\&\leqq\gamma^{-2k}(\gamma^{-2(n-k)}-1)\left\{z^2\int_{|q|>1}d^4q\frac{e^{iqz}}{q^4}+z^2\int_{|q|<1}d^4q\frac{e^{iqz}}{q^2(q^2+\gamma^{-2k})}\right\}\\&\leqq O(\gamma^{-2k})\{z^2|\log|z||+z^2\log\gamma^{2k}\}.\end{aligned} \tag{17b}$$

Proceeding in a similar way one can get explicit expressions for all the other coefficients of (1.12) and also prove that a result analogous to (10b) holds: for k large enough,

$$|T^{(i,j)}(k)-T^{(i,j)}|=O(\gamma^{-2(1-\varepsilon)k}),\qquad \varepsilon>0, \tag{18b}$$

where

$$T^{(i,j)}(k)=\{G^{(i,j)}(k),\gamma^{-2k}M^{(i,j)}(k),A^{(i,j)}(k)\}. \tag{19b}$$

We have

$$\begin{aligned}G^{(4,4)}&=\binom{4}{2}^2 2!c_1\int\frac{d^4z}{z^2}\bar{C}(z),\\G^{(4,2)}&=\binom{4}{1}\binom{2}{1}\int d^4z\,\bar{C}(z),\\M^{(4,4)}&=\frac{3}{2}\binom{4}{3}^2 3!\gamma^2c_1\int\frac{d^4z}{z^2}\bar{C}(z)\left(\sum_0^\infty{}_l\,\gamma^{-2l}\bar{C}(\gamma^{-l}z)\right),\\M^{(4,2)}&=\binom{4}{2}2!2\gamma^2c_1\int\frac{d^4z}{z^2}\bar{C}(z),\\M^{(4,2')}&=-\binom{4}{2}2!2\gamma^2c_1\int\frac{d^4z}{z^2}\Delta\bar{C}(z),\\M^{(2,2)}&=\frac{1}{2}\binom{2}{1}^2\gamma^2\int d^4z\,\bar{C}(z),\\A^{(4,4)}&=\binom{4}{3}^2\frac{1}{4^2}3!3c_1^2\int\frac{d^4z}{z^2}\bar{C}(z),\\A^{(2,2)}&=\frac{1}{4^2}\binom{2}{1}^2\int d^4z\,z^2\bar{C}(z),\\A^{(2',2)}&=\frac{1}{2^3}\binom{2}{1}^2\int d^4z\,\underline{z}\cdot\underline{\partial}\bar{C}(z).\end{aligned} \tag{20b}$$

We discuss now the connection between the asymptotic differential equations (1.12) and the true ones in which we keep the dependence on k in the coefficients, and also the connection between these results and those for the finite difference equations. We want to prove that the solution of (1.12) gives us asymptotically the behaviour of the solution of the true equations

$$\frac{d\lambda}{dt}=\bar{G}^{(4,4)}(t)\lambda^2+\bar{G}^{(4,2)}(t)\lambda\mu\,,$$

$$\frac{d\mu}{dt}=-\mu+(\gamma^{-2t}\bar{M}^{(4,4)}(t))\lambda^2+\dots\,, \tag{1.12'}$$

$$\frac{d\alpha}{dt}=-\bar{A}^{(4,4)}(t)\lambda^2+\bar{A}^{(2',2)}(t)\alpha\mu-\bar{A}^{(2,2)}(t)\mu^2\,.$$

This can be summarized in the following theorem, which we do not discuss here.

Theorem. *Let us call* $\underline{n}_\infty\equiv(\lambda_\infty,\mu_\infty,\alpha_\infty)$ *a solution of* (1.12) *with initial data at* $t=t_0$, $\underline{n}_0\equiv\underline{n}_\infty(t_0)=\underline{g}+O(\underline{g}^2)$, *where* $\underline{g}=(g,m,a)$; *then we have the following result: in a ball with center* n_0 *and radius of order* $\underline{g}^2e^{-\alpha t_0}$, $\alpha>0$, *there exist initial data for Eq.* (1.12)′ *such that the corresponding solution* $\underline{n}(t)$ *tends to* $\underline{n}_\infty(t)$ *as* $t\to\infty$, *and moreover*

$$|\underline{n}(t)-\underline{n}_\infty(t)|=O(g^2)e^{-\alpha t}\,, \tag{21b}$$

where $\alpha=2(1-\varepsilon)\log\gamma$, $0<\varepsilon<1$. □

Therefore for $t\in[t_0,\infty)$ we can write

$$\underline{n}(t)=\underline{n}_\infty(t)+O(\underline{g}^2)e^{-\alpha t}\,. \tag{22b}$$

We finally discuss briefly the connection between Eqs. (1.10) (finite difference equations) and (1.12). We consider the first equation of (1.10),

$$\lambda(k)-\lambda(k-1)=G^{(4,4)}(k)\,\lambda^2(k)+G^{(4,2)}(k)\,\lambda(k)\,\mu(k)=F(\gamma^k;\lambda(k),\mu(k))\,, \tag{23b}$$

and remember that the choice of γ is arbitrary provided $\gamma>1$. Therefore changing γ in $\gamma'=\gamma^{1/n}$, we can repeat the results and computations done before, just substituting γ' for γ. Equation (23b) becomes

$$\begin{aligned}\lambda(k)-\lambda(k-1)&\equiv\lambda(\gamma^k)-\lambda(\gamma^{k-1})=\lambda(\gamma'^{nk})-\lambda(\gamma'^{(nk-n)})\\&=\sum_1^n{}_l\,[\lambda(\gamma'^{(nk-(l-1))})-\lambda(\gamma'^{(nk-l)})]\,.\end{aligned} \tag{24b}$$

Assume that $\lambda(\gamma^k)$ is a C^∞ function of γ^k. Then

$$\begin{aligned}\lambda(\gamma'^{(nk-(l-1))})-\lambda(\gamma'^{(nk-l)})&\approx\left.\frac{\partial\lambda}{\partial x}\right|_{x=\gamma'^{(nk-l)}}\gamma'^{(nk-l)}(\gamma'-1)\\ \approx\left(\frac{\partial\lambda}{\partial x}\right)_{x=\gamma^{k-\tau}}\gamma^{(k-\tau)}\frac{1}{n}\log\gamma&=\frac{1}{n}\left(\frac{\partial}{\partial\varrho}\gamma^\varrho\right)_{\varrho=k-l/n}\left(\frac{\partial\lambda}{\partial x}\right)_{x=\gamma^\varrho}\\ =\frac{1}{n}\left(\frac{\partial\lambda}{\partial\varrho}\right)_{\varrho=k-l/n},\qquad \tau=\frac{l}{n}\,.\end{aligned} \tag{25b}$$

Therefore

$$\sum_{1}^{n} {}_l \left[\lambda(\gamma'^{(nk-(l-1))}) - \lambda(\gamma'^{(nk-l)})\right] \simeq \sum_{1}^{n} {}_l \frac{1}{n}\left(\frac{\partial\lambda}{\partial\varrho}\right)_{\varrho=k-l/n} \Rightarrow$$

$$\text{as} \quad n\to\infty, \int_{k-1}^{k} d\varrho \frac{\partial\lambda}{\partial\varrho}. \tag{26b}$$

The right-hand side of (24b) can be worked out exactly in the same manner, obtaining

$$\int_{k-1}^{k} d\varrho \frac{\partial\lambda}{\partial\varrho} = \int_{k-1}^{k} d\varrho \log\gamma \bar{F}(\varrho, \lambda(\varrho), \mu(\varrho)), \tag{27b}$$

from which

$$\frac{d\lambda}{d\varrho} = \log\gamma \bar{F}(\varrho, \lambda(\varrho), \mu(\varrho)) \tag{28b}$$

follows, where $F(k, \lambda(k), \mu(k)) = (\gamma-1)\bar{F}(k, \lambda(k), \mu(k))$ and

$$\lim_{\gamma\to 1} \bar{F}(k, \lambda(k), \mu(k)) \neq 0. \tag{29b}$$

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