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A Generalization of the Classical Moment Problem on *-Algebras with Applications to Relativistic Quantum Theory. II.

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Abstract. We discuss some properties of a non-commutative generalization of the classical moment problem (the m-problem) previously introduced. It is shown that there is a connexion between the determination of the problem and the self-adjointness properties in the corresponding Hilbert space. This generalizes the well-known connexion between the determination of the measure in the classical moment problem and the self-adjointness properties of the polynomials as operators in the corresponding L^2-space. The dependence of the m-problem on the choice of C^*-semi-norms and on the action of *-homomorphisms is also investigated. As an application, it is shown that if a quantum field (in a very general sense) is essentially self-adjoint then the m-problem for the Wightman functional is determined on the quasi-localizable C^*-algebra and that the corresponding representation of the localizable algebra generates the bounded observables of the field. It is pointed out that (ultraviolet and spatially) cut-off fields fall in this class and, therefore, are in one to one correspondance with states on the quasi-localizable C^*-algebra.

1. Introduction

This paper is a continuation of a preceding one [1] hereafter referred as Part I. Its object is to complete the algebraic discussion of the non-commutative generalization of the classical moment problem (the m-problem) introduced in Part I and to extend the applications to quantum field theory.

Let us first describe an important result on the classical moment problem [3–5] which will be generalized in this paper. Let ϕ be a positive linear form on the *-algebra $\mathbb{C}[X]$ of the complex polynomials with respect to one indeterminate X. Basically, to solve the moment problem for ϕ means to produce a self-adjoint operator $\pi(X)$ in a Hilbert space $\mathfrak{H}$ with a vector $\Omega \in \bigcap_{n\geq 0} \mathrm{dom}(\pi(X)^n)$ in such a way

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that Ω is cyclic for the von Neumann algebra $\{\pi(X)\}''$ and that the equalities $\phi(X^n) = (\Omega|\pi(X)^n\Omega)$ hold for all integers $n \geqq 0$. Indeed the corresponding conventional solution is the positive rapidly decreasing measure μ on $\mathbb{R}$ such that, up to a unitary equivalence, we have: $\mathfrak{H} = L^2(\mathbb{R}, d\mu)$, Ω is the constant function equal to 1 on $\mathbb{R}$ and $\pi(X)$ is the multiplication by the identity mapping of $\mathbb{R}$ on itself. Clearly the problem is determined ($\Leftrightarrow \mu$ unique) if and only if $(\mathfrak{H}, \pi(X), \Omega)$ is unique up to a unitary equivalence under the above conditions.

Let $(\pi_\phi, \mathfrak{H}_\phi, D_\phi, \Omega_\phi)$ be the unbounded cyclic *-representation associated with ϕ [1, 2]; we have canonically $\Omega_\phi = \Omega$, D_ϕ = linear hull of $\{\pi(X)^n\Omega\}$ $\mathfrak{H}_\phi$ = closure of D_ϕ in $\mathfrak{H}$ and $\pi_\phi(X) = \pi(X) \restriction D_\phi$. It is well known, and easy to see, that the moment problem is determined if $\pi_\phi(X)$ is an essentially self-adjoint operator in $\mathfrak{H}_\phi$. Then, the solution is given, of course, by $\mathfrak{H} = \mathfrak{H}_\phi$ and $\pi(X) = \pi_\phi(X)^*$. Furthermore, in this case all the $\pi_\phi(X^n) = \pi_\phi(X)^n$ are also essentially self-adjoint operators ($n \in \mathbb{N}$).

In this paper, this type of result connecting determination and self-adjointness will be generalized. According to Part I, the m-problem on a *-algebra $\mathfrak{A}$ is a generalization of the classical moment problem in the following sense: If $\mathfrak{A}$ is the algebra $\mathbb{C}[X_1, \ldots, X_n]$ of the polynomials with respect to n indeterminates, then the m-problem on $\mathfrak{A}$ is exactly the n-dimensional classical moment problem.

It will be a consequence of the results of this paper that if ϕ is a strongly positive linear form on a *-algebra $\mathfrak{A}$ (i.e. a positive linear form for which the m-problem is soluble [1]) and if $\pi_\phi(h)$ is an essentially self-adjoint operator for any hermitian h in $\mathfrak{A}$, then the m-problem is determined. However, in the applications, it is generally too strong to assume that $\pi_\phi(h)$ is essentially self-adjoint for all the $h = h^*$ in $\mathfrak{A}$. What frequently happens is that essential self-adjointness holds for the $\pi_\phi(h)$ when h runs over a generating subset Σ of hermitian elements of $\mathfrak{A}$, but in contrast to what happens when $\mathfrak{A} = \mathbb{C}[X]$, this does not imply, in general, essential self-adjointness for all the $\pi_\phi(h)$ with $h = h^*$ in $\mathfrak{A}$.

For instance, let $\mathfrak{A}$ be the tensor algebra $T(\mathbb{C}^2)$ over $\mathbb{C}^2$ equipped with the unique involution for which $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ($\in \mathbb{C}^2 \subset T(\mathbb{C}^2)$) are hermitian and let ϕ be the linear form on $\mathfrak{A}$ defined by:

$$\phi((\lambda_1 x + \mu_1 y) \ldots (\lambda_n x + \mu_n y)) = \pi^{-1/2} \int_{\mathbb{R}} dq\, e^{-q^2/2} \left(\lambda_1 q - \mu_1 i \frac{d}{dq}\right) \ldots \left(\lambda_n q - \mu_n i \frac{d}{dq}\right) e^{-q^2/2}.$$

ϕ is a positive linear form on $\mathfrak{A}$ and since $\mathfrak{A}$ is a tensor algebra over an involutive space, we know (see Part I, Section 4, Theorem 2) that ϕ is strongly positive so the m-problem is soluble. On the other hand, we have canonically $\mathfrak{H}_\phi = L^2(\mathbb{R}, dq)$, $\Omega_\phi = \pi^{-1/4} e^{-q^2/2}$, $D_\phi = \{P(q) e^{-q^2/2} | P(X) \in \mathbb{C}[X]\}$, $(\pi_\phi(x)\Psi)(q) = q\Psi(q)$ and $(\pi_\phi(y)\Psi)(q) = -i \dfrac{d\Psi}{dq}(q)$ for $\Psi \in D_\phi$. It is well known that $\pi_\phi(x)$ and $\pi_\phi(y)$ are essentially self-adjoint operators but it is also known[1] that there are hermitian combinations of $\pi_\phi(x)$ and $\pi_\phi(y)$ which are not essentially self-adjoint on D_ϕ; in other words there are hermitian elements h of $\mathfrak{A}$ for which $\pi_\phi(h)$ are not essentially

[1] $\pi_\phi(x)^n \pi_\phi(y) \pi_\phi(x)^n$, $n \geqq 2$

self-adjoint operators so here $\Sigma=\{x, y\}$. In fact Ω_ϕ is an entire vector for $\pi_\phi(tx+ry)$, $\forall t$, $r\in\mathbb{R}$, so one may take $\Sigma=\{tx+ry|t, r\in\mathbb{R}\}=E_2\cong\mathbb{R}^2$ as well as $\Sigma=\{x, y\}$ (=basis of E_2). It is worth noticing here that if $\dot{\phi}$ denotes the positive linear form on the quotient algebra $\dot{\mathfrak{A}}=\mathfrak{A}/\pi_\phi^{-1}(0)$ induced by ϕ, $\dot{\phi}$ is not strongly positive; indeed $\dot{\mathfrak{A}}$ is the *-algebra generated by Heisenberg canonical commutation relations and it is well known that this algebra does only admit unbounded *-representations so there are no C^*-semi-norm on $\dot{\mathfrak{A}}$ and therefore no non-trivial strongly positive linear form on $\dot{\mathfrak{A}}$. The situation is similar if one considers the algebra generated by the free hermitian field and this phenomena has been already pointed out in the Part I of this work [see the Remark 9b) in Part I].

In Part I of this work, we have associated to any *-algebra $\mathfrak{A}$ with a unit a C^*-algebra $\mathfrak{B}(\mathfrak{A})$. $\mathfrak{B}(\mathfrak{A})$ is such that any hermitian element h of $\mathfrak{A}$ determines a *-homomorphism $f\mapsto f(h)$ of the C^*-algebra $\mathscr{C}_{(0)}(\mathbb{R})$ of complex continuous functions vanishing at infinity on $\mathbb{R}$ into $\mathfrak{B}(\mathfrak{A})$. Furthermore, the ranges of these homorphisms generate $\mathfrak{B}(\mathfrak{A})$ as C^*-algebra (when h runs over the set $\mathfrak{A}^h$ of all the hermtian elements of $\mathfrak{A}$) (see Part I, Section 6). Let Σ be a subset of $\mathfrak{A}^h$ and let $\mathfrak{B}(\Sigma)$ be the C^*-subalgebra of $\mathfrak{B}(\mathfrak{A})$ generated by $\{f(h)|h\in\Sigma, f\in\mathscr{C}_{(0)}(\mathbb{R})\}$. In this paper, we shall show the following. If ϕ is a strongly positive linear form on $\mathfrak{A}$, if $(\pi_\phi, \mathfrak{H}_\phi, D_\phi, \Omega_\phi)$ is the associated cyclic *-representation and if Σ is a generating subset of hermitian elements of $\mathfrak{A}$ then, any solution ω of the m-problem for ϕ [ω is a positive linear form on $\mathfrak{B}(\mathfrak{A})$, see Section 7 in Part I] leads to a cyclic representation π_Σ of $\mathfrak{B}(\Sigma)$ in a Hilbert space $\mathfrak{H}_\Sigma$ which contains $\mathfrak{H}_\phi$ as closed subspace and with Ω_ϕ as cyclic vector such that $\omega(f(h))=(\Omega_\phi|\pi_\Sigma(f(h))\Omega_\phi)$; $h\in\Sigma$, $f\in\mathscr{C}_{(0)}(\mathbb{R})$. Furthermore, if the $\pi_\phi(h)$ are essentially self-adjoint when $h\in\Sigma$ then $\mathfrak{H}_\Sigma=\mathfrak{H}_\phi$, π_Σ is unique and we have: $\pi_\Sigma(f(h))=f(\overline{\pi_\phi(h)})$; $\forall h\in\Sigma$, $\forall f\in\mathscr{C}_{(0)}(\mathbb{R})$. In any case, $(\pi_\Sigma, \mathfrak{H}_\Sigma, \Omega_\phi)$ is canonically the G.N.S. triplet associated with $\omega\upharpoonright\mathfrak{B}(\Sigma)$ and ϕ can be reconstructed from $\omega\upharpoonright\mathfrak{B}(\Sigma)$.

For instance, let $\mathfrak{A}$, x, y, and ϕ be as in the above example and let $\Sigma=\{x, y\}$. Then there is a unique representation π_Σ of $\mathfrak{B}(\{x, y\})=\mathfrak{B}(\Sigma)$ in $\mathfrak{H}_\phi$ for which Ω_ϕ is cyclic and $\pi_\Sigma(f(x))=f(\overline{\pi_\phi(x)})$, $\pi_\Sigma(f(y))=f(\overline{\pi_\phi(y)})$ for any $f\in\mathscr{C}_{(0)}(\mathbb{R})$.

For the applications to quantum field theory, it is important to realize that, following H. J. Borchers, we consider that a (scalar hermitian) quantum field is a *-representation of the tensor algebra over the space of complex test functions. Furthermore, we shall be interested in self-adjointness properties of the field operator, so we take $\mathfrak{A}=T(\mathscr{D}(M))$ [=the tensor algebra over the space $\mathscr{D}(M)$ of all complex C^∞ functions with compact support on the space-time M] and $\Sigma=\mathscr{D}(M, \mathbb{R})$=real C^∞ functions with compact supports. It turns out that $\mathfrak{B}(\Sigma)$ is the quasi-localizable C^*-algebra $\hat{\mathfrak{B}}(M)$ introduced in Part I.

We use the above results to show that the bounded observables of cut-off field theories generate concrete C^*-algebras which are images of cyclic representations of the (universal) quasi-localizable C^*-algebra $\hat{\mathfrak{B}}(M)$ with the ground states of the hamiltonians as cyclic vectors. The corresponding states on $\hat{\mathfrak{B}}(M)$ being elements of the weakly compact states' space of $\hat{\mathfrak{B}}(M)$, a program of removing the cut-offs by compactness arguments is suggested. A difficulty connected with general features of the construction described in the work (Part I and this paper) is pointed out: Namely we do not take into account the topologies of our basic spaces (*-algebras, spaces of test functions) with this algebraical construction.

In a forthcoming paper, we shall analyze the restrictions which come from the continuity properties in the m-problem.

In Section 2 we analyse the connexion between the determination of the m-problem and the self-adjointness properties in the corresponding hermitian cyclic representation [1, 2]. We give a generalization of the well-known connexion [3—5] between the determination of the measure in the classical moment problem and the self-adjointness properties of the polynomials as operators in the corresponding L^2-space.

In Section 3 we discuss the dependence of the problem on the choice of sets of C^*-semi-norms. We introduce a notion which generalize the notion of support in the classical moment problem.

In Section 4 the action of *-homomorphisms is investigated. We define C^*-algebras associated with real vector spaces. This generalizes the definition of the quasi-localizable C^*-algebra given in Part I.

In Section 5, we apply the above results to quantum field theory. We show that, if for every real test function the smeared field operator is essentially self-adjoint on the cyclic subspace generated from a unit vector in Hilbert space by the polynomial algebra of the smeared field operators, then the m-problem for the corresponding vector state on the tensor algebra over the space of test functions is determined on the quasi-localizable C^*-algebra and that the corresponding representation of the quasi-localizable C^*-algebra generates the bounded observables of the field.

The last result works for a hermitian scalar field in a very general sense. In particular, it is pointed out in Section 6 that a form of ϕ^4-cut-off field introduced by Jaffe in his thesis [6] fall in this class. The self-adjointness of the space-time smeared cut-off field is obtained as an application of a general method introduced by Glimm and Jaffe in their study of $(\phi^4)_2$ theory [15] (see also [16] and the paper of McBryan [17]). We define, for each cut-off, a state on the quasi-localizable C^*-algebra which correspond to the ground-state-expectation-values of the bounded observables. We allow the mass, the coupling constant and the field normalization to vary with the value of the cut-off.

In conclusion, we discuss some possible pathologies associated with the "limits" obtained (by compactness) when the cut-off is removed. These pathologies were avoided in the work of Glimm and Jaffe on $(\phi^4)_2$ when they proved the "local normality" of the "vacuum state" (see, for instance, the Theorem 4.2.1 in Les Houches 1970, [18]).

Throughout this paper, we use the notation of Part I. If $\mathfrak{B}$ is a C^*-algebra and if ω is a positive linear form on $\mathfrak{B}$, we use the term, G.N.S. triplet associated with ω to denote triplet $(\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega)$ of the cyclic representation π_ω in Hilbert space $\mathfrak{H}_\omega$ with cyclic vector Ω_ω obtained from ω by the Gelfand-Naimark-Segal construction (G.N.S. construction).

$\mathscr{D}_{(0)}(\mathbb{R}^n)$ denote the space of continuous complex functions with compact supports on $\mathbb{R}^n$.

$\mathscr{D}(\mathbb{R}^n)$ denote the space of C^∞-complex functions with compact supports on $\mathbb{R}^n$.

$\mathscr{C}(\mathbb{R}^n)$ is the space of complex continuous functions on $\mathbb{R}^n$.

$\mathscr{C}_{(0)}(\mathbb{R}^n)$ is the C^*-algebra of complex continuous functions vanishing a infinity on $\mathbb{R}^n$.

2. Self-Adjointness and Determination

Throughout this section $\mathfrak{A}$ is a *-algebra with a unit ($\mathbb{1}\in\mathfrak{A}$), Γ is a separating directed set of C^*-semi-norms on $\mathfrak{A}$ and $\mathfrak{B}(\mathfrak{A}, \Gamma)$ is the associated C^*-algebra defined in Part I (Part I, Section 6, Definition 3).

Lemma 1. *Let π be a representation of $\mathfrak{B}(\mathfrak{A}, \Gamma)$ in a Hilbert space $\mathfrak{H}_\pi$ and let h be an arbitrary hermitian element of $\mathfrak{A}$. Then, in the Hilbert subspace $\mathfrak{H}_\pi(h)$ of $\mathfrak{H}_\pi$ spanned by the set $\{\pi(f(h))\,\Phi\,|\,\Phi\in\mathfrak{H}_\pi$ and $f\in\mathscr{C}_{(0)}(\mathbb{R})\}$, there is a unique self-adjoint operator $\pi(h)$ for which we have: $\pi(h)\pi(f(h))\,\Phi=\pi(f^I(h))\,\Phi$, $\forall\,\Phi\in\mathfrak{H}_\pi$ and $\forall f\in\mathscr{D}_{(0)}(\mathbb{R})$, where $f^I\in\mathscr{D}_{(0)}(\mathbb{R})$ is defined by $f^I(t)=tf(t)$ $(\forall t\in\mathbb{R})$. Furthermore, we have $\pi(f(h))\upharpoonright\mathfrak{H}_\pi(h)=f(\pi(h))$, $\forall f\in\mathscr{C}_{(0)}(\mathbb{R})$, and if A is a bounded operator in $\mathfrak{H}_\pi$ which commutes with $\pi(f(h))$ for any $f\in\in\mathscr{C}_{(0)}(\mathbb{R})$ then $\mathfrak{H}_\pi(h)$ is stable by A and the restriction of A to $\mathfrak{H}_\pi(h)$ commutes with $\pi(h)$ (i.e. its spectral projections).*

Proof[2]. Let $D^0_\pi(h)$ be the linear hull in $\mathfrak{H}_\pi$ of the set $\{\pi(f(h))\,\Phi\,|\,\Phi\in\mathfrak{H}_\pi$ and $f\in\mathscr{D}_{(0)}(\mathbb{R})\}$; $D^0_\pi(h)$ is a dense subspace of $\mathfrak{H}_\pi(h)$ and there is a unique linear mapping $\pi^0(h)$ of $D^0_\pi(h)$ into itself satisfying $\pi^0(h)\pi(f(h))\,\Phi=\pi(f^I(h))\,\Phi$, $\forall\,\Phi\in\mathfrak{H}_\pi$ and $\forall f\in\mathscr{D}_{(0)}(\mathbb{R})$. As an operator in $\mathfrak{H}_\pi(h)$, $\pi^0(h)$ is a symmetric operator for which $D^0_\pi(h)$ is a dense domain of entire vectors. Therefore $\pi^0(h)$ is closable and its closure $\pi(h)$ is a self-adjoint operator in $\mathfrak{H}_\pi(h)$ which is clearly unique under the above conditions. Using the definition and the fact that $D^0_\pi(h)$ is dense stable domain of entire and even bounded vectors for $\pi(h)$, it is straightforward to see that the equality $\pi(f(h))\,\Phi=f(\pi(h))\,\Phi$ holds for any $f\in\mathscr{D}_{(0)}(\mathbb{R})$ and any $\Phi\in D^0_\pi(h)$ and therefore, by continuity it also holds for $f\in\mathscr{C}_{(0)}(\mathbb{R})$ and $\Phi\in\mathfrak{H}_\pi(h)$.

Suppose that $A\in\mathscr{L}(\mathfrak{H}_\pi)$ commutes with $\{\pi(f(h))|f\in\mathscr{C}_{(0)}(\mathbb{R})\}$. Then, for any $\Phi\in\mathfrak{H}_\pi$ and for any $f\in\mathscr{D}_{(0)}(\mathbb{R})$, we have: $A\pi(f(h))\,\Phi\in D^0_\pi(h)$, since $A\pi(f(h))\Phi=\pi(f(h))A\Phi$, and, $A\pi(h)\pi(f(h))\,\Phi=A\pi(f^1(h))\,\Phi=\pi(f^1(h))\,A\Phi=\pi(h)\,A\pi(f(h))\,\Phi$. Therefore we have: $AD^0_\pi(h)\subset D^0_\pi(h)$ and $A\pi(h)\,\Phi=\pi(h)\,A\Phi$ for any $\Phi\in D^0_\pi(h)$. Let Φ_α be a net of vectors in $D^0_\pi(h)$ such that (Φ_α) converges to Φ and $(\pi(h)\Phi_\alpha)$ converges to Ψ in $\mathfrak{H}_\pi(h)$; then $A\Phi=\lim A\Phi_\alpha(\in\mathfrak{H}_\pi(h))$ and we also have $A\Psi=\lim(A\pi(h)\Phi_\alpha)=\lim(\pi(h)A\Phi_\alpha)$ (since A is bounded). So, if $D_\pi(h)$ denote the domain of $\pi(h)$ $(D^0_\pi(h)\subset D_\pi(h)\subset\mathfrak{H}_\pi(h))$, we have: $AD_\pi(h)\subset D_\pi(h)$ and $A\pi(h)\,\Phi=\pi(h)\,A\Phi$, $\forall\Phi\in D_\pi(h)$.

This achieves the proof of the Lemma 1. □

Let ϕ be a positive linear form on $\mathfrak{A}$ and let π_ϕ be the corresponding cyclic *-representation [8] in Hilbert space $\mathfrak{H}_\phi$ with cyclic vector $\Omega_\phi\in\mathfrak{H}_\phi$ and domain $D_\phi=\pi_\phi(\mathfrak{A})\,\Omega_\phi$ such that $\phi(x)=(\Omega_\phi|\pi_\phi(x)\,\Omega_\phi)$; it is well known that, for ϕ fixed, $(\pi_\phi, \mathfrak{H}_\phi, D_\phi)$ is unique up to a unitary equivalence under the above conditions. Suppose that ϕ is Γ-strongly positive and let ω be a solution of the $m(\Gamma)$-problem for ϕ. Let $(\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega)$ be the G.N.S. triplet associated with ω and let us use the notation of the Section 7 in Part I (in particular see the Proposition 3 in Part I). In the Proposition 3 of Part I, it was shown that $\pi_\phi(x)\,\Omega_\phi\mapsto\Psi_\omega(x)$ define an isometric inbedding of $\mathfrak{H}_\phi$ in $\mathfrak{H}_\omega$. Therefore it is justified, and we *shall always do so in the following*, to identify $\mathfrak{H}_\phi$ with a closed subspace of $\mathfrak{H}_\omega$, writing: $\Omega_\phi=\Omega_\omega$, $\pi_\phi(x)\,\Psi_\omega(y)=\Psi_\omega(xy)$; $\forall x, y\in\mathfrak{A}$ [whenever ω is a solution of the $m(\Gamma)$-problem for ϕ]. In general, the inclusion $\mathfrak{H}_\phi\subset\mathfrak{H}_\omega$ is strict (see what happens in the classical moment problem).

[2] This proof is influenced by the appendix of a paper of Ruelle [7]

Lemma 2. *Let ϕ be a Γ-strongly positive linear form on $\mathfrak{A}$ and let ω be an arbitrary solution of the m (Γ)-problem for ϕ. Then we have: $D_\phi = \Psi_\omega(\mathfrak{A}) \subset D_{\pi_\omega}(h)$ and $\pi_\phi(h) = \pi_\omega(h) \upharpoonright D_\phi$, $\forall h = h^* \in \mathfrak{A}$ (where $D_{\pi_\omega}(h)$ denote the domain of the self-adjoint operator $\pi_\omega(h)$ in $\mathfrak{H}_{\pi_\omega}(h)$ defined as in Lemma 1 and where we make the above identifactions).*

Proof. Let $\Psi_\omega(x)$ $(x \in \mathfrak{A})$ be an arbitrary element of D_ϕ. We know (Proposition 3 in Part I, Section 7) that ω_x is a solution of the $m(\Gamma)$-problem for ϕ_x, where ω_x and ϕ_x are defined by: $\omega_x(y) = (\Psi_\omega(x) | \pi_\omega(y)\, \Psi_\omega(x))$, $\forall y \in \mathfrak{B}(\mathfrak{A}, \Gamma)$, $\phi_x(y) = \phi(x^* y x)$ $= (\Psi_\omega(x) | \pi_\phi(y)\, \Psi_\omega(x))$, $\forall y \in \mathfrak{A}$.

It follows (see, in Part I, the Theorems 4 and 5) that the positive measure μ_{h,ω_x} on $\mathbb{R}$ defined by $(h = h^* \in \mathfrak{A})$

$$\omega_x(f(h)) = \int f(t)\, d\mu_{h,\omega_x}(t), \forall f \in \mathscr{C}_{(0)}(\mathbb{R}),$$

is a rapidly decreasing measure and that we have:

$$\phi_x(h^n) = \int t^n\, d\mu_{h,\omega_x}(t), \forall n \in \mathbb{N}.$$

Let χ_n be a sequence of continuous function on $\mathbb{R}$ such that $0 \leqq \chi_n \leqq 1$, $\chi_n(t) = 1$ if $|t| < n$, $\chi_n(t) = 0$ if $|t| \geqq n+1$. Then $\pi_\omega(\chi_n(h))\, \Psi_\omega(x)$ is a sequence of element of $D^0_{\pi_\omega}(h)$ and $\pi_\omega(h)\, \pi_\omega(\chi_n(h))\, \Psi_\omega(x) = \pi_\omega(\chi^I_n(h))\, \Psi_\omega(x)$ (with the notation of Lemma 1). We have:

$$\|\pi_\phi(h)\, \Psi_\omega(x) - \pi_\omega(h)\, \pi_\omega(\chi_n(h))\, \Psi_\omega(x)\|^2 = \int t^2 (1 - \chi_n(t))^2\, d\mu_{h,\omega_x}(t)$$
$$\|\Psi_\omega(x) - \pi_\omega(\chi_n(h))\, \Psi_\omega(x)\|^2 = \int (1 - \chi_n(t))^2\, d\mu_{h,\omega_x}(t).$$

Both terms converge to zero when n goes to infinity; so $\Psi_\omega(x) \in D_{\pi_\omega}(h)$ and $\pi_\omega(h)\, \Psi_\omega(x) = \pi_\phi(h)\, \Psi_\omega(x)$. This prove the lemma since $x \in \mathfrak{A}$ and $h = h^* \in \mathfrak{A}$ are arbitrary. □

Corollary 1. *Let ϕ and ω be as in Lemma 2 and let $h = h^*$ be a hermitian element of $\mathfrak{A}$ such that we have: $\pi_\omega(f(h))\, \mathfrak{H}_\phi \subset \mathfrak{H}_\phi$, $\forall f \in \mathscr{C}_{(0)}(\mathbb{R})$. Then $\pi_\omega(h) \upharpoonright (\mathfrak{H}_\phi \cap D_{\pi_\omega}(h))$ is a self-adjoint extension of the symmetric operator $\pi_\phi(h)$ in $\mathfrak{H}_\phi$.*

Corollary 2. *Let ϕ and ω be as above, let Σ be a set of hermitian elements of $\mathfrak{A}$ which generates $\mathfrak{A}$ and let $\mathfrak{B}(\Sigma, \Gamma)$ denote the C^*-subalgebra of $\mathfrak{B}(\mathfrak{A}, \Gamma)$ generated by the set $\{f(h) | h \in \Sigma$ and $f \in \mathscr{C}_{(0)}(\mathbb{R})\}$. Then we have $\mathfrak{H}_\phi \subseteqq \overline{\pi_\omega(\mathfrak{B}(\Sigma, \Gamma))\, \Omega_\omega}$ (where the bar denote the closure in $\mathfrak{H}_\omega$) and if the equality holds $\pi_\omega(h) \upharpoonright (\mathfrak{H}_\phi \cap D_{\pi_\omega}(h))$ is a self-adjoint extension of $\pi_\phi(h)$, $\forall h \in \Sigma$ (these corollaries need not to be proved).*

Theorem 1. *Let ϕ be a Γ-strongly positive linear form on $\mathfrak{A}$ and let Σ be a set of hermitian elements of $\mathfrak{A}$. Suppose that for any $h \in \Sigma$, $\pi_\phi(h)$ is an essentially self-adjoint operator in $\mathfrak{H}_\phi$. Then the $m(\Gamma)$-problem for ϕ is determined on the C^*-subalgebra $\mathfrak{B}(\Sigma, \Gamma)$ of $\mathfrak{B}(\mathfrak{A}, \Gamma)$ generated by the set $\{f(h) \in \mathfrak{B}(\mathfrak{A}, \Gamma) | h \in \Sigma$ and $f \in \mathscr{C}_{(0)}(\mathbb{R})\}$. Furthermore, if ω is an arbitrary solution of the $m(\Gamma)$-problem for ϕ and if $\overline{\pi_\phi(h)}$ denote the closure of $\pi_\phi(h)$, for $h \in \Sigma$ (considered as an operator in $\mathfrak{H}_\phi$), we have: $\pi_\omega(f(h)) \upharpoonright \mathfrak{H}_\phi = f(\overline{\pi_\phi(h)})$, $\forall f \in \mathscr{C}_{(0)}(\mathbb{R})$.*

Proof. It follows from the Lemma 2 that, for any hermitian element h of $\mathfrak{A}$, we have $\mathrm{dom}(\overline{\pi_\phi(h)}) \subset D_{\pi_\omega}(h)$ and $\overline{\pi_\phi(h)} = \pi_\omega(h) \upharpoonright \mathrm{dom}(\overline{\pi_\phi(h)})$ for any solution ω of the $m(\Gamma)$-problem for ϕ. If $h \in \Sigma$, $\overline{\pi_\phi(h)}$ is by assumption a self-adjoint operator in $\mathfrak{H}_\phi$; let $t \mapsto E_h(t)$ be its spectral resolution $(\overline{\pi_\phi(h)} = \int t dE_h(t)$ in $\mathfrak{H}_\phi)$. For any positive number s, let $\mathfrak{H}_s(h)$ be the closed subspace of $\mathfrak{H}_\phi$ defined by: $\mathfrak{H}_s(h) = \int_{-s}^{+s} dE_h(t) \mathfrak{H}_\phi$. $\bigcup_{s \geqq 0} \mathfrak{H}_s(h)$ is a dense subspace of $\mathfrak{H}_\phi$ contained in the domain $\mathrm{dom}(\overline{\pi_\phi(h)})$ of the self-adjoint operator $\overline{\pi_\phi(h)}$ in $\mathfrak{H}_\phi$. We have $\overline{\pi_\phi(h)}\, \mathfrak{H}_s(h) \subset \mathfrak{H}_s(h)$ and $\pi_\omega(h) \upharpoonright \mathfrak{H}_s(h) = \overline{\pi_\phi(h)} \upharpoonright \mathfrak{H}_s(h)$ so $\pi_\omega(f(h)) \upharpoonright \mathfrak{H}_s(h) = f(\overline{\pi_\phi(h)}) \upharpoonright \mathfrak{H}_s(h)$ $\forall f \in \mathscr{C}_{(0)}(\mathbb{R})$ [since $\overline{\pi_\phi(h)} \upharpoonright \mathfrak{H}_s(h)$ is a bounded self-adjoint operator in $\mathfrak{H}_s(h)$ with $\|\overline{\pi_\phi(h)} \upharpoonright \mathfrak{H}_s(h)\| \leqq s$]. So we also have: $\pi_\omega(f(h)) \upharpoonright \mathfrak{H}_\phi = f(\overline{\pi_\phi(h)})$, $\forall f \in \mathscr{C}_{(0)}(\mathbb{R})$ and $\forall h \in \Sigma$ [where ω is an arbitrary solution of the $m(\Gamma)$-problem for ϕ]. This implies in particular that if ω_1 and ω_2 are two solutions of the $m(\Gamma)$-problem then we have:

$$\pi_{\omega_1}(x)\Phi = \pi_{\omega_2}(x)\Phi \in \mathfrak{H}_\phi, \forall \Phi \in \mathfrak{H}_\phi \text{ and } \forall x \in \mathfrak{B}(\Sigma, \Gamma).$$

Therefore (since $\Omega_{\omega_1} = \Omega_{\omega_2} = \Omega_\phi \in \mathfrak{H}_\phi$) we have, $\forall x \in \mathfrak{B}(\Sigma, \Gamma)$ $\omega_1(x) = \omega_2(x)$. This achieves the proof of the theorem. □

Remark 1. a) Replacing Σ by $\Sigma \cup \{\mathbb{1}\}$ it follows that we may replace in the statement $\mathfrak{B}(\Sigma, \Gamma)$ by $\mathfrak{B}(\Sigma \cup \{\mathbb{1}\}, \Gamma)$ [which contains the identity of $\mathfrak{B}(\mathfrak{A}, \Gamma)$].

b) Remembering that if Γ_M is the set of all the C^*-semi-norms on $\mathfrak{A}$ then $\mathfrak{B}(\mathfrak{A}, \Gamma_M)$ is denoted by $\mathfrak{B}(\mathfrak{A})$ (and called the C^*-algebra associated to $\mathfrak{A}$), the $m(\Gamma_M)$-problem is called the m-problem and a Γ_M-strongly positive linear form is called a strongly positive linear form and that, furthermore, every solution of the $m(\Gamma)$-problem is a fortiori (canonically) a solution of the m-problem for ϕ [via the canonical surjective *-homomorphism: $\mathfrak{B}(\mathfrak{A}) \mapsto \mathfrak{B}(\mathfrak{A}, \Gamma)$]; it follows that the condition of Theorem 1 is already a sufficient condition for the determination of the m-problem. Since it may happen that a $m(\Gamma)$-problem is determined and that there are several solutions of the corresponding m-problem (compare Stieltjes problem and Hamburger's problem in the classical moment problem), one cannot expect, in general, that the condition of Theorem 1 is a necessary condition of determination of the $m(\Gamma)$-problem.

c) In the proof of theorem it was shown that the representation π_ω of $\mathfrak{B}(\Sigma, \Gamma)$ leaves $\mathfrak{H}_\phi$ invariant and is unique on $\mathfrak{H}_\phi$ [i.e. independent of the choice of the solution ω of the $m(\Gamma)$-problem and even the m-problem by b), replacing $\mathfrak{B}(\Sigma, \Gamma)$ by $\mathfrak{B}(\Sigma)$]. Finally, let us notice that this implies that, if Σ generates $\mathfrak{A}$, the closure of $\pi_\omega(\mathfrak{B}(\Sigma, \Gamma))\Omega_\omega$ in $\mathfrak{H}_\omega$ is exactly $\mathfrak{H}_\phi$ (compare with Corollary 2).

Lemma 3. *Let ϕ be a pure state*[3] *on $\mathfrak{A}$ which is Γ-strongly positive and let $\mathfrak{S}_\phi$ be the (convex and weakly compact) set of all the solutions of the $m(\Gamma)$-problem for ϕ. Then every extreme point of $\mathfrak{S}_\phi$ is a pure state on $\mathfrak{B}(\mathfrak{A}, \Gamma)$.*

Proof. Let ω be an extreme point of $\mathfrak{S}_\phi$ and let ω_1 be a positive linear form on $\mathfrak{B}(\mathfrak{A}, \Gamma)$ such that $\omega \geqq \omega_1$. If π_ω is the cyclic representation associated with ω, we

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[3] A state on $\mathfrak{A}$ is a positive linear form on $\mathfrak{A}$ for which $\phi(\mathbb{1}) = 1$. A positive linear form ϕ is pure if the only positive linear forms Ψ satisfying $\phi(x^*x) \geqq \Psi(x^*x)$ are the multiples $\Psi = \lambda\phi$ of $\phi (0 \leqq \lambda \leqq 1)$

know that there is a positive $A_1 \in \pi_\omega(\mathfrak{B}(\mathfrak{A},\Gamma))'$ with $0 \leqq A_1 \leqq \mathbb{1}$ for which $\omega_1(x) = (\Omega_\omega | A_1 \pi_\omega(x)\Omega_\omega)$, $\forall x \in \mathfrak{B}(\mathfrak{A},\Gamma)$. From the last part of Lemma 1 and from Lemma 2, it follows that we have $\phi_1(h) = (\Omega_\omega | A_1 \pi_\omega(h)\Omega_\omega) = (\Omega_\omega | \pi_\omega(h) A_1 \Omega_\omega) = (\Omega_\phi | A_1 \pi_\phi(h)\Omega_\phi) = (\pi_\phi(h)\Omega_\omega | A_1 \Omega_\phi)$, $\forall h \in \mathfrak{A}$, and from $0 \leqq A_1 \leqq \mathbb{1}$ it follows that ϕ_1 defines a positive linear form on $\mathfrak{A}$ for which $\phi_1(x^*x) \leqq \phi(x^*x)$. It follows that $\phi_1 = \lambda\phi$. For some constant $0 \leqq \lambda \leqq 1$ (since ϕ is pure). This implies immediately that $\tilde{\omega} = \lambda^{-1}\omega_1$ is a solution of the $m(\Gamma)$-problem for ϕ if $\lambda \neq 0$, and that, if $\lambda \neq 1$ the same is true for $\tilde{\omega}'_1 = (\omega - \omega_1)/(1-\lambda)$ so we have $\omega = \lambda\tilde{\omega}_1 + (1-\lambda)\tilde{\omega}'_1$ which implies (since ω is an extreme point in $\mathfrak{S}_\phi$) either $\tilde{\omega}_1 = \tilde{\omega}'_1 = \omega$ either $\lambda = 0$ or $\lambda = 1$ so in any case $\lambda\omega_1 = \omega$ which means that ω is pure. □

Notice that, quite generally, if ϕ is an arbitrarily Γ-strongly positive linear form on $\mathfrak{A}$, if ω is a solution of the $m(\Gamma)$-problem for ϕ and if P^ω_ϕ denote the orthogonal projection on $\mathfrak{H}_\phi$ (as a subspace of $\mathfrak{H}_\omega$) then the mapping $A \mapsto P^\omega_\phi A P^\omega_\phi$ of $\mathscr{L}(\mathfrak{H}_\omega)$ in $\mathscr{L}(\mathfrak{H}_\phi)$ maps the commutant $\pi_\omega(\mathfrak{B}(\mathfrak{A},\Gamma))'$ of $\pi_\omega(\mathfrak{B}(\mathfrak{A},\Gamma))$ in the weak commutant $\pi_\phi(\mathfrak{A})'_w$ of $\pi_\phi(\mathfrak{A})$ in $\mathfrak{H}_\phi$ (where π_ϕ is considered as a *-representation of $\mathfrak{A}$ in $\mathfrak{H}_\phi$ with domain D_ϕ [2]).

Let $S(\mathfrak{A},\Gamma)$ be the convex cone of all the Γ-strongly positive linear forms on $\mathfrak{A}$ and let $M(\mathfrak{A},\Gamma)$ be the convex cone of all the positive linear forms on $\mathfrak{B}(\mathfrak{A},\Gamma)$ which are solutions of $m(\Gamma)$-problems. Any element ω of $M(\mathfrak{A},\Gamma)$ is solution of the $m(\Gamma)$-problem for a unique element ϕ_ω of $S(\mathfrak{A},\Gamma)$ and we have:

$$\phi_{t_1\omega_1 + t_2\omega_2} = t_1\phi_{\omega_1} + t_2\phi_{\omega_2}; \quad \forall \omega_1, \omega_2 \in M(\mathfrak{A},\Gamma) \quad \text{and} \quad \forall t_1, t_2 \in \mathbb{R}^+.$$

Lemma 4. *Let (ϕ_α) be a net of Γ-strongly positive linear forms on $\mathfrak{A}$ and let us choose, for each α, a solution ω_α of the $m(\Gamma)$-problem for ϕ_α. Suppose that (ϕ_α) converges weakly to ϕ (in the dual space of $\mathfrak{A}$) and let ω be the weak limit (in $\mathfrak{B}(\mathfrak{A},\Gamma)'$) of an arbitrary weakly convergent subnet of (ω_α). Then ω is a solution of the $m(\Gamma)$-problem for ϕ.*

Proof. Let (ω_β) be a subnet of (ω_α) which converges weakly to ω. For each β let ψ_β be the linear form on the subspace $\mathfrak{A} + \mathfrak{B}(\mathfrak{A},\Gamma)$ of $\mathscr{A}(\mathfrak{A},\Gamma)$ which is positive on $(\mathfrak{A} + \mathfrak{B}(\mathfrak{A},\Gamma)) \cap \mathscr{A}^+(\mathfrak{A},\Gamma)$ and satisfies: $\phi_\beta = \psi_\beta \upharpoonright \mathfrak{A}$ and $\omega_\beta = \psi_\beta \upharpoonright \mathfrak{B}(\mathfrak{A},\Gamma)$. The net (ψ_β) is by assumption simply convergent on $\mathfrak{A}$ and on $\mathfrak{B}(\mathfrak{A},\Gamma)$ so it is simply convergent to a linear form ψ on $\mathfrak{A} + \mathfrak{B}(\mathfrak{A},\Gamma)$ which is positive on $(\mathfrak{A} + \mathfrak{B}(\mathfrak{A},\Gamma)) \cap \mathscr{A}^+(\mathfrak{A},\Gamma)$ and such that we have:

$$\phi = \psi \upharpoonright \mathfrak{A} \quad \text{and} \quad \omega = \psi \upharpoonright \mathfrak{B}(\mathfrak{A},\Gamma). \quad \square$$

Remark 2. This lemma implies in particular that if the $m(\Gamma)$-problem for ϕ is determined on $\mathfrak{B}(\Sigma,\Gamma)$ (for some $\Sigma \subset \mathfrak{A}^h$) then the net $\omega_\alpha \upharpoonright \mathfrak{B}(\Sigma,\Gamma)$ is weakly convergent.

3. Stability of the *m*-Problem: Supports

Let $\mathfrak{A}$ be a *-algebra with a unit and let $\mathscr{N}(\mathfrak{A})$ be the set of all the C^*-semi-norms on $\mathfrak{A}$ equipped with the uniformity of the simple convergence on $\mathfrak{A}$. The relations caracterizing the C^*-semi-norms define a closed subset in $\mathbb{R}^{\mathfrak{A}}$ so $\mathscr{N}(\mathfrak{A})$ is a complete uniform space. Furthermore, $\mathscr{N}(\mathfrak{A})$ is canonically an ordered directed set.

Let Γ be a directed set of C^*-semi-norms on $\mathfrak{A}$ [i.e. Γ is a directed subset of $\mathcal{N}(\mathfrak{A})$] and let $\mathcal{T}_\Gamma$ be the locally convex topology on $\mathfrak{A}$ generated by Γ. According to Part I, the directed set $\tilde{\Gamma}$ of all the $\mathcal{T}_\Gamma$-continuous C^*-semi-norms on $\mathfrak{A}$ consists of all the C^*-semi-norms on $\mathfrak{A}$ which are bounded by C^*-semi-norms of Γ and all the concepts and constructions introduced so far do only on $\tilde{\Gamma}(\mathscr{A}(\mathfrak{A},\Gamma)=\mathscr{A}(\mathfrak{A},\tilde{\Gamma})$ etc. ...). It follows that the convenient assumption that Γ is directed may be dropped. Indeed, if Γ is an arbitrary set of C^*-semi-norms ($\Gamma\subset\mathcal{N}(\mathfrak{A})$) on $\mathfrak{A}$, then the locally convex topology $\mathcal{T}_\Gamma$ on $\mathfrak{A}$ generated by Γ is also the locally convex topology generated by the directed set $\tilde{\Gamma}$ of all the $\mathcal{T}_\Gamma$-continuous C^*-semi-norms on $\mathfrak{A}$ so, for instance, we may define $\mathscr{A}(\mathfrak{A},\Gamma)$ to be $\mathscr{A}(\mathfrak{A},\tilde{\Gamma})$ etc. .... Notice that we have: $\tilde{\Gamma}=\{p\in\mathcal{N}(\mathfrak{A})\,|\,p\leqq\sup_{i\in I_p}(p_i)$ for a finite family $(p_i)_{i\in I_p}$ in $\Gamma\}$. Many constructions introduced in Part I are only auxiliary constructions needed to state and to discuss the m-problem so it is natural to look for the dependence on Γ of the $m(\Gamma)$-problem [Γ being now an arbitrary subset of $\mathcal{N}(\mathfrak{A})$]; it is the object of this section.

Lemma 5. *Let x be an arbitrary element of $\mathfrak{A}$ and let p_1 and p_2 be two semi-norms on $\mathfrak{A}$ such that $p_1(\mathbb{1})=p_2(\mathbb{1})=1$ and $p_i(p_i(x)\mathbb{1}-x)\leqq p_i(x)$ for $i=1,2$. Then the semi-norm $p=\sup(p_1,p_2)$ satisfies $p(\mathbb{1})=1$ and $p(p(x)\mathbb{1}-x)\leqq p(x)$.*

Proof. Suppose for instance that $p_1(x)\geqq p_2(x)$. Then, we have $\sup(p_1,p_2)(\sup(p_1,p_2)(x)\mathbb{1}-x)=\sup_{i=1,2}p_i(p_1(x)\mathbb{1}-x)$. By assumption $p_1(p_1(x)\mathbb{1}-x)\leqq p_1(x)=\sup(p_1,p_2)(x)$, and we have $p_2(p_1(x)\mathbb{1}-x)\leqq p_2(p_2(x)\mathbb{1}-x)+p_2([p_1(x)-p_2(x)]\mathbb{1})=p_1(x)-(p_2(x)-p_2(p_2(x)\mathbb{1}-x))\leqq p_1(x)$. So we have $p(p(x)\mathbb{1}-x)\leqq p(x)$ and, on the other hand $p(\mathbb{1})=1$ is obvious. □

As usual, the positive cone $\mathfrak{A}^+$ of $\mathfrak{A}$ is the convex cone in $\mathfrak{A}$ generated by $\{x^*x\,|\,x\in\mathfrak{A}\}$.

Lemma 6. *Let h be an arbitrary hermitian element of $\mathfrak{A}$ and let Γ be a set of C^*-semi-norms on $\mathfrak{A}$. Then h is in the $\mathcal{T}_\Gamma$-closure of $\mathfrak{A}^+$ if and only if we have: $p(p(h)\mathbb{1}-h)\leqq p(h)$, $\forall p\in\Gamma$.*

Proof. Let Γ_1 be the directed set of C^*-semi-norms on $\mathfrak{A}$ generated by Γ. We have $\mathcal{T}_{\Gamma_1}=\mathcal{T}_\Gamma$ and $\mathscr{A}(\mathfrak{A},\Gamma_1)=\mathscr{A}(\mathfrak{A},\Gamma)$. On the other hand, by the Lemma 5, we have: "$p(p(h)\mathbb{1}-h)\leqq p(h)$; $\forall p\in\Gamma$"$\Leftrightarrow$"$p(p(h)\mathbb{1}-h)\leqq p(h)$; $\forall p\in\Gamma_1$". As usual (see in Part I), we may assume that $\mathcal{T}_\Gamma=\mathcal{T}_{\Gamma_1}$ is a Hausdorff topology on $\mathfrak{A}$ (otherwise replace $\mathfrak{A}$ by the factor algebra $\mathfrak{A}/\overline{\{0\}}^{\mathcal{T}_\Gamma}$). So h is in the $\mathcal{T}_\Gamma$-closure of $\mathfrak{A}^+$ if and only if $h\in\mathfrak{A}\cap\mathscr{A}^+(\mathfrak{A},\Gamma_1)$ and this is equivalent to $\pi_p(h)$ positive, $\forall p\in\Gamma_1$ [see Part I, Section 2, Lemma 3d)]. On the other hand, it is known that in a C^*-algebra with a unit, a hermitian element x is positive if and only if it satisfies $\|\,\|x\|\,\mathbb{1}-x\|\leqq\|x\|$, [9]. Therefore $h=h^*\in\mathfrak{A}$ is in the $\mathcal{T}_\Gamma$-closure of $\mathfrak{A}^+$ if and only if we have $p(p(h)\mathbb{1}-h)\leqq p(h)$ for any $p\in\Gamma_1$ [i.e. $\forall p\in\Gamma_1$, $\|\,\|\pi_p(h)\|\,\mathbb{1}-\pi_p(h)\|\leqq\|\pi_p(h)\|$] and, since this is equivalent to $p(p(h)\mathbb{1}-h)\leqq p(h)$ for any $p\in\Gamma$, this proves the Lemma 6. □

Lemma 7. *Let x be an arbitrary element of $\mathfrak{A}$. Then $p\mapsto p(p(x)\mathbb{1}-x)$ is a continuous mapping of $\mathcal{N}(\mathfrak{A})$ in $\mathbb{R}$.*

Proof. Let p_0 be an arbitrary element of $\mathcal{N}(\mathfrak{A})$ and consider, for any $\varepsilon>0$, the neighborhood of p_0 in $\mathcal{N}(\mathfrak{A})$ defined by:

$$\mathcal{V}_{p_0,\varepsilon}=\{p\in\mathcal{N}(\mathfrak{A})\,|\,|p_0(x)-p(x)|\leqq\varepsilon/2,\ |p_0(p_0(x)\mathbb{1}-x)-p(p_0(x)\mathbb{1}-x)|\leqq\varepsilon/2\}\,.$$

For any $p\in\mathcal{V}_{p_0,\varepsilon}$, we have: $|p_0(p_0(x)\mathbb{1}-x)-p(p(x)\mathbb{1}-x)|\leqq|p_0(p_0(x)\mathbb{1}-x)-p(p_0(x)\mathbb{1}-x)|+|p(p_0(x)\mathbb{1}-x)-p(p(x)\mathbb{1}-x)|\leqq\varepsilon$, where we used $|p(a)-p(b)|\leqq p(a-b)$ and $p(\mathbb{1})=1$. This prove that $p\mapsto p(p(x)\mathbb{1}-x)$ is continuous since $\varepsilon>0$ and $p_0\in\mathcal{N}(\mathfrak{A})$ are arbitrary. □

Remark 3. Notice that $p\mapsto p(p(x)\mathbb{1}-x)$ is not uniformly continuous.

It follows from the last lemma and from the Lemma 5 that the set $\Gamma_x=\{p\in\mathcal{N}(\mathfrak{A})\,|\,p(p(x)\mathbb{1}-x)\leqq p(x)\}$ is a closed directed subset of $\mathcal{N}(\mathfrak{A})$ for any $x\in\mathfrak{A}$.

Proposition 1. *Let Γ be a set of C^*-semi-norms on $\mathfrak{A}$ and let $\overline{\mathfrak{A}^+}^{\mathcal{T}_\Gamma}$ denote the closure of $\mathfrak{A}^+$ in $\mathfrak{A}$ equipped with the locally convex topology $\mathcal{T}_\Gamma$ generated by Γ. Then the set $\hat{\Gamma}=\{p\in\mathcal{N}(\mathfrak{A}))\,|\,p(p(h)\mathbb{1}-h)\leqq p(h),\forall h\in\overline{\mathfrak{A}^+}^{\mathcal{T}_\Gamma}\}$ is the greatest set of C^*-semi-norms on $\mathfrak{A}$ for which we have: $\overline{\mathfrak{A}^+}^{\mathcal{T}_{\hat{\Gamma}}}=\overline{\mathfrak{A}^+}^{\mathcal{T}_\Gamma}$. $\hat{\Gamma}$ is a closed directed subset of $\mathcal{N}(\mathfrak{A})$ and we have $\hat{\Gamma}=\hat{\hat{\Gamma}}$.*

Proof. We have: $\hat{\Gamma}=\bigcap_{h\in\overline{\mathfrak{A}^+}^{\mathcal{T}_\Gamma}}\Gamma_h$. So $\hat{\Gamma}$ is a closed directed subset of $\mathcal{N}(\mathfrak{A})$. The rest of the proof follows from the Lemma 6. □

We know (Part I, Section 6, Proposition 2) that, if Γ_1 and Γ_2 are two sets of C^*-semi-norms on $\mathfrak{A}$ such that $\mathcal{T}_{\Gamma_1}$ is finer than $\mathcal{T}_{\Gamma_2}$, the canonical continuous $*$-homomorphism $\pi_{\Gamma_2\Gamma_1}:\mathcal{A}(\mathfrak{A},\Gamma_1)\to\mathcal{A}(\mathfrak{A},\Gamma_2)$ has a restriction to $\mathfrak{B}(\mathfrak{A},\Gamma_1)$ which is a surjective $*$-homomorphism of the C^*-algebra $\mathfrak{B}(\mathfrak{A},\Gamma_1)$ on the C^*-algebra $\mathfrak{B}(\mathfrak{A},\Gamma_2)$ $(\pi_{\Gamma_2\Gamma_1}(\mathfrak{B}(\mathfrak{A},\Gamma_1))=\mathfrak{B}(\mathfrak{A},\Gamma_2))$. Let us simply denote by π_Γ (instead of $\pi_{\Gamma,\mathcal{N}(\mathfrak{A})}$ the canonical $*$-homomorphism of $\mathcal{A}(\mathfrak{A},\mathcal{N}(\mathfrak{A}))$ in $\mathcal{A}(\mathfrak{A},\Gamma)$; $\forall\Gamma\subset\mathcal{N}(\mathfrak{A})$. Then, since $\pi_\Gamma(\mathfrak{B}(\mathfrak{A}))=\mathfrak{B}(\mathfrak{A},\Gamma)$ (remembering that the C^*-algebra associated with $\mathfrak{A}$, $\mathfrak{B}(\mathfrak{A})$, is defined by $\mathfrak{B}(\mathfrak{A},\mathcal{N}(\mathfrak{A}))=\mathfrak{B}(\mathfrak{A})$), it follows that $\mathfrak{B}(\mathfrak{A},\Gamma)$ is $*$-isomorphic with the factor C^*-algebra $\mathfrak{B}(\mathfrak{A})/\pi_\Gamma^{-1}(0)\cap\mathfrak{B}(\mathfrak{A})$. In the following we shall identify $\mathfrak{B}(\mathfrak{A},\Gamma)$ and $\mathfrak{B}(\mathfrak{A})/\pi_\Gamma^{-1}(0)\cap\mathfrak{B}(\mathfrak{A})$ under this isomorphism writing therefore:

$$\mathfrak{B}(\mathfrak{A},\Gamma_1)=\mathfrak{B}(\mathfrak{A},\Gamma_2)\quad\text{if}\quad\pi_{\Gamma_1}^{-1}(0)\cap\mathfrak{B}(\mathfrak{A})=\pi_{\Gamma_2}^{-1}(0)\cap\mathfrak{B}(\mathfrak{A})\,;$$

where

$$\Gamma_1\subset\mathcal{N}(\mathfrak{A})\quad\text{and}\quad\Gamma_2\subset\mathcal{N}(\mathfrak{A})\,.$$

Let Γ be a set of C^*-semi-norms on $\mathfrak{A}$ and let $\gamma_\Gamma:\mathfrak{A}\to\mathcal{A}(\mathfrak{A},\Gamma)$ be the canonical continuous $*$-homomorphism of $\mathfrak{A}$ equipped with $\mathcal{T}_\Gamma$ in the associated complete Hausdorff topological $*$-algebra $\mathcal{A}(\mathfrak{A},\Gamma)$. In the Part I (Section 6), we have defined $f(h)\in\mathcal{A}(\mathfrak{A},\Gamma)$ for any continuous functions f on $\mathbb{R}$ and for any hermitian element h of $\mathcal{A}(\mathfrak{A},\Gamma)$; furthermore, if $\mathcal{C}(\mathbb{R})$ is the $*$-algebra of complex continuous functions on $\mathbb{R}$ equipped with the topology of compact convergence, $f\mapsto f(h)$ is a continuous $*$-homomorphism of $\mathcal{C}(\mathbb{R})$ in $\mathcal{A}(\mathfrak{A},\Gamma)$. It will be convenient in the following to denote the spectrum of $\gamma_\Gamma(x)$ in $\mathcal{A}(\mathfrak{A},\Gamma)$ by $\mathrm{Sp}_\Gamma(x)$, for any $x\in\mathfrak{A}$; $\overline{\mathrm{Sp}_\Gamma(x)}$ will denote its closure in $\mathbb{C}$ (in $\mathbb{R}$ if $x=x^*$).

Theorem 2. *Let Γ_1 and Γ_2 be two sets of C^*-semi-norms on $\mathfrak{A}$; the following conditions are equivalent:*

a) *the Γ_1-strongly positive linear forms on $\mathfrak{A}$ and the Γ_2-strongly positive linear forms on $\mathfrak{A}$ are identical,*

b) *the $\mathcal{T}_{\Gamma_1}$-continuous positive linear forms on $\mathfrak{A}$ are Γ_2-strongly positive and the $\mathcal{T}_{\Gamma_2}$-continuous positive linear forms on $\mathfrak{A}$ are Γ_1-strongly positive,*

c) *the $\mathcal{T}_{\Gamma_1}$-closure of $\mathfrak{A}^+$ coincides with its $\mathcal{T}_{\Gamma_2}$-closure (in $\mathfrak{A}$),*

d) *$\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are identical (in $\mathcal{N}(\mathfrak{A})$),*

e) *$\mathfrak{B}(\mathfrak{A},\Gamma_1)=\mathfrak{B}(\mathfrak{A},\Gamma_2)$ (i.e. $\pi_{\Gamma_1}^{-1}(0)\cap\mathfrak{B}(\mathfrak{A})=\pi_{\Gamma_2}^{-1}(0)\cap\mathfrak{B}(\mathfrak{A})$),*

f) *$\overline{\mathrm{Sp}_{\Gamma_1}(h)}=\overline{\mathrm{Sp}_{\Gamma_2}(h)}$ for any hermitian element h of $\mathfrak{A}$.*

Furthermore, under these conditions, for any linear form ϕ on $\mathfrak{A}$, a positive linear form ω on $\mathfrak{B}(\mathfrak{A},\Gamma_1)=\mathfrak{B}(\mathfrak{A},\Gamma_2)$ is a solution of the $m(\Gamma_1)$-problem (resp. $\tilde{m}(\Gamma_1)$-problem) for ϕ if and only if it is a solution of the $m(\Gamma_2)$-problem (resp. $\tilde{m}(\Gamma_2)$-problem) for ϕ.

Proof. a)$\Rightarrow$b) since, for any $\Gamma\subset\mathcal{N}(\mathfrak{A})$, a $\mathcal{T}_\Gamma$-continuous positive linear form on $\mathfrak{A}$ is positive on the $\mathcal{T}_\Gamma$-closure of $\mathfrak{A}^+$ which means that it is Γ-strongly positive.

b)$\Rightarrow$c) since $\mathcal{T}_{\Gamma_1}$ and $\mathcal{T}_{\Gamma_2}$ are locally convex topology on $\mathfrak{A}$ and since $\mathfrak{A}^+$ is a convex subset of $\mathfrak{A}$ [$\overline{\mathfrak{A}^+}^{\mathcal{T}_\Gamma}$ is the polar of the set of all the $\mathcal{T}_\Gamma$-continuous positive linear form and the set of all the Γ-strongly positive linear forms is the polar of $\overline{\mathfrak{A}^+}^{\mathcal{T}_\Gamma}$ in the algebraic dual of $\mathfrak{A}$; $\forall\Gamma\subset\mathcal{N}(\mathfrak{A})$] [10].

c) $\Rightarrow$a) by the very definition of strong positivity;

c) $\Leftrightarrow$d) is trivial (see the Lemma 6);

d) $\Rightarrow$e) is equivalent with $\mathfrak{B}(\mathfrak{A},\hat{\Gamma})=B(\mathfrak{A},\Gamma)$; $\forall\Gamma\subset\mathcal{N}(\mathfrak{A})$.

Let us prove this equality which is equivalent to $\pi_{\Gamma\hat{\Gamma}}^{-1}(0)\cap\mathfrak{B}(\mathfrak{A},\hat{\Gamma})=\{0\}$; $\forall\Gamma\subset\mathcal{N}(\mathfrak{A})$. Any continuous linear form on $\mathcal{A}(\mathfrak{A},\hat{\Gamma})$ is a finite combination of positive continuous linear forms on $\mathcal{A}(\mathfrak{A},\hat{\Gamma})$ and, on the other hand, $\mathfrak{B}(\mathfrak{A},\hat{\Gamma})\subset\mathcal{A}(\mathfrak{A},\hat{\Gamma})$ implies that the continuous linear forms on $\mathcal{A}(\mathfrak{A},\hat{\Gamma})$ (their restrictions to $\mathfrak{B}(\mathfrak{A},\hat{\Gamma})$) separates the points of $\mathfrak{B}(\mathfrak{A},\hat{\Gamma})$ [$\mathcal{A}(\mathfrak{A},\hat{\Gamma})$ is a Hausdorff locally convex space]. It follows that, in order to prove our statement, it is sufficient to show that for any continuous positive linear form ψ on $\mathcal{A}(\mathfrak{A},\hat{\Gamma})$ there is a positive linear form ω on $\mathfrak{B}(\mathfrak{A},\Gamma)$ for which we have: $\psi(x)=\omega(\pi_{\Gamma\hat{\Gamma}}(x))$, $\forall x\in\mathfrak{B}(\mathfrak{A},\hat{\Gamma})$.

But let ϕ be the defined by $\phi(y)=\psi(\gamma_{\hat{\Gamma}}(y))$, $\forall y\in\mathfrak{A}$; ϕ is Γ-strongly positive [by d) $\Leftrightarrow$b)] since it is $\mathcal{T}_\Gamma$-continuous. So there is a solution ω of the $m(\Gamma)$-problem for ϕ and, since the representation of $\mathfrak{A}$ associated with ϕ is bounded, ω is unique (by the Theorem 1, for instance) and we have: $\psi(f(\gamma_{\hat{\Gamma}}(h))=\omega(f(\gamma_\Gamma(h)))$, $\forall f\in\mathcal{C}_{(0)}(\mathbb{R})$ and $\forall h\in\mathfrak{A}^h$. This implies $\psi(x)=\omega(\pi_{\Gamma\hat{\Gamma}}(x))$, $\forall x\in\mathfrak{B}(\mathfrak{A},\hat{\Gamma})$ [indeed $f(\gamma_\Gamma(h))=\pi_{\Gamma\hat{\Gamma}}(f(\gamma_{\hat{\Gamma}}(h)))$ and $\mathfrak{B}(\mathfrak{A},\Gamma)$ is generated by the $f(\gamma_\Gamma(h))$, $f\in\mathcal{C}_{(0)}(\mathbb{R})$, $h=h^*\in\mathfrak{A}$].

e)$\Rightarrow$f) because if e) is satisfied then $f(\gamma_{\Gamma_1}(h))=0$ is equivalent to $f(\gamma_{\Gamma_2}(h))=0$, $\forall h=h^*\in\mathfrak{A}$ and $\forall f\in\mathcal{C}_{(0)}(\mathbb{R})$; so the greatest closed subset $S_h\subset\mathbb{R}$ such that $f\in\mathcal{C}_{(0)}(\mathbb{R})$ and $f(S_h)=0$ imply $f(\gamma_{\Gamma_1}(h))=0$ is also the greatest closed subset of $\mathbb{R}$ such that $f\in\mathcal{C}_{(0)}(\mathbb{R})$ and $f(S_h)=0$ imply $f(\gamma_{\Gamma_2}(h))=0$. This means $\overline{\mathrm{Sp}_{\Gamma_1}(h)}=\overline{\mathrm{Sp}_{\Gamma_2}(h)}=S_h$ because we have: $\mathfrak{B}(\mathfrak{A},\Gamma)\subset\prod_{p\in\Gamma}\mathfrak{B}_p(\mathfrak{A})$ and $\pi_p(\mathfrak{B}(\mathfrak{A},\Gamma))=\mathfrak{B}_p(\mathfrak{A})$ (see in Part I) and $\mathrm{Sp}_\Gamma(h)=\bigcup_{p\in\Gamma}\mathrm{Sp}(\pi_p(\gamma_\Gamma(h)))$.

f) $\Rightarrow$ c) because $h \in \overline{\mathfrak{A}^+}^{\mathcal{T}_\Gamma}$ is equivalent to $\mathrm{Sp}_\Gamma(h) \subset \mathbb{R}^+$ (see Part I, Section, Lemma 3).

This achieves the proof of the equivalence of the conditions a)—f).

The conditions a), e), and f) clearly imply that any solution ω of the $\tilde{m}(\Gamma_1)$-problem for a linear form ϕ on $\mathfrak{A}$ is also a solution of the $\tilde{m}(\Gamma_2)$-problem and conversely. It remains to show that the same it true for the m-problem. It is clearly sufficient to show that any solution ω of the $m(\hat{\Gamma})$-problem for a linear form ϕ on $\mathfrak{A}$ is also a solution of the $m(\Gamma)$-problem for $\phi (\forall \Gamma \subset \mathcal{N}(\mathfrak{A}))$. The closures of $\{0\}$ in $\mathfrak{A}$ for $\mathcal{T}_{\hat{\Gamma}}$ and for $\mathcal{T}_\Gamma$ obviously coincide so we may suppose without restriction that $\mathcal{T}_{\hat{\Gamma}}$ and $\mathcal{T}_\Gamma$ are Hausdorff topologies on $\mathfrak{A}$ (replace $\mathfrak{A}$ by $\mathfrak{A}/\overline{\{0\}}$ where $\overline{\{0\}} = \overline{\{0\}}^{\mathcal{T}_{\hat{\Gamma}}} = \overline{\{0\}}^{\mathcal{T}_\Gamma}$). Using the definition of the m-problem (Part I, Section 7, Definition 4′), we are reduced to show that $h = h^* \in \mathfrak{A}$, $b = b^* \in \mathfrak{B}(\mathfrak{A}, \Gamma) = \mathfrak{B}(\mathfrak{A}, \hat{\Gamma})$ and $h - b \in \mathcal{A}^+(\mathfrak{A}, \Gamma)$ imply $h - b \in \mathcal{A}^+(\mathfrak{A}, \hat{\Gamma})$. However $h - b \in \mathcal{A}^+(\mathfrak{A}, \hat{\Gamma})$ is equivalent to $\psi(h-b) \geqq 0$ for all positive continuous linear form ψ on $\mathcal{A}(\mathfrak{A}, \hat{\Gamma})$. But then, if ψ is positive and $\mathcal{T}_{\hat{\Gamma}}$-continuous, we know that $\psi \upharpoonright \mathfrak{B}(\mathfrak{A}, \Gamma)$ $(= \mathfrak{B}(\mathfrak{A}, \hat{\Gamma}))$ is the unique solution of the $m(\Gamma)$-problem for $\psi \upharpoonright \mathfrak{A}$ [by the same argument as in d) $\Rightarrow$ e)]. It follows that $\forall h \in \mathfrak{A}$ and $b = b^* \in \mathfrak{B}(\mathfrak{A}, \Gamma)$ such that $h - b \in \mathcal{A}^+(\mathfrak{A}, \Gamma)$ we have $\psi(h-b) \geqq 0$ for any continuous positive linear form ψ on $\mathcal{A}(\mathfrak{A}, \hat{\Gamma})$ so we have $h - b \in \mathcal{A}^+(\mathfrak{A}, \hat{\Gamma})$. □

Definition 1. Two sets Γ_1 and Γ_2 satisfying the equivalent conditions a)—f) of the last theorem will be said to have *the same support and we write*: $\mathrm{Supp}(\Gamma_1) = \mathrm{Supp}(\Gamma_2)$. This is obviously an equivalence relation on $\mathfrak{P}(\mathcal{N}(\mathfrak{A}))$ and the corresponding factor space will be called the set of supports. If $\Gamma_1 \subset \hat{\Gamma}_2 \subset \mathcal{N}(\mathfrak{A})$ we say that *the support of* Γ_1 *is contained in the support of* Γ_2 *and we write*: $\mathrm{Supp}(\Gamma_1) \subset \mathrm{Supp}(\Gamma_2)$ (this is an order relation).

Remark 4. Let h be an arbitrary hermitian element of $\mathfrak{A}(h \in \mathfrak{A}^h)$ and let $\mathbb{C}[X]$ be the *-algebra of complex polynomials with respect to the indeterminate X. Then we may associate to any set Γ of C^*-semi-norms on $\mathfrak{A}$ the set $\Gamma(h)$ of C^*-semi-norms on $\mathbb{C}[X]$ defined by:

$$\Gamma(h) = \{p_h \mid p_h(P(X)) = p(P(h)), \forall P(X) \in \mathbb{C}[X]; p \in \Gamma\}. \tag{1}$$

It is not hard to see [use condition f) in the last theorem] that $\mathrm{Supp}(\Gamma_1) = \mathrm{Supp}(\Gamma_2)$ is equivalent to $\mathrm{Supp}(\Gamma_1(h)) = \mathrm{Supp}(\Gamma_2(h))$, $\forall h \in \mathfrak{A}^h$, and that, for the *-algebras of polynomials, the above definition is consistant with the definition given in the Section 5 of Part I. It follows that the set of supports may be identified with a set of closed subsets of $\mathbb{R}^{\mathfrak{A}^h}$ [$\mathrm{Supp}(\Gamma_1) \subset \mathrm{Supp}(\Gamma_2)$ if and only if the corresponding inclusion holds in $\mathbb{R}^{\mathfrak{A}^h}$].

4. Homomorphisms and Tensor Algebras

We already pointed out in Part I that the tensor algebras over involutive vector spaces are of particular interest since every *-algebra with unit is (in a non unique way however) the quotient of such a tensor algebra by a *-invariant two-sided ideal. An immediate consequence of the last theorem and of the Theorem 2 of Part I (Section 4) is the following proposition.

Proposition 2. *Let E be an involutive vector space, let E' be a $*$-invariant subspace of the dual space of E which separates the points of E and let $\Gamma_{E'}$ be as in the Theorem 2 of Part I. Then $\hat{\Gamma}_{E'}$ is the set of all the C^*-semi-norms on the tensor algebra $T(E)$.*

So we have $\mathfrak{B}(T(E))=\mathfrak{B}(T(E), \Gamma_{E'})$ etc.....

Generally when one considers a $*$-algebra with unit as a factor space of some tensor algebra over an involutive vector space, this means that one is interested on a real vector space of hermitian elements in this algebra which is generating. This is, for instance, typically the case in quantum field theory when one considers the field operator smeared with real test functions. This suggests to generalize the definitions of the quasi-localizable C^*-algebra (Definition 6, Part I, Section 9) by the following one.

Definition 2. Let E be a real vector space and let $T(E+iE)$ be the tensor algebra over the complexified vector space $E+iE$ of E equipped with its canonical structure of $*$-algebra with unit. We define the *C^*-algebra associated with E*, $\mathfrak{B}_0(E)$, to be the C^*-subalgebra of $\mathfrak{B}(T(E+iE))$ generated by $\{f(h)\in\mathfrak{B}(T(E+iE))\,|\,h\in E \text{ and } f\in\mathscr{C}_{(0)}(\mathbb{R})\}$.

It should be clear from this definition and from the Proposition 2 that the quasi-localizable C^*-algebra $\hat{\mathfrak{B}}(M)$ is the C^*-algebra $\mathfrak{B}_0(\mathscr{D}(M;\mathbb{R}))$ associated with the space $\mathscr{D}(M;\mathbb{R})$ of the real C^∞-function with compact supports on M.

In order to complete the discussion, we must describe the invariance of the m-problem under $*$-homomorphisms.

Proposition 3. *Let $\alpha_{12}:\mathfrak{A}_2\to\mathfrak{A}_1$ be a homomorphism of $*$-algebras with units ($\alpha_{12}(\mathbb{1})=\mathbb{1}$), let Γ_1 (resp. Γ_2) be a set of C^*-semi-norms on $\mathfrak{A}_1$ (resp. $\mathfrak{A}_2$) and let us assume that $p\in\Gamma_1$ implies $p\circ\alpha_{12}\in\hat{\Gamma}_2$ (i.e. α_{12} is continuous from $(\mathfrak{A}_2,\mathscr{T}_{\hat{\Gamma}_2})$ in $(\mathfrak{A}_1,\mathscr{T}_{\Gamma_1})$). Then there is a unique $*$-homomorphism $\beta_{\alpha_{12}}:\mathfrak{B}(\mathfrak{A}_2,\Gamma_2)\to\mathfrak{B}(\mathfrak{A}_1,\Gamma_1)$ such that we have:*

$$\beta_{\alpha_{12}}(f(h))=f(\alpha_{12}(h))^4, \forall h\in\mathfrak{A}_2^h \text{ and } \forall f\in\mathscr{C}_{(0)}(\mathbb{R}).$$

If ϕ is a Γ_1-strongly positive linear form on $\mathfrak{A}_1$ and if ω is a solution of the $m(\Gamma_1)$-problem for ϕ, then $\phi\circ\alpha_{12}$ is a Γ_2-strongly positive linear form on $\mathfrak{A}_2$ and $\omega\circ\beta_{\alpha_{12}}$ is a solution of the $m(\Gamma_2)$-problem for $\phi\circ\alpha_{12}$.

If $\alpha_{23}:\mathfrak{A}_3\to\mathfrak{A}_2$ is another homomorphism of $*$-algebras with units and if Γ_3 is a set of C^*-semi-norms on $\mathfrak{A}_3$ such that, for any $p\in\Gamma_2$, $p\circ\alpha_{23}$ is in $\hat{\Gamma}_3$; then $\forall p\in\Gamma_1$, $p\circ\alpha_{12}\circ\alpha_{23}$ is in $\hat{\Gamma}_3$ and we have:

$$\beta_{\alpha_{12}\circ\alpha_{23}}=\beta_{\alpha_{12}}\circ\beta_{\alpha_{23}}.$$

If α is the identity mapping of $\mathfrak{A}_2$ onto itself, β_α is the identity mapping of $\mathfrak{B}(\mathfrak{A}_2,\Gamma_2)$ onto itself.

Proof. Let $\alpha_{12}:\mathfrak{A}_2\to\mathfrak{A}_1$ be as in the proposition. Then α_{12} is continuous if $\mathfrak{A}_1$ is equipped with $\mathscr{T}_{\Gamma_1}$ and if $\mathfrak{A}_2$ is equipped with $\mathscr{T}_{\hat{\Gamma}_2}$; therefore there is a unique continuous $*$-homomorphism $\hat{\alpha}_{12}:\mathscr{A}(\mathfrak{A}_2,\hat{\Gamma}_2)\to\mathscr{A}(\mathfrak{A}_1,\Gamma_1)$ for which $\hat{\alpha}_{12}(\gamma_{\hat{\Gamma}_2}(h))=\gamma_{\Gamma_1}(\alpha_{12}(h))$, where $\gamma_{\hat{\Gamma}_2}:\mathfrak{A}_2\to\mathscr{A}(\mathfrak{A}_2,\hat{\Gamma}_2)$ and $\gamma_{\Gamma_1}:\mathfrak{A}_1\to\mathscr{A}(\mathfrak{A}_1,\Gamma_1)$ are the canonical mappings and where h runs over the set $\mathfrak{A}_2^h$ of all the hermitian elements of $\mathfrak{A}_2$.

4 Here we use the notation $f(h)$ to denote the element $f(\gamma_{\Gamma_2}(h))$ of $\mathfrak{B}(\mathfrak{A}_2,\Gamma_2)$ where $h=h^*\in\mathfrak{A}_2$, $f\in\mathscr{C}_{(0)}(\mathbb{R})$ and γ_{Γ_2} is the canonical mapping of $\mathfrak{A}_2$ in $\mathscr{A}(\mathfrak{A}_2,\Gamma_2)$. (The same convention is applied for $\mathfrak{A}_1,\Gamma_1$)

Then, the restriction $\hat{\alpha}_{12} \upharpoonright \mathfrak{B}_\infty(\mathfrak{A}_2, \hat{\Gamma}_2)$ is easily seen to be a *-homomorphism of $\mathfrak{B}_\infty(\mathfrak{A}_2, \hat{\Gamma}_2)$ in $\mathfrak{B}_\infty(\mathfrak{A}_1, \Gamma_1)$ for which we have $\hat{\alpha}_{12}(f(\gamma_{\hat{\Gamma}_2}(h))) = f(\gamma_{\Gamma_1}(\alpha_{12}(h)))$ for any $h \in \mathfrak{A}_2^h$ and $f \in \mathscr{C}_{(0)}(\mathbb{R})$. It follows that the restriction $\beta_{\alpha_{12}}$ of $\hat{\alpha}_{12}$ to $\mathfrak{B}(\mathfrak{A}_2, \hat{\Gamma}_2)$ is a *-homomorphism of $\mathfrak{B}(\mathfrak{A}_2, \hat{\Gamma}_2)$ in $\mathfrak{B}(\mathfrak{A}_1, \Gamma_1)$ and we have canonically: $\mathfrak{B}(\mathfrak{A}_2, \Gamma_2) = \mathfrak{B}(\mathfrak{A}_2, \hat{\Gamma}_2)$ and $f(\gamma_{\Gamma_2}(h)) = f(\gamma_{\hat{\Gamma}_2}(h))\,(=f(h)$ with our conventions) for any $h \in \mathfrak{A}_2^h$ and $f \in \mathscr{C}_{(0)}(\mathbb{R})$. So $\beta_{\alpha_{12}}$ satisfies the condition of the proposition and is clearly unique under this condition since $\mathfrak{B}(\mathfrak{A}_2, \Gamma_2)$ is generated by $\{f(h) \in \mathfrak{B}(\mathfrak{A}_2, \Gamma_2) \mid h = h^* \in \mathfrak{A}_2$ and $f \in \mathscr{C}_{(0)}(\mathbb{R})\}$ (as C^*-algebra). The rest of the proof of this proposition is completely straightforward (use the Theorem 2). □

Corollary 3. *Let* $\alpha_{12}: \mathfrak{A}_2 \to \mathfrak{A}_1$ *be a homomorphism of *-algebras with units. Then there is a unique *-homomorphism* $\mathfrak{B}(\alpha_{12}): \mathfrak{B}(\mathfrak{A}_2) \to \mathfrak{B}(\mathfrak{A}_1)$ *for which we have:* $\mathfrak{B}(\alpha_{12})(f(h)) = f(\alpha_{12}(h)), \forall h = h^* \in \mathfrak{A}_2$ *and* $f \in \mathscr{C}_{(0)}(\mathbb{R})$. *If* $I_{\mathfrak{A}}$ *is the identity mapping of the *-algebra with unit* $\mathfrak{A}$ *onto itself,* $\mathfrak{B}(I_{\mathfrak{A}}) = I_{\mathfrak{B}(\mathfrak{A})}$. *If* $\alpha_{23}: \mathfrak{A}_3 \to \mathfrak{A}_2$ *is another homomorphism of *-algebras with units, then we have:* $\mathfrak{B}(\alpha_{12}) \circ \mathfrak{B}(\alpha_{23}) = \mathfrak{B}(\alpha_{12} \circ \alpha_{23})$.

Proposition 4. *Let* $u: E \to F$ *be a real-linear mapping of the real vector space E in the real vector space F. Then, there is a unique *-homomorphism* $\mathfrak{B}_0(u): \mathfrak{B}_0(E) \to \mathfrak{B}_0(F)$ *for which we have:* $\mathfrak{B}_0(u)(f(e)) = f(u(e)), \forall e \in E$ *and* $\forall f \in \mathscr{C}_{(0)}(\mathbb{R})$. *If* $v: F \to G$ *is another real-linear mapping, we have:* $\mathfrak{B}_0(v \circ u) = \mathfrak{B}_0(v) \circ \mathfrak{B}_0(u)$. *Furthermore we have:* $\mathfrak{B}_0(I_E) = I_{\mathfrak{B}_0(E)}$.

Proof. Again the uniqueness of $\mathfrak{B}_0(u)$ immediately follows from the definition of $\mathfrak{B}_0(E)$ (Definition 2 above).

Let $\tilde{u}: T(E + iE) \to T(F + iF)$ be the unique homomorphism of *-algebras with units which extends u (with obvious identifications). Then it is easy to see that the restriction $\mathfrak{B}_0(u) = \mathfrak{B}(\tilde{u}) \upharpoonright \mathfrak{B}_0(E)$ of $\mathfrak{B}(\tilde{u}): \mathfrak{B}(T(E + iE)) \to \mathfrak{B}(T(F + iF))$ mapps the C^*-subalgebra $\mathfrak{B}_0(E)$ of $\mathfrak{B}(T(E + iE))$ in the C^*-subalgebra $\mathfrak{B}_0(F)$ of $\mathfrak{B}(T(F + iF))$ and satisfies the condition of the Proposition 4. The rest of the proof is immediate. □

Remark 5. a) The Proposition 4 implies, in particular that there is a canonical group homomorphism of the group $\mathrm{Aut}(E)$ of all the real-linear inversible mapping of E on itself in the group $\mathrm{Aut}(\mathfrak{B}_0(E))$ of all the *-automorphisms of $\mathfrak{B}_0(E)$. This must be compared with the Proposition 5 of Part I.

b) Corollary 3 (resp. Proposition 4) implies that $\mathfrak{B}$ (resp. $\mathfrak{B}_0$) is a covariant functor of the category of *-algebras with units (resp. of the real vector spaces) in the category of C^*-algebras.

c) It would be of some interest to be able to define $\mathfrak{B}(\mathfrak{A})$ and $\mathfrak{B}_0(E)$ as solutions of universal problems (this could bring some light, for instance, on the connexion of the m-problem with the $\tilde{m}$-problem).

5. Application to Essentially Self-adjoint Quantum Fields

It will be convenient in this section to call hermitian scalar field a linear mapping A of the real vector space $\mathscr{D}(M, \mathbb{R})$ of the real C^∞-functions with compact supports on the Minkowski space $M = \mathbb{R}^{s+1}$ in the real vector space of the symmetric operators on a dense domain D in a Hilbert space $\mathfrak{H}$ such that the following conditions are satisfied:

a) $A(h)D \subset D, \forall h \in \mathscr{D}(M, \mathbb{R})$,

b) there is a unit vector $\Omega \in D$ such that D is the linear hull of $A(h_1)\dots A(h_N)\Omega$ when (h_i) runs over the finite families in $\mathscr{D}(M, \mathbb{R})$.

Let A be a hermitian scalar field in the above sense, then of course $\sum \lambda_{i_1 \dots i_n} h_{i_1} \otimes \dots \otimes h_{i_n} \mapsto \sum \lambda_{i_1 \dots i_n} A(h_{i_1}) \dots A(h_{i_n})$ defines a *-representation of the tensor algebra $T(\mathscr{D}(M))$ over the space $\mathscr{D}(M)$ of complex C^∞-functions with compact supports on M. This *-representation is a cyclic representation with cyclic vector Ω and if ϕ_Ω denote the vector state on $T(\mathscr{D}(M))$ corresponding to Ω, we have canonically:

$$\mathfrak{H} = \mathfrak{H}_{\phi_\Omega}, D = D_{\phi_\Omega}, \Omega = \Omega_{\phi_\Omega} \text{ and } A(h) = \pi_{\phi_\Omega}(h), \forall h \in \mathscr{D}(M, \mathbb{R}).$$

Theorem 3. *Let A be hermitian scalar field and let ϕ_Ω be the corresponding state on $T(\mathscr{D}(M))$ (as above). Suppose that for any real test function $h \in \mathscr{D}(M)$ the field operator $A(h)$ is essentially self-adjoint. Then, the m-problem for ϕ_Ω is determined on the quasi-localizable C^*-algebra $\hat{\mathfrak{B}}(M)$ and we have: $\tilde{\omega}(f_1(h_1)\dots f_n(h_n)) = (\Omega | f_1(\overline{A(h_1)}) \dots f_n(\overline{A(h_n)})\Omega), \forall n \geqq 0, \forall f_1, \dots, f_n \in \mathscr{C}_{(0)}(\mathbb{R}), \forall h_1, \dots, h_n \in \mathscr{D}(M, \mathbb{R})$ and for any solution $\tilde{\omega}$ of the m-problem for ϕ_Ω. Let ω be the unique positive linear form on $\mathfrak{B}(M)$ obtained by restriction of an arbitrary solution $\tilde{\omega}$ of the m-problem for ϕ_Ω and let $(\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega)$ be the corresponding G.N.S. triplet. Then, we have canonically: $\mathfrak{H}_\omega = \mathfrak{H}$, $\Omega_\omega = \Omega$ and $\pi_\omega(f(h)) = f(\overline{A(h)}); \forall f \in \mathscr{C}_{(0)}(\mathbb{R})$ and $\forall$ $h \in \mathscr{D}(M, \mathbb{R})$ (where $\overline{A(h)} = A(h)^*$ is the closure of $A(h)$).*

So $\pi_\omega(\hat{\mathfrak{B}}(M))$ generates the bounded observables of the field.

This theorem has not to be proved since it is a specific case of the Theorem 1 (supplemented by the fact that by the Proposition 2 and the Definition 2, the C^*-algebra $\hat{\mathfrak{B}}(M)$ is identical with $\mathfrak{B}_0(\mathscr{D}(M, \mathbb{R}))$.

Remark 6. a) The Part b) of the Theorem 6 in Part I follows from the Part a) of that theorem and from the above Theorem 3. However, even if A is a nice local Wightman field which is essentially self-adjoint on its usual domain, we do not know in general if its spectral projections generate local rings. In other words, the corresponding representation of the localizable algebra may fail to be local.

b) In the above definition of a hermitian scalar field, the field may be not local and since translation invariance does not enter, it may happen that Ω has not the meaning of a vacuum.

c) Notice also that no continuity with respect to the test functions is assumed.

Lemma 8. *Let $(A, \mathfrak{H}, \Omega)$ be a hermitian scalar field and let $(A_\alpha, \mathfrak{H}_\alpha, \Omega_\alpha)_{\alpha \in I}$ be a net of hermitian scalar fields such that we have: $\lim(\Omega_\alpha | A_\alpha(h_1)\dots A_\alpha(h_n)\Omega_\alpha) = (\Omega | A(h_1)\dots A(h_n)\Omega)$ for any finite family $h_1, \dots, h_n$ in $\mathscr{D}(M)$. Let ω_α be, for each $\alpha \in I$, the restriction to $\hat{\mathfrak{B}}(M)$ of an arbitrary solution of the m-problem for ϕ_{Ω_α}. Then there is a weakly convergent subnet of (ω_α) and the limit ω of such a subnet is the restriction to $\hat{\mathfrak{B}}(M)$ of a solution of the m-problem for ϕ_Ω. If the m-problem for ϕ_Ω is determined on $\hat{\mathfrak{B}}(M)$, then (ω_α) is weakly convergent.*

Proof. Let $(\omega_{\alpha'})$ be a weakly convergent subnet of (ω_α) and let ω be its weak limit [such $(\omega_{\alpha'})$ exist since the set of all the states on a C^*-algebra is weakly compact]. Let $(\tilde{\omega}_{\alpha'})$ be the corresponding solutions of the m-problems for the $\phi_{\Omega_{\alpha'}}$. $(\tilde{\omega}_{\alpha'})$ is net of states on $\mathfrak{B}(T(\mathscr{D}(M)))$ so (again by compacity) there is a weakly convergent subnet $(\tilde{\omega}_{\alpha''})$ of $(\tilde{\omega}_{\alpha'})$. By the Lemma 4, its limit $\tilde{\omega}$ is solution of the m-problem for ϕ_Ω and, on the other hand, we clearly have $\omega = \tilde{\omega} \upharpoonright \hat{\mathfrak{B}}(M)$. □

Remark 7. Roughly speaking, this means that one cannot miss the right result if one starts with an approximation $(A_\alpha, \mathfrak{H}_\alpha, \Omega_\alpha)$ then constructs ω_α as above and obtains ω by compactness in the weak dual of the quasi-localizable C^*-algebra $\hat{\mathfrak{B}}(M)$. Indeed, $(A, \mathfrak{H}, \Omega)$ can be reconstructed from ω because by Corollary 2 we have $\mathfrak{H} = \mathfrak{H}_{\phi_\Omega} \subset \mathfrak{H}_\omega$, $\Omega = \Omega_\omega$, and by Lemmas 1 and 2 we have $A(h) = \pi_{\tilde{\omega}}(h) \upharpoonright \pi_{\tilde{\omega}}(T(\mathscr{D}(M)))\,\Omega$ where $\tilde{\omega}$ is any state on $\mathfrak{B}(T(\mathscr{D}(M)))$ extending ω. A does not depend on the choice of the extension $\tilde{\omega}$ of ω.

6. Example: A Class of Jaffe Cut-off Fields [6][5]

We want to discuss the case of interacting cut-off hamiltonian field theory. In order to be explicit, we shall concentrate on $\lambda\phi^4$ theory and on a specific cut-off introduced by Jaffe in his thesis [6]; it must however be clear from the work of Jaffe that the arguments used here may be generalized to any polynomial interaction of degree $d \geqq 2$ which is bounded from below and to other types of cut-off which incorporate both an ultra-violet cut-off and a spatial cut-off.

We work in space-time $M = \mathbb{R}^{1+s} = \{(t, \boldsymbol{r}) \mid t \in \mathbb{R}, \boldsymbol{r} \in \mathbb{R}^s\}$ and we shall deal with a cut-off which consists to replace the interaction picture time zero field $\phi(0, \boldsymbol{r})$ and its conjugate momentum $\pi(0, \boldsymbol{r})$ by two very regular objects on space $(\boldsymbol{r} \in \mathbb{R}^s)$ $\phi_N(0, \boldsymbol{r}) = \phi_N(\boldsymbol{r})$ and $\pi_N(0, \boldsymbol{r}) = \pi_N(\boldsymbol{r})$ which describe a system with N degrees of freedom (for each integer $N \geqq 1$).

So let us consider in the Hilbert space $\mathfrak{H}_N = L^2(\mathbb{R}^N)$ the $2N$ operators (p_n, q_n) defined on the dense domain $D_N^1 = \mathscr{S}(\mathbb{R}^N)$ by:

$$\Phi(q_1, \dots, q_n) \mapsto -i \frac{\partial \Phi}{\partial q_n}(q_1, \dots, q_N) \quad \text{and} \quad \Phi(q_1, \dots, q_N) \mapsto q_n \Phi(q_1, \dots, q_N),$$

$\Phi \in D_N^1$. As it is well known, D_N^1 is a dense stable domain for these operators on which they are essentially self-adjoint.

Let $(e_n)_{n \in \mathbb{N}}$ denote a fixed orthonormal basis of the Hilbert space $L^2(\mathbb{R}^s)$ which consists of real valued function in $\mathscr{S}(\mathbb{R}^s)$ (for instance the Hermite functions). For each positive integer N, we define ϕ_N and π_N by:

$$\phi_N(\boldsymbol{r}) = \phi_N(0, \boldsymbol{r}) = \sum_{n=1}^{n=N} e_n(\boldsymbol{r}) q_n\,; \qquad \pi_N(\boldsymbol{r}) = \pi_N(0, \boldsymbol{r}) = \sum_{n=1}^{n=N} e_n(\boldsymbol{r}) p_n\,. \tag{2}$$

For any real tempered distribution $T \in \mathscr{S}(\mathbb{R}^s)'$, [11], the operators

$$\langle T, \phi_N \rangle = \sum_{n \leqq N} \langle T, e_n \rangle q_n \quad \text{and} \quad \langle T, \pi_N \rangle = \sum_{n \leqq N} \langle T, e_n \rangle p_n$$

are essentially self-adjoint on D_N^1 and map D_N^1 into itself.

[5] Jaffe suggested to us several simplifications for the content of this section. The author thanks him for these suggestions

The sense in which (ϕ_N, π_N) is a cut-off versus for the canonical pair (ϕ, π) must be clear; so for a formal interaction picture hamiltonian, we define the corresponding cut-off hamiltonian to be the operator in $\mathfrak{H}_N = L^2(\mathbb{R}^N)$ obtained by substitution of (ϕ_N, π_N) to (ϕ, π) in the corresponding formal expression. We shall concentrate on $\lambda\phi^4$ theory and, since from perturbation theory for $s=3$ we expect some mass and coupling constant renormalizations, we shall allow a dependence on N for these parameters. Therefore we define H_N on D_N^1 by:

$$H_N = \int d^s r \left\{ \frac{1}{2} [\pi_N(\boldsymbol{r})^2 + (\nabla\phi_N(\boldsymbol{r}))^2 + m^2 \phi_N(\boldsymbol{r})^2] + \lambda_N \phi_N(\boldsymbol{r})^4 - \frac{\delta m_N^2}{2} \phi_N(\boldsymbol{r})^2 \right\} - E_N . \quad (3)$$

Using the formula (2), we obtain: $\left(\Delta_N = \sum_{n=1}^{n=N} \left(\frac{\partial}{\partial q_n} \right)^2 \right)$

$$H_N = \frac{1}{2} \sum_{n=1}^{n=N} p_n^2 + V_N(q_1, \ldots, q_N) = -\frac{1}{2}\Delta_N + V_N(q_1, \ldots, q_N),$$

where

$$\begin{aligned} V_N(q_1, \ldots, q_N) = {} & \lambda_N [\textstyle\int e_{n_1} e_{n_2} e_{n_3} e_{n_4}] q_{n_1} q_{n_2} q_{n_3} q_{n_4} \\ & + \tfrac{1}{2} [\textstyle\int (\nabla e_n)(\nabla e_m) + (m^2 - \delta m_N^2)\delta_{nm}] q_n q_m - E_N . \end{aligned} \quad (4)$$

H_N is a symmetric operator[6] on D_N^1 and $H_N D_N^1 \subset D_N^1$; the closure of H_N will again be denoted by H_N. The "free part"

$$H_N^{(0)} = \int d^s r \{ \tfrac{1}{2} [\pi_N^2 + (\nabla\phi_N)^2 + m^2 \phi_N^2] \} \qquad \text{(its closure)} \quad (5)$$

is a well-known operator which is a semi-bounded (from below) self-adjoint operator with a pure discrete spectrum and lowest eigenvalue of multiplicity one. We shall denote the corresponding normalized eigenvector by $\Omega_N^{(0)}$ and refer to it as the *Fock vacuum*. We have: $\Omega_N^{(0)} \in D_N^1 = \mathscr{S}(\mathbb{R}^N)$. Let $\Delta E_N^{(0)}$ be the difference between the lowest eigenvalue distinct from those corresponding to $\Omega_N^{(0)}$ and this last one, we have:

$$\Delta E_N^{(0)} > m \quad \text{and} \quad \lim_{N\to\infty} \Delta E_N^{(0)} = m = \inf_N (\Delta E_N^{(0)}). \quad (6)$$

In what follows we shall suppose that either $\lambda_N > 0$, or $\lambda_N = 0$ and $\delta m_M = 0$. Under this assumption $V(q_1, \ldots, q_N)$ is a polynomial of degree 4 or 2 in the q_n which is bounded from below and we have the following result [6].

Jaffe's Theorem. *H_N is self-adjoint, bounded from below, has a pure discrete spectrum and its minimum eigenvalue has multiplicity one. The resolvant of H_N is compact and the eigenfunctions of H_N belong to $D_N^1 = \mathscr{S}(\mathbb{R}^N)$ and form a total set in $\mathfrak{H}_N = L^2(\mathbb{R}^N)$. D_N^1 is stable by $e^{itH_N}, \forall t \in \mathbb{R}$ ($e^{itH_N} D_N^1 \subset D_N^1$). Furthermore the cyclic subspace $D_N^{(0)}$ generated from $\Omega_N^{(0)}$ by the polynomials in p_n, q_m is a core for H_N.*

In what follows, we shall suppose that $\lambda_N \geqq 0$, δm_M, and E_N are chosen in such a way that the minimum eigenvalue of H_N is zero (the corresponding normalized

[6] It is of course understood that the parameters m, λ_N, δm_N, and E_N are real

eigenvector, called the *vacuum*, will be denoted by Ω_N) and that the next smallest eigenvalue E_{1N} is equal to $\Delta E_N^{(0)}$:

$$E_{1N}=\Delta E_N^{(0)}>m>0\left(\lim_{N\to\infty} E_{1N}=m\right). \tag{7}$$

Notice that this is a renormalization prescription for the mass of the theory and that it does not determine λ_N (here we do not impose conditions for the coupling constant renormalization).

Now let us define $\phi_N(t,\boldsymbol{r})=\phi_N(x)$ on D_N^1 by:

$$\phi_N(t,\boldsymbol{r})=e^{itH_N}\phi_N(0,\boldsymbol{r})e^{-itH_N}, \tag{8}$$

$$\phi_N(h)=\int dt\, d^s r\, h(t,\boldsymbol{r})\phi_N(t,\boldsymbol{r}). \tag{9}$$

In order to obtain self-adjointness for the space-time smeared field operator, we follow Glimm and Jaffe [15, 16] (see also [17]).

Lemma 9. *There exist positive constants a_N, b_N, and c_N for which we have:*

$$-(\Phi|(H_N+c_N\mathbb{1})\Phi)\leqq(\Phi|\phi_N(t,\boldsymbol{r})\Phi)\leqq(\Phi|(H_N+c_N\mathbb{1})\Phi) \quad \textit{and}$$

$$\|\phi_N(t,\boldsymbol{r})\Phi\|\leqq\|(a_NH_N+b_N\mathbb{1})^{1/2}\Phi\|,\quad \forall\Phi\in D_N^1 \quad \textit{and} \quad \forall(t,\boldsymbol{r})\in M.$$

Proof. From the definition (8) it follows that it is sufficient to prove this inequality for $t=0$, and from (2) it follows that it is sufficient to show that they hold when $\phi(t,\boldsymbol{r})$ is replaced by $q_n(1\leqq n\leqq N)$. But then, the first follows from (4) and the fact that we have $|q_n|\leqq V_N(q_1\dots q_n)+c'_N$ for some positive c'_N and the second from the fact that we have $q_n^2\leqq V_N(q_1,\dots,q_N)+b'_N$ if $\lambda_N\neq 0$ and, if $\lambda_N=0$, $q_n^2\leqq \frac{2}{m^2}V_N(q_1,\dots,q_N)+b'_N$ for some $b'_N>0$. □

Lemma 10. *Let h be an arbitrary real test function of $\mathscr{D}(M)$. Then $\overline{\phi_N(h)}$ is essentially self-adjoint on any core for H_N.*

Proof. The second inequality of Lemma 9 implies that we have:

$$\|\phi_N(h)\Phi\|\leqq\|h\|_{L^1}\|(a_NH_N+b_N\mathbb{1})^{1/2}\Phi\|,\qquad \forall\Phi\in D_N^1. \tag{10}$$

It follows that the domain $\mathrm{dom}(\overline{\phi_N(h)})$ of the closure $\overline{\phi_N(h)}$ of $\phi_N(h)$ contains the completion of D_N^1 for the norm $\Phi\mapsto\left(\left\|\left(H_N+\frac{b_N}{a_N}\mathbb{1}\right)^{1/2}\Phi\right\|^2+\|\Phi\|^2\right)^{1/2}$ which [in fact it is $\mathrm{dom}(H_N{}^{1/2})$] clearly contains the domain $\mathrm{dom}(H_N)$ of the positive self-adjoint operator H_N. So we have:

$$\mathrm{dom}(\overline{\phi_N(h)})\supset\mathrm{dom}(H_N), \tag{11}$$

and the same argument as above shows that for any core C for H_N, $\mathrm{dom}(\overline{\phi_N(h)\restriction C})$ contains $\mathrm{dom}(H_N)$ and therefore that we have $(\overline{\phi_N(h)\restriction C})=\overline{\phi_N(h)}$.

In order to achieve the proof of Lemma 10, it is sufficient to show that $\overline{\phi_N(h)}$ is self-adjoint. Let us first remark that the following identity holds on D_N^1

$$\phi_N(h')=i[H_N,\phi_N(h)],\quad \text{with}\quad h'(t,\boldsymbol{r})=\frac{\partial h}{\partial t}(t,\boldsymbol{r})\quad (\in\mathscr{D}(M)). \tag{12}$$

It follows that we have:

$$\pm i\{(H_N\Phi|\overline{\phi_N(h)}\Phi)-(\overline{\phi_N(h)}\Phi|H_N\Phi)\}=\pm(\Phi|\overline{\phi_N(h')}\Phi)\leqq\|h'\|_{L^1}(\Phi|(H_N+c_N\mathbb{1})\Phi),$$

for any $\Phi\in D_N^1$ (by Lemma 9) and therefore for any $\Phi\in\text{dom}(H_N)$ by the above argument. The self-adjointness of $\overline{\phi_N(h)}$ is then a consequence of the following theorem which may be found in the book of Faris [12] (Theorem 12-1, p. 79; see also in [17] the [Jaffe's] Theorem 2.2).

Theorem. *Let Q be a symmetric operator and H a positive self-adjoint operator. Assume that*

i) $\text{dom}(H)\subset\text{dom}(Q)$

and for some constant c and all Φ in $\text{dom}(H)$,

ii) $\pm i\{(H\Phi|Q\Phi)-(Q\Phi|H\Phi\}\leqq c(\Phi|H\Phi)$.

Then Q is essentially self-adjoint.

So applying this theorem with $Q=\phi_N(h)$, $H=H_N+c_N\mathbb{1}$ and $c=\|h'\|_{L^1}$ we obtain the desired result. □

This lemma implies in particular that $\phi(h)$ is essentially self-adjoint on $D_N^{(0)}$.

Let us define the domain D_N to be the linear hull of the vectors $\phi_N(h_1)\dots\phi_N(h_N)\Omega_N$ when $(h_1,\dots,h_n)$ runs over the finite families in $\mathscr{D}(M)$. Clearly $D_N\subset D_N^1$, $H_ND_N\subset D_N$, $\exp(itH_N)D_N\subset D_N$;$\forall t\in\mathbb{R}$.

There is a choice of the phase such that $\Omega_N(q_1,\dots q_N)$ is strictly positive, $\forall(q_1,\dots q_N)\in\mathbb{R}^N$; furthermore, $\Omega_N\in\mathscr{S}(\mathbb{R}^N)=D_N^1$ decreases faster (strictly faster if $\lambda_N>0$) than a gaussian at infinity. It follows that $\Omega_N^2d^Nq$ is the solution of a determined moment problem and therefore, the polynomials $P(q_1,\dots q_N)$ are dense in $L^2(\mathbb{R}^N,\Omega_N^2d^Nq)$ which implies that the functions $P(q_1,\dots q_N)\,\Omega_N(q_1,\dots,q_N)$ are dense in $\mathfrak{H}_N=L^2(\mathbb{R}^N)$. On the other hand, it is easily seen that the closure of D_N in $\mathfrak{H}_N$ contains these functions; so D_N is dense. Since D_N is dense, contained in $\text{dom}(H_N)$ and invariant by e^{itH_N}, we have the following lemma.

Lemma 11. *D_N is a core for H_N.*

Lemma 11 implies that $(\phi_N,\mathfrak{H}_N,\Omega_N)$ is an essentially self-adjoint scalar field (with our conventions), so we may apply to it the Theorem 3. But since we expect for $s=3$ some "wave function" renormalization, we shall allow a dependence on N of the normalization of the field operator. So we define the field A_N by:

$$A_N(h)=Z_N^{-1/2}[\phi_N(h)-(\Omega_N|\phi_N(h)\Omega_N)]\,, \tag{13}$$

for $h\in\mathscr{D}(M)$ and where Ω_N is the ground state of H_N; Z_N being a strictly positive constant which may be fixed, for instance by the following procedure. Let $A^{(0)}$ be the usual free field of mass m (hermitian scalar free field), let $f\in\mathscr{S}(\mathbb{R}^{s+1})$ be such that $A_N(f)\Omega_N\neq0$ and such that the support of its Fourier transform $\hat{f}$ interset the physical spectrum of $A^{(0)}$ only on the mass shell of $A^{(0)}$, then choose Z_N in such a way that $\|A_N(f)\Omega_N\|=\|A^{(0)}(f)\Omega^{(0)}\|$, where $\Omega^{(0)}$ is the usual vacuum in the Fock space of the free field of mass m ($A_N(t,\boldsymbol{r})$ being well defined on D_N^1 by $A_N(t,\boldsymbol{r})=Z_N^{-1/2}[\phi_N(t,\boldsymbol{r})-(\Omega_N|\phi_N(t,\boldsymbol{r})\Omega_N)]$). In any case (i.e. for any choice of $Z_N>0$) we may summarize the situation by the following statement.

Theorem 4. *The cut-off field* $\mathbf{A}_N = (A_N, \mathfrak{H}_N, \Omega_N)$ *satisfies the assumptions of Theorem* 3.

Therefore, the m-problem for ϕ_{Ω_N} is determined on the quasi-localizable C^*-algebra $\hat{\mathfrak{B}}(M)$. Let π_N be the corresponding unique representation of $\hat{\mathfrak{B}}(M)$ in $\mathfrak{H}_N$ for which $\pi_N(f(h)) = f(\overline{A_N(h)}) \in \mathscr{L}(\mathfrak{H}_N), \forall h = h^* \in \mathscr{D}(M)$ and $\forall f \in \mathscr{C}_{(0)}(\mathbb{R})$; *we define the state* ω_N *on* $\hat{\mathfrak{B}}(M)$ *associated to the above theory by:*

$$\omega_N(x) = (\Omega_N | \pi_N(x) \Omega_N), \qquad \forall x \in \hat{\mathfrak{B}}(M). \tag{14}$$

Clearly (by G.N.S. construction + Lemmas 1 and 2) the knowledge of ω_N is equivalent to the knowledge of the whole theory ($\mathfrak{H}_N, H_N, A_N$ etc. ...). What has been gained in this translation is that the space of states on $\hat{\mathfrak{B}}(M)$ equipped as usual with the weak topology is compact (remembering that this topology is reasonably physically relevant [19]). So from the sequence ω_N, one may extract a convergent subnet. Let $\mathfrak{S}$ be the set of the limits of these convergent subnet. The natural question is then the following one: Is there a choice of the sequence (λ_N, Z_N) for which $\mathfrak{S}$ contains at least an *interesting point*? Clearly an interesting point would be a state ω on $\hat{\mathfrak{B}}(M)$ such that, if $(\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega)$ is the associated G.N.S. triplet and $\mathscr{F}_b$ denote the set of the non-empty open bounded subsets of M, the family $(\pi_\omega(\mathfrak{B}(\mathcal{O}))'')_{\mathcal{O} \in \mathscr{F}_b}$ [7] of von Neumann algebras satisfies the assumptions of the theory of Araki and Haag [13, 14] adapted to the situation where one kind of neutral scalar particle of mass m is present with vacuum Ω_ω. Of course, for the trivial choice $\lambda_N = 0$ [which implies $\delta m_N = 0$ with our convention formula (7)] and $Z_N = 1\, \forall N$, the Wightman distributions of the cut-off fields converge to the Wightman distributions of the free scalar neutral field of mass m which is an essentially self-adjoint hermitian scalar field; so, by Theorem 3 and Lemma 8, ω_N converge weakly to the vacuum expectation values of the bounded observables of the free field. In the case $\lambda_N > 0 (\forall N)$, it will be very difficult (and this is out of the scope of this paper) to recover the locality and the Poincaré invariance when one removes the cut-off. However, since the cut-off fields are time-translation invariant and satisfy the spectrum condition in the time direction with a fixed gap m and a unique (Ω_N), one may expect that the same holds for the "limits" (in the above sense).

7. Conclusion

Suppose that ω is a state on $\hat{\mathfrak{B}}(M)$ which is obtained, by weak compactness argument, from states corresponding to cut-off field theories as in the previous section. If we try to interpret ω as a vacuum state of some "limit" theory, a first difficulty arises because the G.N.S. Hilbert space $\mathfrak{H}_\omega$ may not be separable.

In quantum field theory [20, 21], the separability of the Hilbert space is a consequence of the continuity properties of the corresponding *-representation of the tensor algebra over the space of the test functions (using either the separability or the nuclearity of this topological *-algebra). At this point, it is worth noticing that all the constructions and the results of this work are purely algebraic. Furthermore

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[7] $\mathfrak{B}(\mathcal{O})$ being defined as in Part I, Section 9a); $\mathfrak{B}(\mathcal{O})$ is a C^*-subalgebra of $\hat{\mathfrak{B}}(M), \forall \mathcal{O} \in \mathscr{F}_b$

Theorem 2 and Proposition 2 (above) show that the continuous positive linear forms on the tensor algebra over the test functions which correspond to bounded representations already separate the quasi-localizable C^*-algebra. So continuity will not reduce this C^*-algebra, and at the "C^*-algebraic level" one must use a non-separable C^*-algebra in order to deal with "sufficiently many" field theories. The only way to escape is to remark that since continuity selects a subspace of the algebraic dual of the space of test functions we may expect that correspondingly one can find solutions for the associated m-problems in a (separating) subspace of the topological dual space of the quasi-localizable C^*-algebra such that the quasi-localizable C^*-algebra be separable for the corresponding weak topology. It must be clear that similar considerations apply when we are interested in the m-problem for continuous strongly positive linear forms on locally convex *-algebras.

For instance if E is some locally convex space with topological dual E' and if ϕ is a strongly positive linear form on the symmetric tensor algebra $S(E)$ over E (="polynomials" on E'), then a solution of the m-problem for ϕ will not define a measure on E' but merely a measure on the algebraic dual space E^* of E [22].

One sees that, in order to develop a step further our non-commutative generalization of the moment problem, we have to generalize the part of that problem corresponding to infinite dimensional measures on topological vector spaces. This will be done in a forthcoming paper.

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Note. Soon after the publication of the Part I of this work [1], we realised that the condition of quasi-analicity of the vacuum discussed there had already been introduced by Gachok, V.P.: Nuovo Cimento **45** A, 158 (1966).

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