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On the Discrete Spectrum of the N-Body Quantum Mechanical Hamiltonian. I

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Abstract. We discuss N-body kinematics and study the Berezin-Sigal equations in configuration space. Assuming that the threshold of the continuous spectrum is zero and that the pair potentials satisfy $|V(x)| \leqq C(1+|x|^2)^{-\varrho}$, $x \in \mathbb{R}^3$, $\varrho > 1$ (together with some technical hypotheses), we show that the discrete spectrum of the hamiltonian in the center of mass system is finite. The case of negative threshold will be treated in a further publication.

1. Introduction

The basic theorem on the quantum mechanical hamiltonian in the center of mass system, due to Hunziker [5], van Winter [14], and Zhislin [19], states that under suitable assumptions on the potentials the essential spectrum of this hamiltonian consists of a half-line $[\mu, \infty)$, $\mu \leqq 0$, while the discrete spectrum lies below μ and its only possible accumulation point is μ itself. This immediately suggests the question of determining conditions for finiteness or infinitude of the discrete spectrum. This problem has been attacked successfully by several authors under various conditions on the potentials. Zhislin [19] has shown that atoms have infinite discrete spectrum. Simon [12] proved that if the potentials decay as $|x|^{-2+\delta}$, $\delta \geqq 0$, at infinity then the discrete spectrum may be infinite. In the three body case conditions for finiteness and/or infinitude have been obtained by Combescure and Ginibre [3], Iorio [6], Yafaev [15–17]. The N-body case was analysed by Sigal [10], Yafaev [18], Simon [13] (using geometrical methods). In this article we prove finiteness of the discrete spectrum in case $\mu = 0$ for potentials falling-off as $|x|^{-2-\delta}$, $\delta > 0$, (together with some technical assumptions; see Section 5), using the Berezin-Sigal equations, and working entirely in configuration space. The following notation and definitions will be used throughout this work. If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces we denote by $B(\mathfrak{X}, \mathfrak{Y})$ [resp. $B_0(\mathfrak{X}, \mathfrak{Y})$] the set of all bounded (resp. compact) operators from $\mathfrak{X}$ to $\mathfrak{Y}$. In case $\mathfrak{X} = \mathfrak{Y}$ we write simply $B(\mathfrak{X})$ and

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$B_0(\mathfrak{X})$. If T is a closed operator in X we let $\Sigma(T)$ and $P(T)$ denote its spectrum and resolvent set. If $\mathfrak{X}$ is a Hilbert space and T is self-adjoint we define the discrete spectrum of T, denoted $\Sigma_d(T)$, to be the set of all isolated eigenvalues of finite multiplicity and the essential spectrum of T, denoted $\Sigma_e(T)$ to be $\Sigma(T)/\Sigma_d(T)$. Operator norms will be denoted simply by $\|.\|$, and the letter C will represent various positive constants whose precise values are of no interest. Integrals without explicit domains of integration are to be taken over all of $\mathbb{R}^n$, where n will be clear from the context.

2. N-Body Kinematics

First we introduce some notation. By a cluster decomposition a with clusters $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_s$, we mean a partition of the set $\{1, 2, \ldots, N\}$, and we write $a = \{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_s\}$. We denote by $\mathscr{A}$ the set of all cluster decompositions and by $\mathscr{A}_s$ the set of all cluster decompositions with s clusters. Small script letters are used to denote the elements of $\mathscr{A}$ and in particular the letter ℓ is reserved for the elements of $\mathscr{A}_{N-1}$. If a, $c \in \mathscr{A}$ we say that a is contained in c and write $a \subset c$ in case a can be obtained by breaking up one or more clusters of c. The symbol $a \subseteqq c$ means that either $a \subset c$ or $a = c$. We also let $\# : \mathscr{A} \to Z^+$ be defined by $\#(a) =$ number of clusters in a.

Consider now a system of N particles with masses m_i, position vectors X_i, interacting through pair potentials $V_{ij} \in L^2(\mathbb{R}^3)$, $i, j \in \{1, 2, \ldots, N\}$. This system is described by a total hamiltonian $H_{\text{tot}} = \sum_i (2m_i)^{-1} \Delta_{X_i} + \sum_{i<j} V_{ij}$, acting in $L^2(\mathbb{R}^{3N})$ where the sums are taken over the set $\{1, 2, \ldots, N\}$, Δ_{X_i} is the laplacian with respect to the variable X_i and V_{ij} is the operator of multiplication by $V_{ij}(X_i - X_j)$ (we take Planck's constant to be 1). Let $x = (X_1, \ldots, X_N) \in \mathbb{R}^{3N}$, $a = \{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_s\} \in \mathscr{A}$ and define:

$$U = \left\{ x \in \mathbb{R}^{3N} : \sum_{i=1}^{N} m_i X_i = 0 \right\}, \qquad V = \{x \in \mathbb{R}^{3N} : X_1 = X_2 = \ldots = X_N\}; \tag{2.1}$$

$$U^a = \left\{ x \in U : \sum_{i \in \mathscr{C}_r} m_i X_i = 0, r = 1, 2, \ldots, s \right\}; \tag{2.2}$$

$$U_a = \{x \in U : \text{exists } Y_r \in \mathbb{R}^3 \text{ with } X_i = Y_r, i \in \mathscr{C}_r, r = 1, \ldots, s\}. \tag{2.3}$$

If $d \subseteqq a$ we let $U_d^a = U^a \cap U_d$. We introduce in $\mathbb{R}^{3N}$ the inner product, $[x, y] = \sum_{i=1}^{N} m_i X_i \cdot Y_i$, where the dot denotes the ordinary dot product of $\mathbb{R}^3$. From now on we will always consider $\mathbb{R}^{3N}$ with the structure provided by this inner product. It is easy to see that $U \perp V$, $U^a \perp U_a$, $U^d \perp U_d^a$, $d \subseteqq a$. We have ([11]):

Lemma (2.1). *Let a and d be as above. Then*

a) $\mathbb{R}^{3N} = U \oplus V$, $L^2(\mathbb{R}^{3N}) = L^2(U) \otimes L^2(V)$;

b) $U = U^a \oplus U_a$, $L^2(U) = L^2(U^a) \otimes L^2(U_a)$;

c) $U^a = U^d \oplus U_d^a$, $L^2(U^a) = L^2(U^d) \otimes L^2(U_d^a)$;

d) $\sum_{i=1}^{N} (2m_i)^{-1} \Delta_{X_i} = H_0 \otimes 1' + 1 \otimes T$, $H_0 = -2^{-1}\Delta$, $T = -2^{-1}\Delta'$;

e) $H_0 = H_0^a \otimes 1_a + 1^a \otimes T_a$, $H_0^a = -2^{-1}\Delta^a$, $T_a = -2^{-1}\Delta_a$;

f) $H_0^a = H_0^d \otimes 1_d^a + 1^a \otimes T_d^a$, $T_d^a = -2^{-1}\Delta_d^a$;

where Δ, Δ', Δ^a, Δ_a, Δ_d^a, denote the laplacian in $L^2(U)$, $L^2(V)$, $L^2(U^a)$, $L^2(U_a)$, $L^2(U_d^a)$ respectively, and 1, $1'$, 1^a, 1_a, 1_d^a *stand for the identity operators.*

If P is the orthogonal projection onto U, we have $X_i - X_j = (Px)_i - (Px)_j$, $x \in \mathbb{R}^{3N}$, so that $H_{\text{tot}} = H \otimes 1' + 1 \otimes T$, where $H = H_0 + \sum_{i<j} V_{ij}$. The operator T describes the uniform motion of the center of mass and its properties are well known. The interesting part of H_{tot} is H, which describes the internal structure of the system. If $a \in \mathscr{A}$, we let $V_a = \sum_{i<j}^{a} V_{ij}$, where the symbol $\sum_{i<j}^{a}$ indicates that the sum is taken over all i,j belonging to the same cluster of a. If $d \subseteqq a$, we let V_d^a denote the operator of multiplication by V_d in $L^2(U^a)$, and we define $H_d^a = H_0^a + V_d^a$ and $H^a = H_a^a$. We remark that the operators H_d^a, $d \subseteqq a$, are all self-adjoint, bounded from below, and have domains $D(H_d^a) = D(H_0^a)$ ([9]). We write $R_d^a(z) = (H_d^a - z)^{-1}$ and observe that $R_d^a(z) = R_0^a(z) = (H_0^a - z)^{-1}$ for $d \in \mathscr{A}_N$, and $H = H^a$ for $a \in \mathscr{A}_1$. If P^a denotes the orthogonal projection onto U^a and if the particles i,j belong to the same cluster of a, we have $X_i - X_j = (P^a x)_i - (P^a x)_j$. Hence $H_d^a = H^d \otimes 1_d^a + 1^d \otimes T_d^a$, $d \subseteqq a$. Finally, the following identity holds:

$$1^a + V_d^a R_0^a(z) = (H_d^a - z) R_0^a(z), \; z \notin [0, \infty), \; d \subseteqq a, \; \#(d) \leqq N-1. \tag{2.4}$$

With the notation established above, the Hunziker-van Winter-Zhislin theorem can be stated as follows (for a proof see [13]):

Theorem (2.2). *Suppose $V_{ij} \in L^2(\mathbb{R}^3)$. Then $\Sigma_e(H) = [\mu, \infty)$, $\mu = \min\{\inf \Sigma(H^a) : 2 \leqq \#(a) \leqq N-1\} = \min\{\inf \Sigma(H^a) : a \in \mathscr{A}_2\}$. Moreover, $\Sigma_d(H) = \Sigma(H) \cap (-\infty, \mu)$ and its only possible accumulation point is μ itself*

3. The Berezin-Sigal Equations

The equations which we employ in the study of $\Sigma_d(H)$ were first introduced by Berezin [2] and extensively used by Sigal [9, 10]. We assume for the time being that $\operatorname{Im}(z) \neq 0$. Latter on [see Theorem (3.2)] we will extend our definitions for $z \notin [\mu, \infty)$. Let $\mathscr{O}$ be a total order in $\mathscr{A}$ extending the partial ordering defined by the inclusion relation (for an explicit example of such an order relation see [9]), and let $a \cup c$ denote the smallest element of $\mathscr{A}$ containing both a and c. The Berezin-Sigal equations are defined by induction as follows: let $a, d \in \mathscr{A}$, $d \subseteqq a$, $\#(a)$, $\#(d) \leqq N-1$. If $\#(d) = N-1$, define:

$$L_d^a(z) = V_d^a R_0^a(z), \qquad F_d^a(z) = R_0^a(z). \tag{3.1}$$

From (2.4) we conclude that $1^a + L_d^a(z) = (H_d^a - z) F_d^a(z)$ and also that $(1^a + L_d^a(z))^{-1} \in B(L^2(U^a))$. Assume that we have defined $L_f^a(z)$ for all $f \subset d$ and that $(1^a + L_f^a(z))^{-1} \in B(L^2(U^a))$. Then let,

$$\begin{aligned} 1^a + L_d^a(z) &= (1^a + V_d^a R_0^a(z)) G_d^a(z), \\ F_d^a(z) &= R_0^a(z) G_0^a(z), \end{aligned} \tag{3.2}$$

$$G_d^a(z) = \prod_{f \subset d} (1^a + L_f^a(z))^{-1}, \tag{3.3}$$

where the product is taken in the order $\mathcal{O}$. By (2.4) we $(1^a + V_d^a R_0^a(z))^{-1} \in B(L^2(U^a))$. Therefore, $(1^a + L_d^a(z))^{-1} \in B(L^2(U^a))$. Define:

$$L^a(z) = L_a^a(z), \quad F^a(z) = F_a^a(z), \quad G^a(z) = G_a^a(z), \tag{3.4}$$

$$L(z) = L^a(z), \quad F(z) = F^a(z), \quad G(z) = G^a(z) \quad \text{for} \quad a \in \mathcal{A}_1. \tag{3.5}$$

From (2.4) it follows that $1^a + L_d^a(z) = (H_d^a - z)F_d^a(z)$ and we have:

$$G_d^a(z)^{-1}(1^a + L_d^a(z)) = 1^a + G_d^a(z) V_d^a F_d^a(z). \tag{3.6}$$

In order to use the Berezin-Sigal equations one must expand $L_d^a(z)$ into products of resolvents and potentials. In particular we need the following result to prove that $\Sigma_d(H)$ is finite in case $\mu = 0$:

Lemma (3.1). *$L_d^a(z)$ can be written as a linear combination of terms of the form*

$$V_{b_1}^a R_{c_1}^a V_{b_2}^a R_{c_2}^a \dots V_{b_k}^a R_{c_k}^a, \; c_i \subset d, \; b_i \subseteq d, \; b_i \in \mathcal{A}_{N-1}, \; 1 \leqq i \leqq k, \; \bigcup_{i=1}^{k} b_i = d.$$

The proof of this lemma can be found in [9] [note that the equations used there are adjoints in $L^2(U^a)$ of the ones used in the present work]. The main tool in the analysis of the Berezin-Sigal equations is the following representation of $R_c^a(z)$, $c \subset a$, in terms of spectral integrals (see Appendix 2):

Theorem (3.2). *Suppose $z \notin [\mu, \infty)$. Then $z \in P(H_c^a)$, $c \subset a$, and,*

$$R_c^a(z) = \int_{\mu}^{\infty} E^c(d\xi) \otimes R_{T_c^a}(z - \xi) = \int_{0}^{\infty} R^c(z - \xi) \otimes E_{T_c^a}(d\xi), \tag{3.7}$$

where $R_{T_c^a}(z) = (T_c^a - z)^{-1}$ and E^c, $E_{T_c^a}$ denote the spectral families of H^c and T_c^a respectively. In particular the Berezin-Sigal equations hold for $z \notin [\mu, \infty)$.

4. Properties of $L_d^a(z)$ and $F_d^a(z)$—Case $\mu = 0$

From now on we will assume that $\mu = 0$ and that the potentials are short-range, i.e., $|V_{ij}(y)| \leqq C(1 + |y|^2)^{-\varrho}$, $\varrho > 1$, $i, j \in \{1, 2, \dots, N\}$. It should be noted that the results in this work hold for potentials $V_{ij}(y) = f(y)(1 + |y|^2)^{-\varrho}$, $f \in L^2(\mathbb{R}^3) + L\infty(\mathbb{R}^3)$. We do not prove this explicitly because it would lengthen considerably the present article without introducing any new ideas. We will now define the function spaces in which we analyse the operators $L_d^a(z)$ and $F_d^a(z)$. Let $\mathcal{A}_{N-1}(a) = \{b \in \mathcal{A}_{N-1} : b \subseteq a\}$, $\#(a) \leqq N - 1$. If $b = \{(i, j), (k), \dots, (n)\} \in \mathcal{A}_{N-1}(a)$, we let q_b^a denote the operator of multiplication by $q(X_i - X_j)$ in $L^2(U^a)$, where $q(y) = (1 + |y|^2)^{-\varrho/2}$, $y \in \mathbb{R}^3$. In particular we may write $V_b^a = f_b(q_b^a)^2$, $f_b \in L^\infty(U^b)$. Let $\| \quad \|_{0,a}$ and $(|)_{0,a}$ denote the norm and inner product of $L^2(U^a)$, and define:

$$H^a = \sum \oplus H_b^a, \quad \|g\|_0^a = (\sum \|g_b\|_{0,a}^2)^{1/2}, \quad Q^a g = \sum q_b^a g_b, \tag{4.1}$$

where $H_b^a = L^2(U^a)$, $b \in \mathcal{A}_{N-1}(a)$, $g = (g_b)_{b \in \mathcal{A}_{N-1}(a)} \in H^a$ and the sums are over $\mathcal{A}_{N-1}(a)$.

Definition (4.1). Let $S_\varrho(U^a)$ be the set of all functions $u = Q^a g$, $g \in H^a$, provided with the norm $\|u\|_a = \inf \|g\|_0^a : u = Q^a g\}$, and let $S_\varrho^*(U^a)$ be the set of all complex valued measurable functions v such that $\|v\|_a^* = \sup\{\|q_b^a v\|_{0,a} : b \in \mathcal{A}_{N-1}\} < \infty$.

The spaces $S_\varrho(U^a)$ and $S_\varrho^*(U^a)$ are Banach spaces and it is easy to see that $S_\varrho(U^a) \subset L^2(U^a) \subset S_\varrho^*(U^a)$, the inclusions being continuous. Moreover if we let

$$\Gamma^a(u,v) = \int_{U^a} u(x)\overline{v(x)}dx, \quad u \in S_\varrho(U^a), \quad v \in S_\varrho^*(U^a), \tag{4.2}$$

it follows at once that $|\Gamma^a(u,v)| \leqq \|u\|_a \|v\|_a^*$, and also $\Gamma^a(u,v) = (u|v)_{0,a}$ if $v \in L^2(U^a)$. It is convenient to introduce the following definition:

Definition (4.2). Let D be an open set in $\mathbb{R}$ or $\mathbb{C}$ and let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces. An operator valued function $z \in D \mapsto A(z) \in B(\mathfrak{X}, \mathfrak{Y})$ is said to be WB (for well behaved) in $B(\mathfrak{X}, \mathfrak{Y})$ if it is uniformly bounded, uniformly continuous with respect to the norm topology of $B(\mathfrak{X}, \mathfrak{Y})$. If in addition $A(z) \in B_0(\mathfrak{X}, \mathfrak{Y})$ we say that $A(z)$ is CWB (for compactly WB) in $B(\mathfrak{X}, \mathfrak{Y})$.

Theorem (4.3). *Let $b \in \mathscr{A}_{N-1}(a)$. Then $z \notin [0,\infty) \mapsto L^b(z)$ (resp. $z \notin [0,\infty) \mapsto F^b(z)$) is CWB in $B(S_\varrho(U^a))$ (resp. WB in $B(S_\varrho(U^a), S_\varrho^*(U^a))$). Moreover, $L^b(z)$ tends to zero in the norm of $B(S_\varrho(U^b)$ as $\mathrm{Re}(z) \mapsto -\infty$.*

Proof. By Definition (4.1) it is enough to show: $z \notin [0,\infty) \mapsto f_b q_b^b R_0^b(z) q_b^b$ is CWB in $B(L^2(U^b))$ and tends to zero in norm as $\mathrm{Re}(z) \mapsto -\infty$. Let $b = \{(k,j), (m), \ldots, (n)\}$, $k<j$. Identifying U^b with $\mathbb{R}^3$ through the isomorphism $\xi = \Phi^b(x) = X_k - X_j$, $x \in U^b$, we see that $f_b q_b^b R_0^b(z) q_b^b$ has kernel $C f_b(\xi) q(\xi) |\xi - \eta|^{-1} \exp(-i\sqrt{z}|\xi-\eta|) q(\eta)$, $\mathrm{Im}(z) > 0$. Since $\varrho > 1$, $q^2 \in L^p(\mathbb{R}^3) \cap L^s(\mathbb{R}^3)$, with $p = (3/2) - \delta$, $s = (3/2) + \delta$, $\delta \in (0, 3\varrho - 3)$. The CWB properties then follow by combining the Sobolev inequality ([8], p. 31) and the estimate $|\exp(itz) - \exp(itz')| \leqq 2^{1-\gamma} t^\gamma |z - z'|^\gamma$, $\mathrm{Im}(z)$, $\mathrm{Im}(z') \geqq 0$, $\gamma \in [0,1]$. Since $\|R_0(z)\| \leqq |\mathrm{Re}(z)|^{-1}$ for $\mathrm{Re}(z) < 0$ and $f_b, q_b^b \in L^\infty(U^b)$ we are done. Q.E.D.

Theorem (4.4). *Let $\#(a) < N-1$. Then $L^a(z) \in B_0(S_\varrho(U^a))$, $z \notin [0,\infty)$ and tends to zero in the norm of $B(S_\varrho(U^a))$ as $\mathrm{Re}(z) \to -\infty$.*

Proof. The proof of compactness is long and technical and can be found in Appendix 3. As for the second statement, it is enough to show that operators of the form

$$(f_{b_1} q_{b_1}^a R_{c_1}^a(z) q_{c_2}^a)(f_{b_2} q_{b_2}^a R_{c_2}^a(z) q_{b_3}^a) \ldots (f_{b_k} q_{b_k}^a R_{c_k}^a(z) q_b^a),$$

$$b_i, b \in \mathscr{A}_{N-1}(a), \qquad c_i \subset a, \qquad \bigcup_{i=1}^k b_i = a,$$

tend to zero in the norm of $B(L^2(U^a))$. Since $\mu = 0$, we have $\|R_c^a(z)\| \leqq |\mathrm{Re}(z)|^{-1}$ for $c \subset a$, $\mathrm{Re}(z) < 0$, the result follows. Q.E.D.

Theorem (4.5). *Let $d \subsetneqq a$, $\#(a) < N-1$, $\#(d) \leqq N-1$, and assume that for all $f \subset d$ we have: i) $\lambda \in (-\infty, 0) \mapsto L^f(\lambda)$ is WB in $B(S_\varrho(U^f))$ (and therefore CWB by the preceding theorem); ii) $\lambda \in (-\infty,0) \mapsto F^f(\lambda)$ is WB in $B(S_\varrho(U^f), S_\varrho^*(U^f))$; iii) $0 \notin \Sigma L^f(0)$. Then the map $\lambda \in (-\infty, 0) \mapsto L_d^a(\lambda)$ is WB in $B(S_\varrho(U^a))$.*

Proof. It is enough to show that $\lambda \in (-\infty, 0) \mapsto f_b q_b^a R_c^a(\lambda) q_{b'}^a$, $b, b' \in \mathscr{A}_{N-1}(a)$, $c \subset d$, is WB in $B(L^2(U^a))$. The proof of this fact with $c \in \mathscr{A}_N$ [in which case $R_c^a(\lambda) = R_0^a(\lambda)$], can be found in Appendix 4. If $c \notin \mathscr{A}_N$ there are several cases:

Case 1. $\mathscr{b},\mathscr{b}'\subset c$. By i), iii) and the preceding theorem, $\lambda\in(-\infty,0)\mapsto(1+L^c(\lambda))^{-1}$ is WB in $B(S_\varrho(U^c))$. Combining this with ii) and Theorem (3.2) we obtain,

$$f_{\mathscr{b}}q_{\mathscr{b}}^a R_c^a(\lambda)q_{\mathscr{b}'}^a=\int_0^\infty f_{\mathscr{b}}q_{\mathscr{b}}^c R^c(\lambda-\xi)q_{\mathscr{b}'}^c\otimes E_{T_c}{}^a(d\xi)$$

$$=\int_0^\infty f_{\mathscr{b}}q_{\mathscr{b}}^c F^c(\lambda-\xi)(1+L^c(\lambda-\xi))^{-1}\otimes E_{T_c}{}^a(d\xi),$$

and the result follows in this case from Lemma (A.2.1) of Appendix 2.

Case 2. $\mathscr{b}\subset c$, $\mathscr{b}'\not\subset c$. This follows from Case 1 and the resolvent equation $R_c^a(\lambda)=R_0^a(\lambda)-R_0^a(\lambda)V_c^aR_c^a(\lambda)$. Indeed, $V_c^a=\sum_{\mathscr{b}''\in\mathscr{A}_{N-1}(c)} f_{\mathscr{b}''}(q_{\mathscr{b}''}^a)^2$ and we may write,

$$f_{\mathscr{b}}q_{\mathscr{b}}^a R_c^a(\lambda)q_{\mathscr{b}'}^a=f_{\mathscr{b}}q_{\mathscr{b}}^a R_0^a(\lambda)q_{\mathscr{b}'}^a-\sum_{\mathscr{b}''\in\mathscr{A}_{N-1}(c)} \cdot(f_{\mathscr{b}}q_{\mathscr{b}}^a R_0^a(\lambda)q_{\mathscr{b}''}^a)(f_{\mathscr{b}''}q_{\mathscr{b}''}^a R_c^a(\lambda)q_{\mathscr{b}'}^a).$$

Case 3. $\mathscr{b}\subset c$, $\mathscr{b}'\not\subset c$. This case is similar to Case 2.

Case 4. $\mathscr{b}\not\subset c$, $\mathscr{b}'\not\subset c$. Follows from Cases 2, 3 and the resolvent equation. Q.E.D.

Corollary (4.6). *Suppose that the conditions of Theorem (4.5) are satisfied. Then* $\lambda\in(-\infty,0)\mapsto G_d^a(\lambda)^{-1}$, $\lambda\in(-\infty,0)\mapsto V_d^aF_d^a(\lambda)$ *are WB in* $B(S_\varrho(U^a))$ *and* $\lambda\in(-\infty,0)\mapsto F_d^a(\lambda)$ *is WB in* $B(S_\varrho(U^a), S_\varrho^*(U^a))$.

Proof. This result follows by observing that $V_d^a\in B(S_\varrho^*(U^a), S_\varrho(U^a))$, $G_d^a(\lambda)^{-1}=\prod'_{f\subset d}(1^a+L_f^a(\lambda))$, $F_d^a(\lambda)=\prod_{f\subset d}\prod'_{h\subset f}(1^a+L_d^a(\lambda))(1^a-V_f^aR_f^a(\lambda))$, where the prime indicates that the product is taken in the inverse order of the order $\mathscr{O}$, and by using Definition (4.1), Theorem (4.5) and the results of Appendix 4. Q.E.D.

5. Finiteness of $\Sigma_d(H)$—Case $\mu=0$

We will now prove that $\Sigma_d(H)$ is finite. Similar proofs have appeared in [3.10]. A proof of the following lemma can be found in these references.

Lemma (5.1). *Let* $\lambda\in(-\infty,0)\mapsto A(\lambda)\in B(X)$ *be CWB and* $\{\lambda_n\}$, $\{x_n\}$, $n=1,2,\ldots,$ *be such that* $-x_n=A(\lambda_n)x_n$, $\|x_n\|=1$. *Then there are subsequences* $\{x_{n_k}\}$, $\{\lambda_{n_k}\}$, *and* $x\in X$ *such that* $x=\lim_{k\to\infty}x_{n_k}$, $-x=A(0)x$.

Theorem (5.2). *Suppose that* $(-1)\notin\Sigma L^a(0)$, $2\leqq\#(a)\leqq N-1$. *Then* $\Sigma_d(H)$ *is finite.*

Proof. Assume that $\Sigma_d(H)$ is infinite and define $W=\Sigma q_{\mathscr{b}}Mq_{\mathscr{b}}$, $H(t)=H+tW$, $L(\lambda,t)=L(\lambda)+tWF(\lambda)$, where $t\in[0,1]$, $\lambda<0$, $M\in B_0(L^2(U))$, $M=M^*$, M strictly positive, $q_{\mathscr{b}}=q_{\mathscr{b}}^a$, $a\in A_1$, and the sum is taken over $\mathscr{A}_{N-1}$. It follows that $1+L(\lambda,t)=(H(t)-\lambda)F(\lambda)$ and that $\lambda\in(-\infty,0)\mapsto L(\lambda,t)$ is CWB in $B(S_\varrho(U))$. From Theorem (2.2), the mini-max principle and the invariance of the essential spectrum under relatively compact perturbations we conclude that $\Sigma_e(H(t))=\Sigma_e(H)$, $\Sigma_d(H(t))$ is infinite and accumulates at zero. Using Lemma (5.1) twice we construct sequences $\{t_n\}\downarrow 0$, $\{\varphi_n\}$, and $\varphi\in S_\varrho(U)$ such that $-\varphi_n=L(0,t_n)\varphi_n$, $\varphi=\lim_n\varphi_n$, $\|\varphi_n\|_\varrho=1$,

$-\varphi = L(0)\varphi$. Hence, $(t_n)^{-1}\Gamma((1+L(0))\varphi_n, \ F(0)\varphi) = \Gamma(WF(0)\varphi_n, \ F(0)\varphi)$, where $\Gamma = \Gamma^a$, $a \in \mathscr{A}_1$ is defined in (4.2). But

$$0 = \overline{\Gamma((1+L(0))\varphi, F(0)\varphi_n)} = \lim_{\lambda \to 0} (F(\lambda)\varphi_n | (H-\lambda)F(\lambda)\varphi)_0$$
$$= \lim_{\lambda \to 0} ((1+L(\lambda))\varphi_n | F(\lambda)\varphi)_0 = \Gamma((1+L(0))\varphi_n, F(0)\varphi).$$

Therefore, $\Gamma(WF(0)\varphi, F(0)\varphi) = 0$. Applying the definition of W, and Corollary (4.6), we conclude that $F(0)\varphi = 0$. By (3.6), $G(\lambda)^{-1}(1+L(\lambda))\varphi = \varphi - G(\lambda)^{-1}VF(\lambda)\varphi$, $V = V^a_a$, $a \in \mathscr{A}_1$, $\lambda < 0$. Letting $\lambda \to 0$, we get $\varphi = 0$, a contradiction. Q.E.D.

Appendix 1. Structure of U^a in Terms of Relative Coordinates

Let $a \in \mathscr{A}$ consist of a cluster $\mathscr{C}$ with K particles, $K \geqq 2$ and $(N-K)$ one-particle clusters. To simplify the notation we will write $a = \{\mathscr{C}\}$. Let $\mathscr{B} = \{b_1, b_2, \ldots, b_k\} \subseteqq \mathscr{A}_{N-1}$ be such that $a = \bigcup_{j=1}^{k} b_j$. If $K=2$ we must have $a = b_1 = \ldots = b_k$. If $K > 2$ we may assume that $b_i \neq b_j$ if $i \neq j$ because $b \cup b' = b' \cup b$ for all $b, b' \in \mathscr{A}_{N-1}$. Let $b_1 = \{(i_1, i_2)\}$ and set $b'_1 = b_1$, $\mathscr{Q}(b'_1) = \{b \in \mathscr{B} : b \not\subseteq b'_1\}$. This set must be non-empty for otherwise $a = b'_1$, a contradiction since $K > 2$. Moreover there is a $b = \{(i,j)\} \in \mathscr{Q}(b'_1)$ such that either i or j belongs to the set $\{i_1, i_2\}$. Otherwise every element of $\mathscr{Q}(b'_1)$ must have the form $\{(k,n), \ldots, (i_1), (i_2)\}$ so that (i_1, i_2) will be a cluster of a, a contradiction because $K > 2$ and a has only one cluster with more than one particle. Let b'_2 be the smallest such b and note that the two particle clusters of b'_1 and b'_2 have exactly one particle, say i_2, in common. Hence $b'_2 = \{(i_2, i_3)\}$ with $i_3 \in \mathscr{C}$ and $b'_1 \cup b'_2 = \{(i_1, i_2, i_3)\}$. Moreover if $k=3$ we must have $a = b'_1 \cup b'_2$. If $K > 3$ we set $\mathscr{Q}(b'_1, b'_2) = \{b \in \mathscr{B} : b \not\subseteq b'_1 \cup b'_2\}$. Arguments similar to those used above show that there is a $b'_3 \in \mathscr{Q}(b'_1, b'_2)$ with the property that two particle cluster of b'_3 and the three particle cluster of $b'_1 \cup b'_2$ have exactly one particle, say i_3, in common. Thus $b'_3 = \{(i_3, i_4)\}$, $i_4 \in \mathscr{C}$ and $b'_1 \cup b'_2 \cup b'_3 = \{(i_1, i_2, i_3, i_4)\}$. If $K=4$ we must have $a = b'_1 \cup b'_2 \cup b'_3$. Otherwise proceed as above [i.e., define $\mathscr{Q}(b'_1, b'_2, b'_3)$ and so on] until we exhaust the particles in $\mathscr{C}$. In this way we obtain the first part of the following lemma:

Lemma (A.1.1). *Let a and $\mathscr{B}$ be as above. Then $k \geqq K-1$ and there is a subsequence $b'_1, b'_2, \ldots, b'_{K-1}$ such that, $b'_j = \{(i_j, i_{j+1})\}$, $i_j, i_{j+1} \in \mathscr{C}$, $\bigcup_{m=1}^{j} b'_m = \{(i_1, i_2, \ldots, i_{j+1})\}$, $j = 1, 2, \ldots, K-1$ and $a = \bigcup_{m=1}^{K-1} b'_m$. Moreover the map,*

$$\Phi^a = (\Phi_{b'_1}, \ldots, \Phi_{b'_{K-1}}) : U^a \to \mathbb{R}^{3(K-1)}, \ \Phi_{b'_j}(x) = X_{s_j} - X_{S_j}, \tag{A.1.1}$$

where $s_j = \min\{i_j, i_{j+1}\}$, $S_j = \max\{i_j, i_{j+1}\}$ is an isomorphism of U^a onto $\mathbb{R}^{3(K-1)}$. In particular U^a and $L^2(U^a)$ can be identified with $\mathbb{R}^3$ and $L^2(\mathbb{R}^{3(K-1)})$.

Proof. Let $x = (X_1, \ldots, X_N) \in U^a$. Then $\sum_{n=1}^{K} m_{i_n} X_{i_n} = 0$, $X_i = 0$, $i \notin \mathscr{C}$. Now, if $\Phi^a(x) = 0$, we must have $X_{i_1} = X_{i_2} = \ldots = X_{i_K}$. Taking this into the preceding equations we conclude that $x = 0$. Q.E.D.

Now let $a=\{\mathscr{C}_1,\mathscr{C}_2,\ldots,\mathscr{C}_t,(j_1),\ldots,(j_r)\}$, where each cluster $\mathscr{C}_m$ contains K_m particles, $K_m\geqq 2$, $1\leqq m\leqq t$, $t\geqq 1$ and the remaining clusters contain one particle each. Let $a_m=\{\mathscr{C}_m\}$. Then, $a=\bigcup_{m=1}^{t} a_m$ and $U^a=\sum_{m=1}^{t}\oplus U^{a_m}$. Let $\mathscr{B}=\{b_1,b_2,\ldots,b_k\}$ be such that $a=\bigcup_{j=1}^{t} b_j$. Without loss of generality we may assume that $b_1\subseteqq\{\mathscr{C}_1\}$ and that the elements of $\mathscr{B}$ are all distinct. Let $\mathscr{B}_m=\{b=\{(i,j)\}:i,j\in\mathscr{C}_m\}$. It follows that $\mathscr{B}_m$ and $\mathscr{B}_{m'}$ are disjoint if $m\neq m'$ and $a_m=\bigcup_{b\in\mathscr{B}_m} b$. Applying Lemma (A.1.1) to each a_m we obtain subsequences $b_1^m,\ldots,b_{K_m-1}^m$ of $\mathscr{B}_m$ and isomorphisms $\Phi^{a_m}:U^a m\to\mathbb{R}^{3(K_m-1)}$. Then $\Phi^a=\sum_{m=1}^{t}\oplus\Phi^{a_m}:U^a\to\mathbb{R}^{3s}$, $s=\sum_{m=1}^{t}(K_m-1)$ is an isomorphism of U^a onto $\mathbb{R}^{3s}$. It may be written as $\Phi^a=(\Phi_{b_{j_m}^m})$, $\Phi_{b_{j_m}^m}(x)=X_{s_{j_m}}-X_{S_{j_m}}$, $s_{j_m}=\min\{i_{j_m}^m,\ i_{j_m}^m\}$, $S_{j_m}=\max\{i_{j_m},i_{j_m+1}\}$, $j_m=1,2,\ldots,K_{m-1}$, $1\leqq m\leqq t$ and $\mathscr{C}_m=(i_1^m,i_2^m,\ldots,i_{K_m}^m)$.

Appendix 2. Spectral Integrals

Let Z be a Hilbert space and $F(\xi)$, $\xi\in\mathbb{R}$ a (right continuous) spectral family in Z. If $A:[a,\infty)\to B(Z)$ is a function, we define $\int_a^b A(\xi)F(d\xi)$ to be the norm limit of Riemman-Stieltjes sums. The integral over $[a,\infty)$ is then defined as the strong limit of integrals over $[a,b]$ as $b\to\infty$. In Theorem (3.2) we used the notation $E^c(d\xi)\otimes R_{T_c^a}(z-\xi)=(1\otimes R_{T_c^a}(z-\xi))(E^c(d\xi)\otimes 1)$ and $R^c(z-\xi)\otimes E_{T_c^a}(d\xi)=(R^c(z-\xi)\otimes 1)(1\otimes E_{T_c^a}(d\xi))$. It can be shown that,

Lemma (A.2.1). *If $A:[a,\infty)\to B(Z)$ is continuous, $A(\xi)F(\eta)=F(\eta)A(\xi)$, for all $\xi,\eta\in[a,\infty)$ and $\|A(\xi)\|\to 0$ as $\xi\to\infty$, then $\int_a^\infty A(\xi)F(d\xi)$ exists, the strong limit may be replaced by norm limit and* $\left\|\int_a^\infty A(\xi)F(d\xi)\right\|\leqq\sup\{\|A(\xi)\|:\xi\in[a,\infty)\}$.

For a proof of this lemma, that of Theorem (3.2) and for further properties and applications of spectral integrals see [1, 4, 6, 7].

Appendix 3. Compactness of $L^a(z)$, $z\notin[0,\infty)$

By Lemma (3.1) and Definition (4.1) it is enough to show that any operator of the form $Q(z)f_{b_k}q_{b_k}R_{c_k}^a(z)q_b^a$ belongs to $B_0(L^2(U^a))$, where

$$\begin{aligned} Q(z)&=\prod_{i=1}^{k-1} f_{b_i}q_{b_i}^a R_{c_i}^a(z)q_{b_{i+1}}^a, \qquad b,b_i\in\mathscr{A}_{N-1}(a),\\ c_i&\subset a=\bigcup_{i=1}^{k} b_i, \qquad f_{b_i}\in L(U^{b}i). \end{aligned}\tag{A.3.1}$$

Since $z\notin[0,\infty)\mapsto f_b q_b^a R_c^a(z)q_{b'}^a\in B(L^2(U^a))$, $b,b'\in\mathscr{A}_{N-1}(a)$, $c\subset a$, is analytic, it suffices to show that $Q(z)$ is compact for $z=\lambda\in\mathbb{R}$ sufficiently negative ([12], Appendix 3). Let $N_j(\lambda)=(q_{b_{j+1}}^a\cdots q_{b_k}^a)^{-1}R_{c_j}^a(\lambda)q_{b_{j+1}}^a\cdots q_{b_k}^a$. Then,

$$Q(\lambda)=f_{b_1}q_{b_1}^a R_{c_1}^a(\lambda)q_{b_2}^a\cdots q_{b_k}^a\prod_{j=2}^{k-1} f_{b_j}N_j(\lambda). \tag{A.3.2}$$

Lemma (A.3.1). *$N_j^0(\lambda)=(q^a_{\ell_{j+1}}\dots q^a_{\ell_k})^{-1}R_0^a(\lambda)q^a_{\ell_{j+1}}\dots q^a_{\ell_k}\in B(L^2(U^a))$ for $\lambda<0$, and tends to zero in norm as $\lambda\to-\infty$.*

Proof. Let $x,y\in U^a$, $\ell=\{(i,k),(m),\dots,(n)\}\in\mathscr{A}_{N-1}(a)$. Using the triangle inequality and the fact that the function $u\in[0,\infty)\mapsto(1+u^2)(1+u)^{-2}$ is bounded with bounded inverse it is easy to see that $q(X_i-X_k)^{-1}\leqq(1+[x-y]^2)^\varrho q(Y_i-Y_k)^{-1}$. Using the notation and results of Appendix 1 we identify U^a with $\mathbb{R}^{3s}$. Consider,

$$\psi(\xi)=((q^a_{\ell_{j+1}}\dots q^a_{\ell_k})^{-1}\exp(-tH_0^a)q^a_{\ell_{j+1}}\dots q^a_{\ell_k}\varphi)(\xi),\quad \varphi\in L^2(\mathbb{R}^{3s}). \tag{A.3.3}$$

The operator $\exp(-tH_0^a)$ has kernel $g(\xi-\eta)=\alpha t^{-3s/2}\exp(-\beta t^{-1}|\xi-\eta|^2)$, where α,β are positive constants. Using the estimate for $q(X_i-X_k)^{-1}$, we obtain:

$$|\psi(\xi)|\leqq Ct^{-3s/2}\int d\eta(1+|\xi-\eta|^2)^\gamma g(\xi-\eta)|\varphi(\eta)|, \tag{A.3.4}$$

where γ is a positive integer larger or equal to $\varrho(k-j+1)$. Now, $(1+|\cdot|^2)^\gamma g(\cdot)$ belongs to $L^1(\mathbb{R}^{3s})$ for $t>0$ so that by Young's theorem on the convolution ([8], p. 28) we get $\|\psi\|_2\leqq Ct^{-3s/2}\|(1+|\cdot|^2)^\gamma g(\cdot)\|_1\|\varphi\|_2$, $t>0$, where $\|\cdot\|_p$ denotes the L^p norm. Using spherical coordinates and the binomial theorem it is easy to show that $\|(1+|\cdot|^2)^\gamma g(\cdot)\|_1\leqq Ct^{3s/2}w(t)$, $w(t)$ a polynomial with positive coefficients. Since,

$$R_0^a(\lambda)=-\int_0^\infty dt\exp(\lambda t)\exp(-tH_0^a),\quad \lambda<0, \tag{A.3.5}$$

it follows that $\|N_j^0(\lambda)\|\leqq C\int_0^\infty dt\exp(\lambda t)w(t)$. Q.E.D.

Lemma (A.3.2). *There is a $\lambda_j<0$ such that $N_j(\lambda)\in B(L^2(U^a))$ for all $\lambda\leqq\lambda_j$.*

Proof. Since $R^a_{c_j}(\lambda)=R_0^a(\lambda)-R^a_{c_j}(\lambda)V^a_{c_j}R_0^a(\lambda)$ and V^a_{cj} is a bounded function, the preceeding lemma implies that $\|V^a_{c_j}N_j^0(\lambda)\|\leqq1$ for $\lambda\leqq\lambda_j$, for some $\lambda_j<0$. Thus,

$$R^a_{c_j}(\lambda)q^a_{\ell_{j+1}}\dots q^a_{\ell_k}=R_0^a(\lambda)q^a_{\ell_{j+1}}\dots q^a_{\ell_k}(1+V^a_{c_j}N_j^0(\lambda))^{-1},\quad\lambda\leqq\lambda_j.$$

Multiplying both sides by $(q^a_{\ell_{j+1}}\dots q^a_{\ell_k})^{-1}$ the lemma follows. Q.E.D.

Similar arguments show that,

$$f_{\ell_1}q^a_{\ell_1}R^a_{c_1}(\lambda)q^a_{\ell_2}\dots q^a_{\ell_k}=f_{\ell_1}q^a_{\ell_1}R_0^a(\lambda)q^a_{\ell_2}\dots q^a_{\ell_k}(1+V^a_{c_1}N_1^0(\lambda))^{-1}, \tag{A.3.6}$$

for sufficiently negative. Thus to prove that $Q(\lambda)\in B_0(L^2(U^a))$ for λ sufficiently negative it is enough to show that $q^a_{\ell_1}R_0^a(\lambda)q^a_{\ell_2}\dots q^a_{\ell_k}\in B_0(L^2(U^a))$, $\lambda\in(-\infty,0)$. Let $a=\{\mathscr{C}_1,\mathscr{C}_2,\dots,\mathscr{C}_t,(j_1),\dots,(j_r)\}$, K_m be as in Appendix 1. Recall that $a=\bigcup_{i=1}^k\ell_i$ and let $\ell^m_{j_m}$, $j_m=1,\dots,K_m-1$ be the sequence constructed after the proof of Lemma (A.1.1) (note that $\ell_1=\ell_1^1$ by construction). Since $q(y)\leqq1$, it suffices to prove that $T(\lambda)=q^a_{\ell_1^1}R_0^a(\lambda)q^a_{\ell_2^1}\dots q^a_{\ell_{K_t-1}}\in B_0(L^2(U^a))$. Identifying U^a with $\mathbb{R}^{3s}$ we see that:

$$\begin{aligned}(T(\lambda)\varphi)(\xi)=q(\xi_1^1)\int d\eta K_\lambda^0(\xi-\eta)&\\ \cdot q(\eta_2^1)\dots q(\eta^t_{K_t-1})\varphi(\eta),&\quad\varphi\in L^2(\mathbb{R}^{3s}),\end{aligned} \tag{A.3.7}$$

where $K_\lambda^0\in L^1(\mathbb{R}^{3s})$ ([8, p. 58, 59) is the resolvent kernel. Compactness then follows from the next lemma. In order to state it let $C_\infty(\mathbb{R}^n)$ denote the set of continuous complex valued functions defined on $\mathbb{R}^n$ that vanish at infinity. It is well known that such functions can be uniformly approximated by functions in $C_0^\infty(\mathbb{R}^n)$.

Lemma (A.3.3). *Let* $F\in C_\infty(\mathbb{R}^3)$, $G\in C_\infty(\mathbb{R}^{3s-3})$ *and* $S\in L^1(\mathbb{R}^{3s})$. *Let* M_F, M_G *denote the operators of multiplication by* $F(\xi_1^1)$ *and* $G(\xi_1^2,\dots,\xi_{K_t-1}^t)$ *in* $L^2(\mathbb{R}^{3s})$. *Let* $B=S*\varphi$, $\varphi\in L^2(\mathbb{R}^{3s})$. *Then* $M^F BM_G\in B^0(L^2(\mathbb{R}^{3s}))$.
Proof. It is easy to see that $\|M_F BM_G\|\leqq\|F\|_\infty\|G\|_\infty\|S\|_1$. Thus if F_n, G_n, S_n, $n=1,2,\dots$ converging to F, G, S in $L^\infty(\mathbb{R}^3)$, $L^\infty(\mathbb{R}^{3s-3})$, $L^1(\mathbb{R}^{3s})$, the sequence $M_{F_n}B_nM_{G_n}$ converges to M_FBM_G in the norm of $B(L^2(\mathbb{R}^{3s}))$. Finally it is easy to see that $M_{F_n}B_nM_{G_n}$ is Hilbert-Schmidt for all n since all functions involved have compact support. Q.E.D.

Appendix 4. The Map $\lambda\in(-\infty,0)\mapsto f_{\mathscr{b}}q_{\mathscr{b}}^a R_0^a(\lambda)$, $q_{\mathscr{b}'}^a$, $\mathscr{b},\mathscr{b}'\in\mathscr{A}_{N-1}(a)$, $\#(a)\leqq N-1$, $f_{\mathscr{b}}\in L(U^{\mathscr{b}})$

We will show that this map is WB in $B(L^2(U^a))$. If $\mathscr{b}=\mathscr{b}'$, it follows from Theorem (3.2) that,

$$f_{\mathscr{b}}q_{\mathscr{b}}^a R_0^a(\lambda)q_{\mathscr{b}}^a=\int\limits_c^\infty f_{\mathscr{b}}q_{\mathscr{b}}^{\mathscr{b}}R_0^{\mathscr{b}}(\lambda-\xi)q_{\mathscr{b}}^{\mathscr{b}}\otimes E_{T_c^a}(d\xi)\,. \tag{A.4.1}$$

The result then follows Theorem (4.3) and Lemma (A.2.1). Now consider $\mathscr{b}\neq\mathscr{b}'$, and let $d=\mathscr{b}\cup\mathscr{b}'$. It follows that:

$$f_{\mathscr{b}}q_{\mathscr{b}}^a R_0^a(\lambda)q_{\mathscr{b}'}^a=\int\limits_0^\infty f_{\mathscr{b}}q_{\mathscr{b}}^d R_0^d(\lambda-\xi)q_{\mathscr{b}'}^d\otimes E_{T_c^a}(d\xi)\,. \tag{A.4.2}$$

Using (A.3.6) with a replaced by d we see that it is enough to show,

$$\int\limits_0^\infty dt\|q_{\mathscr{b}}^d\exp(-tH_0^d)q_{\mathscr{b}'}^d\|<\infty\,. \tag{A.4.3}$$

Case 1. The two particle clusters of $\mathscr{b}$ and $\mathscr{b}'$ have one particle in common. Let $\mathscr{b}=\{(i,j),(m),\dots,(n)\}$ and $b'=\{(j,k),(m'),\dots,(n')\}$. Then, if $x\in U^d$ we have $m_iX_i+m_jX_j+m_kX_k=0$, $X_s=0$, $s\neq i,j,k$. Define,

$$\zeta=X_i-X_j\,,\qquad \eta=X_k-(m_i+m_j)^{-1}(m_iX_i+m_jX_j)\,. \tag{A.4.4}$$

Then $x\in U^d\mapsto(\zeta,\eta)\in\mathbb{R}^6$ is an isomorphism and $X_i-X_j=a\zeta+b\eta$, a, $b\in\mathbb{R}$. Consider:

$$\begin{aligned}\psi(\zeta,\eta)&=(q_{\mathscr{b}}^d\exp(-t(H_0^{\mathscr{b}}\otimes 1_{\mathscr{b}'}^d)q_{\mathscr{b}'}^d\varphi)(\zeta,\eta)\\&=\alpha t^{-3/2}q(\zeta)\int d\zeta'\exp(-\beta t^{-1}|\zeta-\zeta'|^2)q(a\zeta+b\zeta)\varphi(\zeta',\eta)\end{aligned}$$

where α,β are positive constants and $\varphi\in L^2(\mathbb{R}^6)$. Let v satisfy $0v<\min\{1,(3-\varrho)\varrho^{-1}\}$ and let $p=3-v$. Then $p>2$ and $q\in L^p(\mathbb{R}^3)$ since $(3-v)\varrho>3$. Choose r,s such that $2^{-1}+p^{-1}=r^{-1}$, $r^{-1}+s^{-1}=1$, $s^{-1}+p^{-1}=2^{-1}$ (note that $1\leqq r\leqq 2$). Using the Riesz-Thorin interpolation theorem ([8], p. 27), it is easy to show that $\psi(\cdot,\eta)$ belongs to $L^2(\mathbb{R}^3,d\xi)$ and $\|\psi(\cdot,\eta)\|_{L_2(d\xi)}$

$\leqq Ct^{-3/p}\|q\|_p^2\|\varphi(\cdot,\eta)\|_{L_2(d\xi)}$. Thus $\|q_{\ell}^d \exp(-t(H_0^{\ell}\otimes 1_{\ell'}^d)q_{\ell'}^d\| \leqq Ct^{-3/p}$. Since $H_0^d = H_0^{\ell}\otimes 1_{\ell'}^d + 1^{\ell}\otimes T_{\ell'}^d$ we have:

$$\|q_{\ell}^d \exp(-tH_0^d)q_{\ell'}^d\| \leqq \|q_{\ell}^d \exp(-t(H_0^{\ell}\otimes 1_{\ell'}^d)q_{\ell'}^d\| \leqq C\min\{1, t^{-3/p}\},$$

from which (A.4.3) follows in this case.

Case 2. The two particle clusters of ℓ and ℓ' have no particle in common. Let $\ell = \{(i,j),(m),\ldots,(n)\}$ and $\ell' = \{(k,r),(m'),\ldots,(n')\}$ with $\{i,j\}$, $\{k,r\}$ disjoint. If $x\in U^d$ we have $m_iX_i + m_jX_j = m_kX_k + m_rX_r = 0$, $X_s = 0$, $s \neq i,j,k,r$. Assuming without loss of generality that $i<j$, $k<r$ we may write $x = (0,\ldots,X_i,\ldots,X_j,0,\ldots,0) + (0,\ldots,0,X_k,0,\ldots,X_r,0,\ldots,0)$. Note that the first vector in the right hand side belongs to U^{ℓ} while the second is in $U^{\ell'}$. Also, it is easy to see that $U^{\ell}\perp U^{\ell'}$, so that $U^d = U^{\ell}\oplus U^{\ell'}$, the direct sum being orthogonal. From Lemma (2.1) it follows that $U_{\ell}^d = U^{\ell'}$, and we have,

$$q_{\ell}^d \exp(-tH_0^d)q_{\ell'}^d = q_{\ell}^d \exp(-t(H_0^{\ell}\otimes 1^{\ell'}))\exp(-t(1^{\ell}\otimes H^{\ell'}))q_{\ell'}^d. \tag{A.4.5}$$

We start by estimating $q_{\ell}^d \exp(-t(H_0^{\ell}\otimes 1^{\ell'})$. Let $\zeta = X_i - X_j$, $\eta = X_k - X_r$. Identifying U^d with $\mathbb{R}^6$ through the isomorphism $x\in U^d \to (\zeta,\eta)\in\mathbb{R}^6$ we have:

$$\begin{aligned}\psi(\zeta,\eta) &= (q_{\ell}^d \exp(-t(H_0^{\ell}\otimes 1^{\ell'}))\varphi)(\zeta,\eta)\\ &= \alpha t^{-3/2} q(\zeta)\int d\zeta' K(\zeta-\zeta')\varphi(\zeta',\eta),\ \varphi\in L^2(\mathbb{R}^6),\end{aligned}$$

where $K(\zeta) = \exp(-\beta t^{-1}|\zeta|^2)$, α, $\beta>0$. Since $\varrho>1$ there is a $\delta>0$ such that $q\in L^p(\mathbb{R}^3)$, $p=3-\delta$. Let $p'^{-1}+p^{-1}=1$. Then $p'>3/2$ and $K\in L^{p'}(\mathbb{R}^3)$. Using Young's theorem on the convolution ([8], p. 28) in the variable ζ and then integrating with respect to η we find $\|\psi\|_2 \leqq Ct^{-3/2}\|K\|_{p'}\|\varphi\|_2$. Using spherical coordinates it follows easily that $\|K\|_{p'} \leqq Ct^{3/2p}$ so that $\|q_{\ell}^d \exp(-t(H_0^{\ell}\otimes 1^{\ell'}))\| \leqq Ct^{-3(1-p^{-1})2^{-1}}$. Similarly, $\|\exp(-t(1^{b}\otimes H_0^{b'}))q_{\ell'}^d\| \leqq Ct^{-3(1-p^{-1})2^{-1}}$. From (A.4.5) we obtain $\|q_{\ell}^d \exp(-t(H_0^d))q_{\ell'}^d\| \leqq C\min 1,\ t^{-3(1-p^{-1})}$ which implies (A.4.3) in this case. The proof is now complete.

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