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ON NORMALIZERS OF C^*-SUBALGEBRAS IN THE CUNTZ ALGEBRA $\mathcal{O}_n$. II

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Abstract. We investigate subalgebras A of the Cuntz algebra $\mathcal{O}_n$ that arise as finite direct sums of corners of the UHF-subalgebra $\mathcal{F}_n$. For such an A, we completely determine its normalizer group inside $\mathcal{O}_n$.

1. Introduction.

This note is a continuation of the investigations of C^*-subalgebras of the Cuntz algebra $\mathcal{O}_n$ carried out by the first named author in [4]. The main results therein pertain C^*-subalgebras A of the core UHF-subalgebra $\mathcal{F}_n$ of $\mathcal{O}_n$ with finite-dimensional relative commutant $A' \cap \mathcal{F}_n$. In particular, [4, Theorem 1.2] says that for such an A, the index of the subgroup $\{\mathrm{Ad}\, u|_A : u \in \mathcal{N}_{\mathcal{F}_n}(A)\}$ in $\{\mathrm{Ad}\, W|_A : W \in \mathcal{N}_{\mathcal{O}_n}(A)\}$ is finite. The main purpose of the present note is to completely determine the structure of normalizer $\mathcal{N}_{\mathcal{O}_n}(A)$ in the case

$$A = \bigoplus_{j=1}^{k} e_j \mathcal{F}_n e_j, \tag{1.1}$$

where $e_1, \ldots, e_k$ are projections in $\mathcal{F}_n$ such that $\sum_{j=1}^k e_j = 1$. The interesting and non-trivial aspects of our analysis stem from the fact, observed already in [4, Example 1.18], that $\mathcal{N}_{\mathcal{O}_n}(A)$ is not contained in $\mathcal{F}_n$ in general.

In addition to its intrinsic interest, our work is motivated by its close relation to index theory in the context of endomorphisms of the Cuntz algebras, e.g. see

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[6, 5, 1]. In a more recent paper on this subject, [2], endomorphisms of $\mathcal{O}_n$ globally preserving $\mathcal{F}_n$ are investigated, and we hope that the results of the present paper may help shed light on some of the outstanding questions raised therein.

Notation. For an integer $n \geq 2$, $\mathcal{O}_n$ is the C^*-algebra generated by isometries $S_1, \ldots, S_n$ such that $\sum_{i=1}^n S_i S_i^* = 1$, [3]. If $\mu = \mu_1\mu_2 \cdots \mu_k$ is a word on alphabet $\{1, \ldots, n\}$ then we denote $S_\mu = S_{\mu_1} S_{\mu_2} \cdots S_{\mu_k}$, an isometry in $\mathcal{O}_n$. The range projections $P_\mu := S_\mu S_\mu^*$ corresponding to all words generate a MASA $\mathcal{D}_n$. For a word $\mu = \mu_1 \cdots \mu_k$ we denote by $|\mu| = k$ its length. Also, we use symbol $\prec$ to denote the lexicographic order on words.

The circle group $U(1)$ acts on $\mathcal{O}_n$ by $\gamma_z(S_i) = zS_i$. The fixed-point algebra for this action, denoted $\mathcal{F}_n$, is a UHF-algebra of type n^∞. Averaging γ yields a faithful conditional expectation $E : \mathcal{O}_n \to \mathcal{F}_n$. We denote by $\tau : \mathcal{F}_n \to \mathbb{C}$ the unique normalized trace on $\mathcal{F}_n$. We also let $\varphi : \mathcal{O}_n \to \mathcal{O}_n$ to be the canonical shift endomorphism, that is

$$\varphi(x) = \sum_{i=1}^n S_i x S_i^*. \tag{1.2}$$

For each $x \in \mathcal{O}_n$ and each generator S_i we have $S_i x = \varphi(x) S_i$.

If B is a unital C^*-algebra then $\mathcal{U}(B)$ denotes the group of its unitary elements. If A is a C^*-subalgebra of B then $\mathcal{N}_B(A) := \{u \in \mathcal{U}(B) : uAu^* = A\}$ is the normalizer of A in B.

2. The main results.

The following lemma and its proof are motivated by Examples 1.17 and 1.18 of [4]. It constitutes a technical basis for our further considerations. In the proof of the lemma we need the following simple fact: for every projection $e \in \mathcal{F}_n$ we have $(e\mathcal{F}_n e)' \cap e\mathcal{O}_n e = \mathbb{C}e$. For otherwise, there are non-zero elements a and b in this relative commutant such that $ab = 0$ and $a, b \geq 0$. Now, $E(a)$ and $E(b)$ are non-zero, positive elements in the center of the simple algebra $e\mathcal{F}_n e$. Thus, there are positive scalars λ and μ such that $E(a) = \lambda e$ and $E(b) = \mu e$. Now, take a finite sum $A = \sum z_j S_{\alpha_j} S_{\beta_j}^*$ (with z_j scalars) in $e\mathcal{O}_n e$ such that $||a - A|| < \lambda/2$. By [3, Lemma 1.8], there exists a non-zero projection f in $e\mathcal{F}_n e$ such that $fAf = fE(A)f$. We have $||af - \lambda f|| \leq ||af - fAf|| + ||fE(A)f - \lambda f|| < \lambda$. Thus af is invertible in $f\mathcal{O}_n f$. Similarly, we can find a non-zero projection g in $e\mathcal{F}_n e$ such that bg is invertible in $g\mathcal{F}_n g$. Now, take a partial isometry v in $e\mathcal{F}_n e$ such that $v^*v \leq g$ and $vv^* \leq f$. Then $0 = vab = avb \neq 0$, a contradiction which proves the claim.

Lemma 2.1. *Let e, f be non-zero projections in $\mathcal{F}_n$, and let $U \in \mathcal{U}(\mathcal{O}_n)$ be such that $Ue\mathcal{F}_n eU^* = f\mathcal{F}_n f$. Then there exists an integer m such that*

$$\frac{\tau(f)}{\tau(e)} = n^m.$$

Proof. For each $z \in U(1)$ and $x \in \mathcal{F}_n$, we have $\gamma_z(UexeU^*) = UexeU^*$. Hence $exeU^*\gamma_z(U) = U^*\gamma_z(U)exe$ and thus $U^*\gamma_z(U)e$ belongs to $(e\mathcal{F}_n e)' \cap e\mathcal{O}_n e$. Since this relative commutant is trivial, for each $z \in U(1)$ there exists a scalar $t(z)$ such that $\gamma_z(Ue) = t(z)Ue$. It follows that the mapping $t : U(1) \to \mathbb{C}$ is a continuous character and consequently there exists an $m \in \mathbb{Z}$ such that $t(z) = z^m$. We consider the following three cases, depending on the sign of m.

(i) If $m = 0$ then $Ue \in \mathcal{F}_n$ and hence $\tau(f) = \tau((Ue)(Ue)^*) = \tau((Ue)^*(Ue)) = \tau(e)$.

(ii) If $m > 0$ then set $V := UeS_1^{*m}$. Since V belongs to $\mathcal{F}_n$, we have $\tau(f) = \tau(VV^*) = \tau(V^*V) = \tau(S_1^m e S_1^{*m}) = \tau(\varphi^m(e)S_1^m S_1^{*m}) = \tau(e)/n^m$.

(iii) If $m < 0$ then set $V := S_1^{-m}Ue$. Again $V \in \mathcal{F}_n$ and thus $\tau(e) = \tau(V^*V) = \tau(VV^*) = \tau(S_1^{-m} f S_1^{*-m}) = \tau(f)/n^{-m}$. □

The following lemma is quite obvious but we give details since it allows us to reduce investigations of subalgebras of the general form (1.1) to some special cases with conveniently chosen projections e_j.

Lemma 2.2. *Let $e_1, \ldots, e_k$ and $f_1, \ldots, f_k$ be projections in $\mathcal{F}_n$ such that $\sum_{j=1}^k e_j = 1 = \sum_{j=1}^k f_j$ and $\tau(e_j) = \tau(f_j)$ for all j. Let $A = \bigoplus_{j=1}^k e_j \mathcal{F}_n e_j$ and $B = \bigoplus_{j=1}^k f_j \mathcal{F}_n f_j$. Then there exists a $u \in \mathcal{U}(\mathcal{F}_n)$ such that $uAu^* = B$. Hence we have $\mathcal{N}_{\mathcal{O}_n}(A) \cong \mathcal{N}_{\mathcal{O}_n}(B)$.*

Proof. For each $j = 1, \ldots, k$ there exists a partial isometry $v_j \in \mathcal{F}_n$ such that $v_j^* v_j = e_j$ and $v_j v_j^* = f_j$. Set $u := \sum_{j=1}^k v_j$. Then $ue_j \mathcal{F}_n e_j u^* = f_j \mathcal{F}_n f_j$ for each j, and the conclusion follows. □

In view of Lemma 2.2, it suffices to consider those subalgebras A of the form (1.1) that all projections e_j belong to the diagonal MASA $\mathcal{D}_n$. Each projection in $\mathcal{D}_n$ is a finite sum of projections P_μ for some words μ. Before treating the general case, we note the following slight generalization of [4, Example 1.18].

Example 2.3. Let $\mu_1, \ldots, \mu_k$ be words such that $\sum_{j=1}^k P_{\mu_j} = 1$. Put $A = \bigoplus_{j=1}^k P_{\mu_j} \mathcal{F}_n P_{\mu_j}$. Then there is a natural isomorphism

$$\mathcal{N}_{\mathcal{O}_n}(A) \cong \mathcal{U}(A) \rtimes \mathcal{S}_k,$$

where $\mathcal{S}_k$ is the symmetric group on k letters. Indeed, for each permutation $\sigma \in \mathcal{S}_k$ set $U_\sigma := \sum_{j=1}^k S_{\mu_{\sigma(j)}} S_{\mu_j}^*$. It is easy to see that each U_σ is unitary normalizing A, and that they form a group acting on A by permuting the direct summands $P_{\mu_j} \mathcal{F}_n P_{\mu_j}$. Thus we have an inclusion $\mathcal{U}(A) \rtimes \mathcal{S}_k \subseteq \mathcal{N}_{\mathcal{O}_n}(A)$. For the reverse inclusion, take a $V \in \mathcal{N}_{\mathcal{O}_n}(A)$. Considering $\mathrm{Ad}\, V$ action, we see that there exists a $\sigma \in \mathcal{S}_k$ such that VU_σ^* acts trivially on the center of A. Since $\mathcal{N}_{P_{\mu_j}\mathcal{O}_n P_{\mu_j}}(P_{\mu_j} \mathcal{F}_n P_{\mu_j}) = \mathcal{U}(P_{\mu_j} \mathcal{F}_n P_{\mu_j})$ ([4] Lemma 1.15), for each j we have $VU_\sigma^* P_{\mu_j} \in \mathcal{U}(P_{\mu_j} \mathcal{F}_n P_{\mu_j})$, and the claim easily follows. □

Now, we consider the general case of a C^*-subalgebra A of the form (1.1). Define an equivalence relation $\sim$ on the set $\{e_1, \ldots, e_k\}$ by

$$e_i \sim e_j \Leftrightarrow \frac{\tau(e_i)}{\tau(e_j)} \in n^{\mathbb{Z}}. \tag{2.1}$$

We denote by $\mathcal{S}_\sim$ the subgroup of the permutation group of $\{e_1, \ldots, e_k\}$ consisting of those permutations which leave each of the equivalence classes of $\sim$ globally invariant.

After this preparation, we are ready to prove our main result.

Theorem 2.4. *Let $e_1, \ldots, e_k$ be non-zero projections in $\mathcal{F}_n$ such that $\sum_{j=1}^k e_j = 1$, and let $A = \bigoplus_{j=1}^k e_j \mathcal{F}_n e_j$. Let $\mathcal{S}_\sim$ be the corresponding subgroup of the permutation group of $\{e_1, \ldots, e_k\}$. Then there exists a natural group isomorphism*

$$\mathcal{N}_{\mathcal{O}_n}(A) \cong \mathcal{U}(A) \rtimes \mathcal{S}_\sim.$$

Proof. By Lemma 2.2, we may assume that each projection e_j belongs to the diagonal MASA $\mathcal{D}_n$. Thus, there exist words $\mu_1, \mu_2, \ldots, \mu_N$, all of the same length and such that $\sum_{j=1}^N P_{\mu_j} = 1$, and there exist positive integers $m_1, m_2, \ldots, m_k$ such that $\sum_{j=1}^k m_j = N$ and $e_j = \sum_{i=m_{j-1}+1}^{m_j} P_{\mu_i}$ (here we put $m_0 = 0$) for each j. Relabelling, if necessary, we may assume that $\mu_{i_1} \prec \mu_{i_2}$ whenever $m_{j-1} + 1 \leq i_1 \leq i_2 \leq m_j$.

Now, let $\sigma \in \mathcal{S}_\sim$. Take a $j \in \{1, \ldots, k\}$ and let $\sigma(e_j) = e_h$. There is an $m \in \mathbb{Z}$ such that $m_j - m_{j-1} = (m_h - m_{h-1})n^m$. Suppose $m \geq 0$ (the case $m \leq 0$ being treated analogously). We note that $e_h = \sum_{i=m_{h-1}+1}^{m_h} \sum_{|\nu|=m} P_{\mu_i \nu}$. There is a unique $\prec$ order-preserving bijection

$$\psi : \{m_{h-1} + 1, \ldots, m_h\} \times \{\nu : |\nu| = m\} \to \{m_{j-1} + 1, \ldots, m_j\}, \tag{2.2}$$

that is,

$$\mu_{i_1}\nu_1 \prec \mu_{i_2}\nu_2 \ \Rightarrow\ \mu_{\psi(i_1,\nu_1)} \prec \mu_{\psi(i_2,\nu_2)}. \tag{2.3}$$

We set

$$u_j := \sum_{i=m_{h-1}+1}^{m_h} \sum_{|\nu|=m} S_{\mu_{\psi(i,\nu)}} S^*_{\mu_i \nu}. \tag{2.4}$$

By construction, we have $u_j^* u_j = e_h$ and $u_j u_j^* = e_j$. Observe that

$$S_{\mu_i \nu} S^*_{\mu_{\psi(i,\nu)}} \mathcal{F}_n S_{\mu_{\psi(i',\nu')}} S^*_{\mu'_i \nu'} \subseteq \mathcal{F}_n,$$

since $|\mu_i \nu| - |\mu_{\psi(i,\nu)}| + |\mu_{\psi(i',\nu')}| - |\mu'_i \nu'| = 0$. It follows that

$$u_j^* e_j \mathcal{F}_n e_j u_j = e_h \mathcal{F}_n e_h. \tag{2.5}$$

Now, we define

$$U_\sigma := \sum_{j=1}^k u_j^*. \tag{2.6}$$

Then U_σ is an element of $\mathcal{O}_n$ with the following properties:

[U1]: U_σ is a unitary normalizing A.
[U2]: $U_\sigma e_j U_\sigma^* = \sigma(e_j)$ for each $j = 1, \ldots, k$.
[U3]: For each $j = 1, \ldots, k$ there exist words α_i, β_i, $i = 1, \ldots, r$ (for some $r \in \mathbb{N}$) such that

- $U_\sigma e_j = \sum_{i=1}^r S_{\alpha_i} S^*_{\beta_i}$,
- $\alpha_i \prec \alpha_{i'}$ whenever $\beta_i \prec \beta_{i'}$, and
- $|\alpha_i| - |\beta_i| = |\alpha_{i'}| - |\beta_{i'}|$ for all $i, i' = 1, \ldots, r$.

The conditions $[U1]$–$[U3]$ characterize U_σ uniquely. First we will show that if a unitary $V \in \mathcal{O}_n$ satisfies the conditions $[U1]$–$[U3]$ for $\sigma = id$, then V must be 1. By [U1] and [U2] for $\sigma = id$, we know that $V \in A$. Then we have

- $Ve_j = \sum_{i=1}^r S_{\alpha_i} S_{\beta_i}^*$ with $|\alpha_i| = |\beta_i| =$ constant,
- $\alpha_i \prec \alpha_{i'}$ whenever $\beta_i \prec \beta_{i'}$.

Since the lexicographic order is a total order for words with a fixed length, we see that $\alpha_i = \beta_i$ and hence $V = 1$. Next we will consider the general case. If a unitary $V \in \mathcal{O}_n$ satisfies the conditions $[U1]$–$[U3]$ for σ, then it is easy to see that $V^* U_\sigma$ satisfies the conditions $[U1]$–$[U3]$ for $\sigma = id$. So we have $V^* U_\sigma = 1$ and hence $V = U_\sigma$. This uniqueness easily implies that $\{U_\sigma : \sigma \in \mathcal{S}_\sim\}$ is a subgroup of $\mathcal{U}(\mathcal{O}_n)$. Indeed, given σ and σ' in $\mathcal{S}_\sim$, both $U_\sigma U_{\sigma'}$ and $U_{\sigma\sigma'}$ satisfy conditions $[U1]$–$[U3]$. Since the group $\{U_\sigma : \sigma \in \mathcal{S}_\sim\}$ is isomorphic to $\mathcal{S}_\sim$ and acts on $\mathcal{U}(A)$ by Ad, we have an inclusion $\mathcal{U}(A) \rtimes \mathcal{S}_\sim \subseteq \mathcal{N}_{\mathcal{O}_n}(A)$.

To see that the reverse inclusion $\mathcal{N}_{\mathcal{O}_n}(A) \subseteq \mathcal{U}(A) \rtimes \mathcal{S}_\sim$ holds as well, take a V in $\mathcal{N}_{\mathcal{O}_n}(A)$. The action of V on the center of A yields a permutation σ of $\{e_1, \ldots, e_k\}$. By Lemma 2.1, this permutation belongs to $\mathcal{S}_\sim$. But then VU_σ^* fixes each projection e_j, and thus it normalizes $e_j \mathcal{F}_n e_j$. Hence, for each j there is a $w_j \in \mathcal{U}(e_j \mathcal{F}_n e_j)$ such that $VU_\sigma e_j = w_j$ ([4] Lemma 1.15). Putting $W := \sum_{j=1}^k w_j$ we get a unitary in A such that $V = WU_\sigma$, and the proof is complete. □

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