Communications in Mathematical Analysis
Volume 9, Number 1, pp. 84–92 (2010)
ISSN 1938-9787
www.commun-math-anal.org

ON A HILBERT-TYPE INEQUALITY WITH A HYPERGEOMETRIC FUNCTION

BING HE*
Department of Mathematics
Guangdong Education College
Guangzhou, 510275 P. R. China

BICHENG YANG†
Department of Mathematics
Guangdong Education College
Guangzhou, 510275 P. R. China

(Communicated by Themistocles Rassias)

Abstract

By applying the method of weight function and techniques from Real Analysis, a Hilbert-type inequality depending upon a multi-parameter with a best constant factor is studied. The best constant is formulated in terms of a hypergeometric function. Furthermore, the inverse inequality is studied.

AMS Subject Classification: 26D15.

Keywords: Hilbert's inequality, Weight coefficient, Hölder's inequality.

1 Introduction

If $a_n, b_n \geq 0, 0 < \sum\limits_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum\limits_{n=1}^{\infty} b_n^2 < \infty$, then(see [1])

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left\{\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2\right\}^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is well known as Hilbert's inequality. Soon after, inequality (1.1) had been generalized by Hardy-Riesz as(see [1]): If

*E-mail address: hebing@gdei.edu.cn
†E-mail address: bcyang@pub.guangzhou.gd.cn

$a_n, b_n \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{1/q}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (1.2) is named of Hardy-Hilbert's inequality (see [1]). It is important in analysis and its applications. It was studied extensively and refinements, generalizations and numerous variants appeared in the literature (see [1]- [5]). Under the same condition of (1.2), we obtained the Hardy-Hilbert's type inequality (see [1], Th. 341, Th. 342)

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m,n\}} < pq \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{1/q}; \tag{1.3}$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\log(m/n)}{m-n} a_m b_n < \pi^2 \csc^2 \frac{\pi}{p} \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{1/q}, \tag{1.4}$$

where the constant factors pq and $\pi^2 \csc^2 \frac{\pi}{p}$ are both the best possible.

In 2008, Yang (see [6]) gave a bilateral inequality as follows: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1, 0 < \lambda \leq 2, a, b, c \geq 0, a + bc > 0$, $a_n, b_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{2})-1} a_n^p < \infty, 0 < \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} b_n^q < \infty$. Then

$$H := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{a \max\{m^\lambda, n^\lambda\} + bm^\lambda + cn^\lambda}$$

$$< C_\lambda(a,b,c) \left\{ \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{2})-1} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} b_n^q \right\}^{1/q}, \tag{1.5}$$

where the constant factor $C_\lambda(a,b,c)$ is the best possible. In addition, for $0 < p < 1$, Yang got the reverse inequality as follows

$$H := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{a \max\{m^\lambda, n^\lambda\} + bm^\lambda + cn^\lambda}$$

$$> C_\lambda(a,b,c) \left\{ \sum_{n=1}^{\infty} [1 - \theta_\lambda(a,b,c,n)] n^{p(1-\frac{\lambda}{2})-1} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} b_n^q \right\}^{1/q}, \tag{1.6}$$

where $\theta_\lambda(a,b,c,m) := \frac{1}{C_1(a,b,c)} \int_0^{1/m^\lambda} \frac{1}{a+b+cu} u^{-1/2} \mathrm{d}u = O(\frac{1}{m^{\lambda/2}}) \in (0,1)$, and the constant factor $C_\lambda(a,b,c)$ is the best possible. During the recent years the reverse form of the Hardy-Hilbert's inequality has been studied by a number of mathematicians ([7]), ([8],[9]).

Very recently, Huang(see [9]) obtained the following inequality: If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda \leq 1, a_n, b_n \geq 0$ such that $0 < \sum_{n=1}^{\infty} n^{p-1} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{q-1} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\min\{m,n\})^\lambda}{m^\lambda + n^\lambda} a_m b_n < \frac{2}{\lambda} \ln 2 \left\{ \sum_{n=1}^{\infty} n^{p-1} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{q-1} b_n^q \right\}^{1/q}, \tag{1.7}$$

where the constant factor $\frac{2}{\lambda}\ln 2$ is the best possible.

In the above considerations we have focus our attention to the study of Hilbert's inequality with negative number kernel. In the following our goal is to investigate the real number kernel.

2 Some Lemmas

The hypergeometric function $F(\alpha,\beta,\gamma,z)$ is defined [10] by

$$F(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n}z^n, \tag{2.1}$$

where $(\alpha)_n=\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1), n\geq 1$ and $(\alpha)_0=1, \alpha\neq 0$.

The integral form of it is [10] (see also [11])

$$F(\alpha,\beta,\gamma,z)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\mathrm{d}t, \tag{2.2}$$

where $\mathrm{Re}(\gamma)>\mathrm{Re}(\beta)>0, |\arg(1-z)|<\pi$ and Γ is the gamma function. In particular, when $\alpha=1$ and $\gamma=\beta+1$, we get

$$\int_0^1\frac{t^{\beta-1}}{1-zt}\mathrm{d}t=\frac{1}{\beta}F(1,\beta,\beta+1,z). \tag{2.3}$$

Lemma 2.1. *Let $\alpha\in\mathbb{R}, \lambda>|\alpha|$ and $A>-1$. Define the weight function $\widetilde{\varphi}_\lambda(\alpha,x)$ and $\widetilde{\psi}_\lambda(\alpha,y)$ as*

$$\widetilde{\varphi}_\lambda(\alpha,x):=\int_0^\infty\frac{(\min\{x,y\})^\lambda}{x^\lambda+y^\lambda+A(\min\{x,y\})^\lambda}\cdot\frac{x^\alpha}{y^{1+\alpha}}\mathrm{d}y, x\in(0,\infty), \tag{2.4}$$

$$\widetilde{\psi}_\lambda(\alpha,y):=\int_0^\infty\frac{(\min\{x,y\})^\lambda}{x^\lambda+y^\lambda+A(\min\{x,y\})^\lambda}\cdot\frac{y^{-\alpha}}{x^{1-\alpha}}\mathrm{d}x, y\in(0,\infty), \tag{2.5}$$

then we obtain

$$\widetilde{\varphi}_\lambda(\alpha,x)=\widetilde{\psi}_\lambda(\alpha,y)=C_\lambda(\alpha,A), \tag{2.6}$$

where

$$C_\lambda(\alpha,A)=\frac{F(1,1+\frac{\alpha}{\lambda},2+\frac{\alpha}{\lambda},-1-A)}{\lambda+\alpha}+\frac{F(1,1-\frac{\alpha}{\lambda},2-\frac{\alpha}{\lambda},-1-A)}{\lambda-\alpha}.$$

Proof. Setting $t=y/x$, we obtain

$$\begin{aligned}\widetilde{\varphi}_\lambda(\alpha,x) &= \int_0^\infty\frac{(\min\{x,y\})^\lambda}{x^\lambda+y^\lambda+A(\min\{x,y\})^\lambda}\cdot\frac{x^\alpha}{y^{1+\alpha}}\mathrm{d}y\\ &= \int_0^\infty\frac{(\min\{1,t\})^\lambda}{1+t^\lambda+A(\min\{1,t\})^\lambda}\cdot t^{-\alpha-1}\mathrm{d}t\\ &= \int_0^1\frac{t^{\lambda-\alpha-1}}{1+(1+A)t^\lambda}\mathrm{d}t+\int_1^\infty\frac{t^{-\alpha-1}}{1+A+t^\lambda}\mathrm{d}t\\ &= \int_0^1\frac{t^{\lambda-\alpha-1}}{1+(1+A)t^\lambda}\mathrm{d}t+\int_0^1\frac{t^{\lambda+\alpha-1}}{1+(1+A)t^\lambda}\mathrm{d}t.\end{aligned}$$

Setting $x = t^\lambda$ for the above equality and in view of (2.3), we get

$$\begin{aligned}\widetilde{\varphi}_\lambda(\alpha,x) &= \int_0^1 \frac{x^{-\alpha/\lambda}}{1+(1+A)x}\mathrm{d}x + \int_0^1 \frac{x^{\alpha/\lambda}}{1+(1+A)x}\mathrm{d}x = C_\lambda(\alpha,A) \\ &= \frac{F(1,1+\frac{\alpha}{\lambda},2+\frac{\alpha}{\lambda},-1-A)}{\lambda+\alpha} + \frac{F(1,1-\frac{\alpha}{\lambda},2-\frac{\alpha}{\lambda},-1-A)}{\lambda-\alpha}.\end{aligned}$$

Similarly, we can calculate that

$$\widetilde{\psi}_\lambda(\alpha,y) = C_\lambda(\alpha,A).$$

The Lemma is proved. □

Lemma 2.2. *Let* $\alpha \in \mathbb{R}, |\alpha| < \lambda \le 1+|\alpha|$ *and* $A > -1$. *Define* $\varphi_\lambda(\alpha,m)$ *and* $\psi_\lambda(\alpha,n)$ *as*

$$\varphi_\lambda(\alpha,m) := \sum_{n=1}^{\infty} \frac{(\min\{m,n\})^\lambda}{m^\lambda+n^\lambda+A(\min\{m,n\})^\lambda}\cdot\frac{m^\alpha}{n^{1+\alpha}}, m\in\mathbb{N}, \tag{2.7}$$

$$\psi_\lambda(\alpha,n) := \sum_{m=1}^{\infty} \frac{(\min\{m,n\})^\lambda}{m^\lambda+n^\lambda+A(\min\{m,n\})^\lambda}\cdot\frac{n^{-\alpha}}{m^{1-\alpha}}, n\in\mathbb{N}, \tag{2.8}$$

then

$$C_\lambda(\alpha,A)\,([1-\theta_\lambda(\alpha,m,A)] < \varphi_\lambda(\alpha,m) < C_\lambda(\alpha,A), \tag{2.9}$$

$$\psi_\lambda(\alpha,n) < C_\lambda(\alpha,A), \tag{2.10}$$

where

$$0 < \theta_\lambda(\alpha,m,A) := \frac{1}{C_\lambda(\alpha,A)}\int_0^{\frac{1}{m}} \frac{t^{\lambda-\alpha-1}}{1+(1+A)t^\lambda}\mathrm{d}t = O\left(\frac{1}{m^{\lambda-\alpha}}\right) \in (0,1), m\to\infty.$$

Proof. On one hand, setting $t = y/m$, by monotonicity and in view of (2.6), we obtain

$$\varphi_\lambda(\alpha,m) < \widetilde{\varphi}_\lambda(\alpha,m) = \int_0^\infty \frac{(\min\{m,y\})^\lambda}{m^\lambda+y^\lambda+A(\min\{m,y\})^\lambda}\cdot\frac{m^\alpha}{y^{1+\alpha}}\mathrm{d}y = C_\lambda(\alpha,A).$$

Similarly, we obtain

$$\psi_\lambda(\alpha,n) < C_\lambda(\alpha,A),$$

thus (2.10) is valid.

On the other hand, setting $t = y/m$, we get

$$\begin{aligned}\varphi_\lambda(\alpha,m) &> \int_1^\infty \frac{(\min\{m,y\})^\lambda}{m^\lambda+y^\lambda+A(\min\{m,y\})^\lambda}\cdot\frac{m^\alpha}{y^{1+\alpha}}\mathrm{d}y \\ &= \int_{\frac{1}{m}}^\infty \frac{(\min\{1,t\})^\lambda}{1+t^\lambda+A(\min\{1,t\})^\lambda}\cdot t^{-\alpha-1}\mathrm{d}t \\ &= C_\lambda(\alpha,A) - \int_0^{\frac{1}{m}} \frac{t^{\lambda-\alpha-1}}{1+(1+A)t^\lambda}\mathrm{d}t \\ &= C_\lambda(\alpha,A)\left[1-\frac{1}{C_\lambda(\alpha,A)}\int_0^{\frac{1}{m}}\frac{t^{\lambda-\alpha-1}}{1+(1+A)t^\lambda}\mathrm{d}t\right] \\ &= C_\lambda(\alpha,A)[1-\theta_\lambda(\alpha,m,A)].\end{aligned}$$

Obviously, $0<\theta_\lambda(\alpha,m,A):=\frac{1}{C_\lambda(\alpha,A)}\int_0^{\frac{1}{m}}\frac{t^{\lambda-\alpha-1}}{1+(1+A)t^\lambda}\mathrm{d}t<1$. Since

$$0<\theta_\lambda(\alpha,m,A)<\frac{1}{C_\lambda(\alpha,A)}\int_0^{\frac{1}{m}}t^{\lambda-\alpha-1}=\frac{1}{(\lambda-\alpha)C_\lambda(\alpha,A)}\cdot\frac{1}{m^{\lambda-\alpha}},$$

then $\theta_\lambda(\alpha,m,A)=O(\frac{1}{m^{\lambda-\alpha}})$. Hence (2.9) is valid. The Lemma is proved. □

Lemma 2.3. *If $p>0, p\neq 1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha\in\mathbb{R}, |\alpha|<\lambda\leq 1+|\alpha|$ and $A>-1$, setting*

$$J(\varepsilon)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^\lambda}{m^\lambda+n^\lambda+A(\min\{m,n\})^\lambda}\cdot m^{\alpha-1-\frac{\varepsilon}{p}}n^{-\alpha-1-\frac{\varepsilon}{q}}, \tag{2.11}$$

where ε is sufficiently small and positive, then we obtain

$$[C_\lambda(\alpha,A)-o(1)]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}<J(\varepsilon)<[C_\lambda(\alpha,A)+\widetilde{o}(1)]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}, \varepsilon\to 0^+. \tag{2.12}$$

Proof. Setting $t=\frac{x}{n}$ in the following, in view of Lemma 2.2, we get

$$\begin{aligned}
J(\varepsilon) &< \sum_{n=1}^{\infty}n^{-\alpha-1-\frac{\varepsilon}{q}}\left(\int_0^\infty\frac{(\min\{x,n\})^\lambda}{x^\lambda+n^\lambda+A(\min\{x,n\})^\lambda}\cdot x^{\alpha-1-\frac{\varepsilon}{p}}\mathrm{d}x\right)\\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left(\int_0^\infty\frac{(\min\{t,1\})^\lambda}{t^\lambda+1+A(\min\{t,1\})^\lambda}\cdot t^{\alpha-1-\frac{\varepsilon}{p}}\mathrm{d}t\right)\\
&= [C_\lambda(\alpha,A)+\widetilde{o}(1)]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\quad(\varepsilon\to 0^+);
\end{aligned}$$

$$\begin{aligned}
J(\varepsilon) &> \sum_{n=1}^{\infty}n^{-\alpha-1-\frac{\varepsilon}{q}}\left(\int_1^\infty\frac{(\min\{x,n\})^\lambda}{x^\lambda+n^\lambda+A(\min\{x,n\})^\lambda}\cdot x^{\alpha-1-\frac{\varepsilon}{p}}\mathrm{d}x\right)\\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left(\int_{\frac{1}{n}}^\infty\frac{(\min\{t,1\})^\lambda}{t^\lambda+1+A(\min\{t,1\})^\lambda}\cdot t^{\alpha-1-\frac{\varepsilon}{p}}\mathrm{d}t\right)\\
&> \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left(\int_0^\infty\frac{(\min\{t,1\})^\lambda}{t^\lambda+1+A(\min\{t,1\})^\lambda}\cdot t^{\alpha-1-\frac{\varepsilon}{p}}\mathrm{d}t\right)\\
&\quad -\sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^{\frac{1}{n}}t^{\lambda+\alpha-1-\frac{\varepsilon}{p}}\mathrm{d}t\right)\\
&= [C_\lambda(\alpha,A)+\widetilde{o}(1)]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}-\sum_{n=1}^{\infty}\left(\frac{1}{n}\int_0^{\frac{1}{n}}t^{\lambda+\alpha-1-\frac{\varepsilon}{p}}\mathrm{d}t\right)\quad(\varepsilon\to 0^+).
\end{aligned}$$

Since

$$\begin{aligned}
&0<\sum_{n=1}^{\infty}\left(\frac{1}{n}\int_0^{\frac{1}{n}}t^{\lambda+\alpha-1-\frac{\varepsilon}{p}}\mathrm{d}t\right)=\sum_{n=1}^{\infty}\left[\frac{1}{n}\cdot\frac{1}{(\lambda+\alpha-\frac{\varepsilon}{p})}\cdot\frac{1}{n^{\lambda+\alpha-\frac{\varepsilon}{p}}}\right]\\
&= \sum_{n=1}^{\infty}\left[\frac{1}{(\lambda+\alpha-\frac{\varepsilon}{p})}\cdot\frac{1}{n^{\lambda+\alpha+1-\frac{\varepsilon}{p}}}\right]=\sum_{n=1}^{\infty}O\left(\frac{1}{n^{\lambda+\alpha+1-\frac{\varepsilon}{p}}}\right).
\end{aligned}$$

In view of the above inequalities, we obtain

$$\begin{aligned} J(\varepsilon) &> [C_\lambda(\alpha,A)+\widetilde{o}(1)]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}-\sum_{n=1}^{\infty}O\left(\frac{1}{n^{\lambda+\alpha+1-\frac{\varepsilon}{p}}}\right)\\ &= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left[(C_\lambda(\alpha,A)+\widetilde{o}(1))-\sum_{n=1}^{\infty}O\left(\frac{1}{n^{\lambda+\alpha+1-\frac{\varepsilon}{p}}}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right)^{-1}\right]\\ &= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}[C_\lambda(\alpha,A)-o(1)]\ (\varepsilon\to 0^+). \end{aligned}$$

The Lemma is proved. □

3 Main Results

Theorem 3.1. *If $p>1, \frac{1}{p}+\frac{1}{q}=1$, $\alpha\in\mathbb{R}, |\alpha|<\lambda\leq 1+|\alpha|$, $A>-1$, $a_n,b_n\geq 0$ such that $0<\sum_{n=1}^{\infty}n^{p(1-\alpha)-1}a_n^p<\infty$ and $0<\sum_{n=1}^{\infty}n^{q(1+\alpha)-1}b_n^q<\infty$. Then we obtain the following inequality*

$$\begin{aligned} I &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^\lambda}{m^\lambda+n^\lambda+A(\min\{m,n\})^\lambda}a_mb_n\\ &< C_\lambda(\alpha,A)\left\{\sum_{n=1}^{\infty}n^{p(1-\alpha)-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q(1+\alpha)-1}b_n^q\right\}^{1/q}, \end{aligned}\tag{3.1}$$

where the constant factor $C_\lambda(\alpha,A)$ is the best possible.

Proof. By Hölder's inequality with weight[12], we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^\lambda}{m^\lambda+n^\lambda+A(\min\{m,n\})^\lambda}a_mb_n\\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^\lambda}{m^\lambda+n^\lambda+A(\min\{m,n\})^\lambda}\left[\frac{m^{(1-\alpha)/q}}{n^{(1+\alpha)/p}}a_m\right]\left[\frac{n^{(1+\alpha)/p}}{m^{(1-\alpha)/q}}b_n\right]\\ &\leq \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^\lambda}{m^\lambda+n^\lambda+A(\min\{m,n\})^\lambda}\frac{m^{(1-\alpha)(p-1)}}{n^{1+\alpha}}a_m^p\right\}^{1/p}\\ &\quad\times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^\lambda}{m^\lambda+n^\lambda+A(\min\{m,n\})^\lambda}\frac{n^{(1+\alpha)(q-1)}}{m^{1-\alpha}}b_n^q\right\}^{1/q}\\ &= \left\{\sum_{m=1}^{\infty}\varphi_\lambda(\alpha,m)m^{p(1-\alpha)-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\psi_\lambda(\alpha,n)n^{q(1+\alpha)-1}b_n^q\right\}^{\frac{1}{q}}. \end{aligned}$$

In view of (2.9) and (2.10), we have (3.1).

Let ε be positive and sufficiently small, setting $\widetilde{a}_m=m^{\alpha-1-\frac{\varepsilon}{p}}$, $\widetilde{b}_n=n^{-\alpha-1-\frac{\varepsilon}{q}}$ $(m,n\in\mathbb{N})$, by (2.11), we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^\lambda}{m^\lambda+n^\lambda+A(\min\{m,n\})^\lambda}\cdot m^{\alpha-1-\frac{\varepsilon}{p}}n^{-\alpha-1-\frac{\varepsilon}{q}}=J(\varepsilon).$$

Assuming that there exists a positive number k with $0 < k \le C_\lambda(\alpha, A)$, such that (3.1) is still correct by changing $C_\lambda(\alpha, A)$ to k, then, in particular, by (2.12), we have

$$[C_\lambda(\alpha,A) - o(1)] \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} < J(\varepsilon)$$

$$< k \left\{ \sum_{n=1}^{\infty} n^{p(1-\alpha)-1} \widetilde{a}_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{q(1+\alpha)-1} \widetilde{b}_n^q \right\}^{1/q} = k \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}.$$

It follows that $C_\lambda(\alpha,A) - o(1) < k$, so $C_\lambda(\alpha,A) \le k (\varepsilon \to 0^+)$. Hence the constant factor $k = C_\lambda(\alpha,A)$ in (3.1) is the best possible. This completes the proof. □

Remark For $A = 0$ and $\alpha = 0$, (3.1) turns into (1.7). Hence (3.1) is a generalization of (1.7).

Theorem 3.2. *If* $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in \mathbb{R}, |\alpha| < \lambda \le 1 + |\alpha|$, $A > -1$, $a_n, b_n \ge 0$ *such that* $0 < \sum_{n=1}^{\infty} n^{p(1-\alpha)-1} a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{q(1+\alpha)-1} b_n^q < \infty$, *then we obtain the following inverse inequality*

$$I = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\min\{m,n\})^\lambda}{m^\lambda + n^\lambda + A(\min\{m,n\})^\lambda} a_m b_n$$

$$> C_\lambda(\alpha,A) \left\{ \sum_{n=1}^{\infty} [1 - \theta_\lambda(\alpha,n,A)] n^{p(1-\alpha)-1} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{q(1+\alpha)-1} b_n^q \right\}^{1/q}, \tag{3.2}$$

where $0 < \theta_\lambda(\alpha,m,A) := \frac{1}{C_\lambda(\alpha,A)} \int_0^{\frac{1}{m}} \frac{t^{\lambda-\alpha-1}}{1+(1+A)t^\lambda} \mathrm{d}t = O(\frac{1}{m^{\lambda-\alpha}}) \in (0,1), m \to \infty$, *and the constant factor* $C_\lambda(\alpha,A)$ *is the best possible.*

Proof. By the reverse Hölder's inequality with weight[12], in view of (2.7) and (2.8), we obtain

$$\begin{aligned} I &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\min\{m,n\})^\lambda}{m^\lambda + n^\lambda + A(\min\{m,n\})^\lambda} \left[\frac{m^{(1-\alpha)/q}}{n^{(1+\alpha)/p}} a_m \right] \left[\frac{n^{(1+\alpha)/p}}{m^{(1-\alpha)/q}} b_n \right] \\ &\ge \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\min\{m,n\})^\lambda}{m^\lambda + n^\lambda + A(\min\{m,n\})^\lambda} \frac{m^{(1-\alpha)(p-1)}}{n^{(1+\alpha)}} a_m^p \right\}^{1/p} \\ &\quad \times \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\min\{m,n\})^\lambda}{m^\lambda + n^\lambda + A(\min\{m,n\})^\lambda} \frac{n^{(1+\alpha)(q-1)}}{m^{(1-\alpha)}} b_n^q \right\}^{1/q} \\ &= \left\{ \sum_{m=1}^{\infty} \varphi_\lambda(\alpha,m) m^{p(1-\alpha)-1} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \psi_\lambda(\alpha,n) n^{q(1+\alpha)-1} b_n^q \right\}^{\frac{1}{q}}. \end{aligned}$$

By (2.9) and (2.10), in view of $q < 0$, we have (3.2).

Let ε be positive and small enough , setting $\widetilde{a}_m = m^{\alpha-1-\frac{\varepsilon}{p}}$, $\widetilde{b}_n = n^{-\alpha-1-\frac{\varepsilon}{q}}$ $(m,n \in \mathbb{N})$, by (2.11), we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\min\{m,n\})^\lambda}{m^\lambda + n^\lambda + A(\min\{m,n\})^\lambda} \cdot m^{\alpha-1-\frac{\varepsilon}{p}} n^{-\alpha-1-\frac{\varepsilon}{q}} = J(\varepsilon).$$

Assuming that there exists a positive number k with $k \geq C_\lambda(\alpha,A)$, such that (3.2) is still correct by changing $C_\lambda(\alpha,A)$ to k, then, in particular, by (2.12), we have

$$(C_\lambda(\alpha,A)+\widetilde{o}(1))\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} > J(\varepsilon)$$

$$> k\left\{\sum_{n=1}^{\infty}[1-\theta_\lambda(\alpha,n,A)]n^{p(1-\alpha)-1}\widetilde{a}_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q(1+\alpha)-1}\widetilde{b}_n^q\right\}^{1/q}$$

$$= k\left\{\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}-\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^{\lambda-\alpha}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p}\left\{\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right\}^{1/q}$$

$$= k\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left\{1-\left(\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right)^{-1}\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^{\lambda-\alpha}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p}.$$

It follows that

$$C_\lambda(\alpha,A)+\widetilde{o}(1) > k\left\{1-\left(\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right)^{-1}\sum_{n=1}^{\infty}\left[O\left(\frac{1}{n^{\lambda-\alpha}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p},$$

and then $C_\lambda(\alpha,A) \geq k(\varepsilon \to 0^+)$. Thus the constant factor $k = C_\lambda(\alpha,A)$ in (3.2) is the best possible. The theorem is proved. □

Acknowledgments

The work was partially supported by the Emphases Natural Science Foundation of Guangdong Institution of Higher Learning, College and University (No. 05Z026). The authors thank the referees for their careful reading of the manuscript and insightful comments.

References

[1] G. H. Hardy, J. E. Littlewood and G. Polya, Inequalities. Cambridge University Press, 1934.

[2] D. S. Mitrinovic, J. E. Pecaric and A. M. Fink, Inequalities involving functions and their integrals and derivatives. Kluwer Academic, Boston, 1991.

[3] M. Gao, On Hilbert's inequality and its applications. *J. Math. Anal. Appl.* **212** (1997), pp 316-323.

[4] B. G. Pachpatte, Inequalities similar to the integral analogue of Hilbert's inequality. *Tamkang J. Math.* **30**(1999), pp 139-146.

[5] B. Yang, The Norm of Operator and Hilbert-type Inequalities. Science Press, Beijing, 2009.

[6] B. Yang, On a relation between Hilberts inequality and a Hilbert-type inequality. *Appl. Math. Lett.*, **21**(5)(2008), pp 483-488.

[7] W. Zhong, A reverse Hilbert's type integral inequality. *Internat. J. Pure Appl. Math.* **36**(3)(2007), pp 353-360.

[8] C. Zhao and L. Debnath, Some new inverse type Hilbert integral inequalities. *J. Math. Anal. Appl.*, **262**(1)(2001), pp 411-418.

[9] Q. Huang, A Hilbert-type Series Inequality with the Homogeneous Kernel of 0-degree. *J. Guangdong Edu. Inst.*, **29**(5)(2009), pp 20-23.

[10] E.D. Rainville, Special Functions. Chelsea Publishing Co., Bronx, N,Y., 1971.

[11] Z. Wang and D. Guo, Introduction to Special Functions. Science Press, Beijing, 1979.

[12] J. Kuang, Applied Inequalities, Shangdong Science Press, Jinan, 2004.