Banach J. Math. Anal. 9 (2015), no. 3, 85–97
http://doi.org/10.15352/bjma/09-3-6
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

ON THE DIFFERENCE PROPERTY OF HIGHER ORDERS FOR DIFFERENTIABLE FUNCTIONS

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Communicated by P. K. Sahoo

Abstract. In this paper, inspired by some results concerning the double difference property, we show that the class $C^p(\mathbb{R},\mathbb{R})$ of p-times continuously differentiable functions has the difference property of p-th order, i.e. if a function $f\colon \mathbb{R} \to \mathbb{R}$ is such that $\Delta_h^p f \in C^p(\mathbb{R}\times\mathbb{R},\mathbb{R})$, where $\Delta_h^p f$ is the p-th iterate of the well-known difference operator $\Delta_h f(x) := f(x+h) - f(x)$, then there exists a polynomial function $\Gamma_{p-1}\colon \mathbb{R} \to \mathbb{R}$ of $(p-1)$-th order such that $f - \Gamma_{p-1} \in C^p(\mathbb{R},\mathbb{R})$. Moreover, some new equalities connected with the difference operator are also presented.

1. Introduction

Throughout this paper, $\mathbb{N}_0$, $\mathbb{N}$ and $\mathbb{R}$ will always denote the sets of all non-negative integers, positive integers and real numbers, respectively. Moreover, let $\mathcal{F}$, $\mathcal{F}^2$ be classes of real valued functions defined on $\mathbb{R}$ and $\mathbb{R}^2$, respectively.

The notion of the so-called difference property for various classes of real functions was investigated by de Bruijn in [3]. He considered the following problem:

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Assume that for each $h \in \mathbb{R}$ the function $\Delta_h f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\Delta_h f(x) := f(x+h) - f(x), \quad x \in \mathbb{R},$$

belongs to a given class $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$. What can be said about the function f?

Date: Received: Jun. 5, 2014; Revised: Sep. 10, 2014; Accepted: Oct. 8, 2014.
2010 *Mathematics Subject Classification.* Primary 47B39; Secondary 39A70, 39B22.
Key words and phrases. Difference property, difference operator, polynomial functions.

In [3] and [4] it was shown that for a great number of important classes $\mathcal{F}$ (e.g. the classes of continuous, differentiable, analytic, absolute continuous, Riemann-integrable functions) the function f may be written in the form of

$$f = g + A, \tag{1.1}$$

where $g \in \mathcal{F}$ and $A\colon \mathbb{R} \to \mathbb{R}$ is an additive function, i.e. it satisfies the Cauchy functional equation

$$A(x+y) = A(x) + A(y), \quad x, y \in \mathbb{R}.$$

Thus we can formulate the following definition: the class $\mathcal{F}$ is said to have the difference property if any function $f\colon \mathbb{R} \to \mathbb{R}$ with all differences $\Delta_h f$ belonging to $\mathcal{F}$ admits a decomposition (1.1).

The results of de Bruijn have been extended and generalized in various directions (cf., e.g., [5], [6], [7], [8], [11], [13]). However, the class of all measurable functions fails to have this property. This class has the so-called weak difference property (see [8] and [15]). An extensive source of information on the difference property is Laczkovich's survey paper [16].

For a given function $f\colon \mathbb{R} \to \mathbb{R}$ and an $h \in \mathbb{R}$ we define inductively the difference operator Δ_h^p with increment h as follows:

$$\Delta_h^p f(x) = \Delta_h \left(\Delta_h^{p-1} f(x)\right), \quad x \in \mathbb{R},\ p \in \mathbb{N}.$$

Additionally, we put $\Delta_h^0 f(x) = f(x)$. In particular, we have $\Delta_h^1 f(x) = \Delta_h f(x)$.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be a polynomial function of p-th order if and only if

$$\Delta_h^{p+1} f(x) = 0$$

for all $x, h \in \mathbb{R}$ (see [14] for more details).

It is well known (cf. [9]) that $f\colon \mathbb{R} \to \mathbb{R}$ is a polynomial function of p-th order if and only if it has a unique representation

$$f = f_0 + f_1 + \ldots + f_p,$$

where f_0 is a constant, and $f_k\colon \mathbb{R} \to \mathbb{R}$ for $k = 1, 2, \ldots, p$ are diagonalizations of k-additive symmetric functions $F_k\colon \mathbb{R}^k \to \mathbb{R}$, i.e.,

$$f_k(x) := F_k(\underbrace{x, \ldots, x}_{k}), \quad x \in \mathbb{R},\ k = 1, 2, \ldots, p.$$

Moreover (see [14], for example), for any $p \in \mathbb{N}$ the following equality holds true:

$$\Delta_h^p f(x) = \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} f(x + ih), \quad x, h \in \mathbb{R}. \tag{1.2}$$

Following J. H. B. Kamperman [12] we say that a class $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ have the difference property of p-th order ($p \in \mathbb{N}$) if and only if any function $f\colon \mathbb{R} \to \mathbb{R}$ such that $\Delta_h^p f \in \mathcal{F}$ for each $h \in \mathbb{R}$ admits a decomposition $f = g + \Gamma$, where $g \in \mathcal{F}$ and $\Gamma\colon \mathbb{R} \to \mathbb{R}$ is a polynomial function of p-th order.

Difference properties of higher orders for various classes of functions (e.g. the classes of all continuous functions on a locally compact Abelian group and all

Riemann integrable functions on a compact second countable Abelian group) have been proved by Z. Gajda (see [10]).

M. Laczkovich in [15] considered the so-called double difference property. He proved, in particular, that for any function $f\colon \mathbb{R} \to \mathbb{R}$ if the Cauchy difference $Cf(x,y) := f(x+y)-f(x)-f(y)$ is Lebesgue measurable, then f can be expressed as a sum of a Lebesgue measurable function and an additive function. Formally, a pair of classes $\left(\mathcal{F}, \mathcal{F}^2\right)$ is said to have the double difference property if every function $f\colon \mathbb{R} \to \mathbb{R}$ such that $Cf \in \mathcal{F}^2$ is of the form $f = g + A$, where $g \in \mathcal{F}$ and A is an additive function. For the sake of brevity we shall use the improper version in which the class $\mathcal{F}$ has the double difference property.

In [18] J. Tabor and J. Tabor proved, in particular, that the class $C^n(X,Y)$ of n-times continuously differentiable functions defined on a real normed space X and taking values in a real Banach space Y has the double difference property. Their result was applied to prove that the Cauchy and Jensen functional equations are stable in Ulam–Hyers sense with respect to large class of seminorms defined by means of derivatives. A similar problem concerning the so-called double quadratic difference property in connectedness with the Ulam–Hyers stability of the quadratic functional equation was considered in [1] and [2].

The purpose of this paper is to show how the difference property of higher orders for the class p-times continuously differentiable functions may be proved by means of other tools. More precisely, we give an alternative proof of this result using the technique associated with the notion of the double difference property [18].

2. Preliminaries

Let $f\colon \mathbb{R} \to \mathbb{R}$ be p-times differentiable function. By $D^p f$, $p \in \mathbb{N}$, we de note p-th derivative of f and $D^0 f$ stands for f. The space of all functions $f\colon \mathbb{R} \to \mathbb{R}$ that are p-times differentiable on $\mathbb{R}$ will be denoted by $D^p(\mathbb{R},\mathbb{R})$. By $C^\infty(\mathbb{R},\mathbb{R})$ we denote the space of infinitely many times continuously differentiable functions. By $\partial_k^p f$, $k=1,2$, we denote, as usual, the p-th partial derivative of $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ with respect to the k-th variable.

In the sequel we will adopt the following notation:

$$\Delta^p f(x,h) := \Delta_h^p f(x), \quad x,h \in \mathbb{R},\ p \in \mathbb{N}.$$

To avoid distinguishing some cases and to shorten some considerations we will also use the following convention. If $m,n \in \mathbb{N}_0$, $m > n$, then by $\sum\limits_{i=m}^{n} a_i$ we mean zero. Moreover, let $0^0 := 1$.

In the present section we prove some lemmas concerning the difference operator $\Delta^p f$ $(p \in \mathbb{N})$ which will be useful in the proof of the main theorem of this paper.

Let $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$. In the sequel we will use the following equality (see, e.g., [17], Lemma 1)

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^k = \begin{cases} (-1)^n \cdot n!, & k = n, \\ 0, & 0 \le k \le n-1. \end{cases} \tag{2.1}$$

From these equalities we easily deduce that

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}i^n = n!. \tag{2.2}$$

Lemma 2.1. *Let $(X,+)$ be a commutative semigroup and let $(Y,+)$ be a commutative group. Let $p \in \mathbb{N}$ be fixed and let $f\colon X \to Y$ be a given function. Then the difference operator $\Delta^p f$ satisfies the following functional equation*

$$\sum_{i=0}^{p}(-1)^i\binom{p}{i}\Delta^p f(x+is, y+it) = \sum_{i=0}^{p}(-1)^i\binom{p}{i}\Delta^p f(x+iy, s+it) \tag{2.3}$$

for all $x, y, s, t \in X$.

Proof. We have by (1.2)

$$\begin{aligned}
&\sum_{i=0}^{p}(-1)^i\binom{p}{i}\Delta^p f(x+is, y+it)\\
&= \sum_{i=0}^{p}(-1)^i\binom{p}{i}\sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}f\big(x+is+k(y+it)\big)\\
&= \sum_{k=0}^{p}(-1)^k\binom{p}{k}\sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i}f\big(x+ks+i(y+kt)\big)\\
&= \sum_{i=0}^{p}(-1)^i\binom{p}{i}\sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}f\big(x+iy+k(s+it)\big)\\
&= \sum_{i=0}^{p}(-1)^i\binom{p}{i}\Delta^p f(x+iy, s+it),
\end{aligned}$$

due to the equality $(-1)^k = (-1)^{-k}$, $k \in \mathbb{N}_0$. □

Lemma 2.2. *Let $(X,+)$ be a commutative semigroup and let $(Y,+)$ be a commutative group. Let $p \in \mathbb{N}$ be fixed and let $f\colon X \to Y$ be a given function. Then the difference operator $\Delta^p f$ satisfies the following functional equation*

$$\Delta^p f(x, 2y) = \sum_{i=0}^{p}\binom{p}{i}\Delta^p f(x+iy, y) \tag{2.4}$$

for all $x, y \in X$.

Proof. For $p = 1$ formula (2.4) becomes

$$\begin{aligned}
\Delta f(x, 2y) &= \sum_{i=0}^{1}\binom{1}{i}\Delta f(x+iy, y) = \Delta f(x,y) + \Delta f(x+y, y)\\
&= \big[f(x+y) - f(x)\big] + \big[f(x+2y) - f(x+y)\big] = f(x+2y) - f(x),
\end{aligned}$$

which is obviously true. Now assume that (2.4) holds for some $p \in \mathbb{N}$. Then we have by (1.2), the recurrence $\Delta^{p+1} f(x,y) = \Delta^p f(x+y, y) - \Delta^p f(x,y)$, $p \in \mathbb{N}_0$, and induction hypothesis

$$
\begin{aligned}
& \Delta^{p+1} f(x,2y) - \sum_{i=0}^{p+1} \binom{p+1}{i} \Delta^{p+1} f(x+iy,y) \\
= \; & \Delta^{p} f(x+2y,2y) - \Delta^{p} f(x,2y) \\
- \; & \sum_{i=0}^{p+1} \binom{p+1}{i} \Delta^{p} f\big(x+(i+1)y,y\big) + \sum_{i=0}^{p+1} \binom{p+1}{i} \Delta^{p} f(x+iy,y) \\
= \; & \sum_{i=0}^{p} \binom{p}{i} \Delta^{p} f\big(x+(i+2)y,y\big) - \sum_{i=0}^{p} \binom{p}{i} \Delta^{p} f(x+iy,y) \\
- \; & \sum_{i=0}^{p+1} \binom{p+1}{i} \Delta^{p} f\big(x+(i+1)y,y\big) + \sum_{i=0}^{p+1} \binom{p+1}{i} \Delta^{p} f(x+iy,y) \\
= \; & \sum_{i=1}^{p+1} \binom{p}{i-1} \Delta^{p} f\big(x+(i+1)y,y\big) - \sum_{i=0}^{p+1} \binom{p+1}{i} \Delta^{p} f\big(x+(i+1)y,y\big) \\
+ \; & \sum_{i=0}^{p+1} \binom{p+1}{i} \Delta^{p} f(x+iy,y) - \sum_{i=0}^{p} \binom{p}{i} \Delta^{p} f(x+iy,y) \\
= \; & \sum_{i=1}^{p+1} \left[\binom{p}{i-1} - \binom{p+1}{i} \right] \Delta^{p} f\big(x+(i+1)y,y\big) - \Delta^{p} f(x+y,y) \\
+ \; & \sum_{i=1}^{p} \left[\binom{p+1}{i} - \binom{p}{i} \right] \Delta^{p} f(x+iy,y) + \Delta^{p} f\big(x+(p+1)y,y\big) \\
= \; & \sum_{i=1}^{p} \binom{p}{i-1} \Delta^{p} f(x+iy,y) - \sum_{i=1}^{p} \binom{p}{i} \Delta^{p} f\big(x+(i+1)y,y\big) \\
+ \; & \Delta^{p} f\big(x+(p+1)y,y\big) - \Delta^{p} f(x+y,y) \\
= \; & \sum_{i=1}^{p} \binom{p}{i-1} \Delta^{p} f(x+iy,y) - \sum_{i=2}^{p+1} \binom{p}{i-1} \Delta^{p} f\big(x+iy,y\big) \\
+ \; & \Delta^{p} f\big(x+(p+1)y,y\big) - \Delta^{p} f(x+y,y) = 0,
\end{aligned}
$$

due to the equality $\binom{p+1}{i} = \binom{p}{i-1} + \binom{p}{i}$. So we obtain formula (2.4) for $p+1$. Induction completes the proof. $\square$

Lemma 2.3. *Let $p \in \mathbb{N}$ be fixed and let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that $\Delta^p f \in D^p(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. Then we have*

$$
\partial_1^{p-k} \partial_2^{k} (\Delta^p f)(x,y) = \frac{1}{p!} \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} i^k \partial_2^p (\Delta^p f)(x+iy,0), \quad x,y \in \mathbb{R}, \quad (2.5)
$$

where $k \in \mathbb{N}_0$ and $k \le p$.

Proof. Let $p \in \mathbb{N}$ and $x,y \in \mathbb{R}$ be fixed. Differentiating both sides of the equality (2.3) $(p-k)$-times with respect to s and k-times with respect to t at the point

$s = t = 0$, we obtain

$$\sum_{i=0}^{p}(-1)^i\binom{p}{i}i^p\partial_1^{p-k}\partial_2^k(\Delta^p f)(x,y) = \sum_{i=0}^{p}(-1)^i\binom{p}{i}i^k\partial_2^p(\Delta^p f)(x+iy,0), \quad x,y\in\mathbb{R}.$$

The cases where $k = p$ and $k = 0$ mean that the equality (2.3) is not differentiated with respect to s and t, respectively. Applying (2.1) to the left-hand side of the above equality, we have

$$(-1)^p p!\partial_1^{p-k}\partial_2^k(\Delta^p f)(x,y) = \sum_{i=0}^{p}(-1)^i\binom{p}{i}i^k\partial_2^p(\Delta^p f)(x+iy,0), \quad x,y\in\mathbb{R}.$$

Dividing both sides by $(-1)^p p!$ and observing that $(-1)^{i-p} = (-1)^{p-i}$ for $p\in\mathbb{N}$ and $i\in\mathbb{N}_0$, we obtain (2.5), which completes the proof. □

Lemma 2.4. *Let $p\in\mathbb{N}$, $p\geq 2$ be fixed and let $f\colon\mathbb{R}\to\mathbb{R}$ be a function such that $\Delta^p f\in D^p(\mathbb{R}\times\mathbb{R},\mathbb{R})$. Then we have*

$$\partial_1^{m-k}\partial_2^k(\Delta^p f)(x,0) = 0, \quad x\in\mathbb{R}, \tag{2.6}$$

where $1\leq m\leq p-1$ and $0\leq k\leq m$.

Proof. Let $p\geq 2$ and $y\in\mathbb{R}$ be fixed. Differentiating both sides of the equality (2.4) m-times with respect to x, where $1\leq m\leq p-1$, we obtain

$$\partial_1^m(\Delta^p f)(x,2y) = \sum_{i=0}^{p}\binom{p}{i}\partial_1^m(\Delta^p f)(x+iy,y), \quad x,y\in\mathbb{R}.$$

Setting $y = 0$ we get

$$\partial_1^m(\Delta^p f)(x,0) = \sum_{i=0}^{p}\binom{p}{i}\partial_1^m(\Delta^p f)(x,0), \quad x\in\mathbb{R},$$

hence

$$\left[\sum_{i=0}^{p}\binom{p}{i} - 1\right]\partial_1^m(\Delta^p f)(x,0) = 0, \quad x\in\mathbb{R},$$

i.e.

$$(2^p-1)\partial_1^m(\Delta^p f)(x,0) = 0, \quad x\in\mathbb{R}.$$

Therefore

$$\partial_1^m(\Delta^p f)(x,0) = 0, \quad x\in\mathbb{R}. \tag{2.7}$$

Thus we have proved (2.6) for $1\leq m\leq p-1$ and $k=0$.

Differentiating both sides of the equality (2.3) $(m-k)$-times with respect to x and k-times with respect to y, where $1\leq m\leq p-1$ and $1\leq k\leq m$, we have

$$\begin{aligned} &2^k\partial_1^{m-k}\partial_2^k(\Delta^p f)(x,2y) \\ =\ &\sum_{i=0}^{p}\binom{p}{i}\left[i^k\partial_1^m(\Delta^p f)(x+iy,y) + \partial_1^{m-k}\partial_2^k(\Delta^p f)(x+iy,y)\right] \end{aligned}$$

for all $x, y \in \mathbb{R}$. Taking $y = 0$ in the above equality we get

$$2^k \partial_1^{m-k} \partial_2^k (\Delta^p f)(x, 0) = \sum_{i=0}^{p} \binom{p}{i} \left[i^k \partial_1^m (\Delta^p f)(x, 0) + \partial_1^{m-k} \partial_2^k (\Delta^p f)(x, 0) \right], \quad x \in \mathbb{R}.$$

From (2.7) it follows that

$$\left[\sum_{i=0}^{p} \binom{p}{i} - 2^k \right] \partial_1^{m-k} \partial_2^k (\Delta^p f)(x, 0) = 0, \quad x \in \mathbb{R},$$

i.e.

$$(2^p - 2^k) \partial_1^{m-k} \partial_2^k (\Delta^p f)(x, 0) = 0, \quad x \in \mathbb{R}.$$

Therefore

$$\partial_1^{m-k} \partial_2^k (\Delta^p f)(x, 0) = 0, \quad x \in \mathbb{R}$$

for $1 \leq m \leq p - 1$ and $1 \leq k \leq m$, which together with (2.7) completes the proof. □

3. Main results

In this section we are going to formulate and prove the main result of this paper. It is worth mentioning that we are able to give an explicit expression for the polynomial function Γ_{p-1} occuring in the statement of the main theorem. In the case where $p = 1$ by $\int_0^1 \int_0^{t_1} \int_0^{t_2} \dots \int_0^{t_{p-1}} f(u) du dt_{p-1} \dots dt_1$ we mean the integral $\int_0^1 f(u) du$.

Theorem 3.1. *Let $p \in \mathbb{N}$ be fixed and let $f \colon \mathbb{R} \to \mathbb{R}$ be a function such that $\Delta^p f \in C^p(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. Then there exists a polynomial function $\Gamma_{p-1} \colon \mathbb{R} \to \mathbb{R}$ of $(p-1)$-th order given by the formula*

$$\Gamma_{p-1}(x) = f(x) - \frac{1}{p!} \int_0^1 \int_0^{t_1} \int_0^{t_2} \dots \int_0^{t_{p-1}} \partial_2^p (\Delta^p f)(ux, 0)(x^p) du dt_{p-1} \dots dt_1, \quad x \in \mathbb{R}$$

such that $f - \Gamma_{p-1} \in C^p(\mathbb{R}, \mathbb{R})$.

Proof. Let $p \in \mathbb{N}$ be fixed. Since by (1.2)

$$\Delta^p f(x, y) = \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} f(x + iy), \quad x, y \in \mathbb{R},$$

then from (2.1) we easily get

$$\Delta^p f(0, 0) = \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} f(0) = 0.$$

Let us fix arbitrarily $x, y \in \mathbb{R}$ and define the function $\varphi_p \colon \mathbb{R} \to \mathbb{R}$ by the formula

$$\varphi_p(t) := \Delta^p f(tx, ty), \quad t \in \mathbb{R}.$$

Obviously $\varphi_p(0) = 0$, $\varphi_p(1) = \Delta^p f(x,y)$ and $\varphi_p \in C^p(\mathbb{R},\mathbb{R})$. Differentiating p-times the function φ_p we have

$$D^p\varphi_p(t) = \sum_{k=0}^{p} \binom{p}{k} \partial_1^{p-k}\partial_2^{k}(\Delta^p f)(tx,ty)\left(x^{p-k}y^k\right). \tag{3.1}$$

It follows from (2.6) that

$$D^m\varphi_p(0) = \sum_{k=0}^{m} \binom{m}{k} \partial_1^{m-k}\partial_2^{k}(\Delta^p f)(0,0)\left(x^{m-k}y^k\right) = 0 \tag{3.2}$$

for $1 \le m \le p-1$. Then for fixed $p \in \mathbb{N}$ we obtain

$$\begin{aligned}
\Delta^p f(x,y) &= \varphi_p(1) - \varphi_p(0) = \int_0^1 D\varphi_p(t)dt = \int_0^1\int_0^{t_1} D^2\varphi_p(u)dudt_1 + D\varphi_p(0) \\
&= \int_0^1\int_0^{t_1}\int_0^{t_2} D^3\varphi_p(u)dudt_2dt_1 + D\varphi_p(0) + \frac{1}{2}D^2\varphi_p(0) \\
&= \ldots = \int_0^1\int_0^{t_1}\int_0^{t_2}\ldots\int_0^{t_{p-1}} D^p\varphi_p(u)dudt_{p-1}\ldots dt_1 + \sum_{i=1}^{p-1}\frac{1}{i!}D^i\varphi_p(0) \\
&= \int_0^1\int_0^{t_1}\int_0^{t_2}\ldots\int_0^{t_{p-1}} D^p\varphi_p(u)dudt_{p-1}\ldots dt_1,
\end{aligned}$$

since from (3.2) we have $\sum\limits_{i=1}^{p-1}\frac{1}{i!}D^i\varphi_p(0) = 0$. Finally, making use of (3.1) we obtain

$$\Delta^p f(x,y) = \int_0^1\int_0^{t_1}\int_0^{t_2}\ldots\int_0^{t_{p-1}} \sum_{k=0}^{p}\binom{p}{k}\partial_1^{p-k}\partial_2^{k}(\Delta^p f)(ux,uy)\left(x^{p-k}y^k\right)dudt_{p-1}\ldots dt_1. \tag{3.3}$$

We define the function $\Gamma_{p-1}\colon \mathbb{R} \to \mathbb{R}$ by the formula

$$\Gamma_{p-1}(x) := f(x) - \frac{1}{p!}\int_0^1\int_0^{t_1}\int_0^{t_2}\ldots\int_0^{t_{p-1}} \partial_2^p(\Delta^p f)(ux,0)(x^p)dudt_{p-1}\ldots dt_1, \quad x \in \mathbb{R}.$$

We show that Γ_{p-1} is a polynomial function of $(p-1)$-th order. In virtue of (1.2) and (3.3) we obtain for fixed $x, y \in \mathbb{R}$

$$\begin{aligned}
&\Delta^p \Gamma_{p-1}(x,y) = \Delta^p f(x,y) \\
&- \frac{1}{p!}\int_0^1\int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_{p-1}} \sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i}\partial_2^p(\Delta^p f)\big(u(x+iy),0\big)(x+iy)^p du dt_{p-1}\dots dt_1 \\
&= \int_0^1\int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_{p-1}} \sum_{k=0}^{p}\binom{p}{k}\partial_1^{p-k}\partial_2^k(\Delta^p f)(ux,uy)\left(x^{p-k}y^k\right) du dt_{p-1}\dots dt_1 \\
&- \frac{1}{p!}\int_0^1\int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_{p-1}} \sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i}\partial_2^p(\Delta^p f)\big(u(x+iy),0\big)(x+iy)^p du dt_{p-1}\dots dt_1 \\
&= \int_0^1\int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_{p-1}} \Bigg[\sum_{k=0}^{p}\binom{p}{k}\partial_1^{p-k}\partial_2^k(\Delta^p f)(ux,uy)\left(x^{p-k}y^k\right) \\
&- \frac{1}{p!}\sum_{k=0}^{p}\binom{p}{k}\sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i} i^k\partial_2^p(\Delta^p f)\big(u(x+iy),0\big)\left(x^{p-k}y^k\right)\Bigg] du dt_{p-1}\dots dt_1 \\
&= \int_0^1\int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_{p-1}} \sum_{k=0}^{p}\binom{p}{k}\Bigg[\Big(\partial_1^{p-k}\partial_2^k(\Delta^p f)(ux,uy) \\
&- \frac{1}{p!}\sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i} i^k\partial_2^p(\Delta^p f)\big(u(x+iy),0\big)\Big)\Bigg]\left(x^{p-k}y^k\right) du dt_{p-1}\dots dt_1 = 0,
\end{aligned}$$

since on account of (2.5) all the integrands above are equal to zero.

Now we prove that the function $f - \Gamma_{p-1}$ is differentiable. Fix arbitrarily $p \in \mathbb{N}$ and $x, h \in \mathbb{R}$, $h \neq 0$. Then we have

$$\begin{aligned}
&\frac{1}{h}\Big[f(x+h) - \Gamma_{p-1}(x+h) - \big(f(x) - \Gamma_{p-1}(x)\big)\Big] \\
&= \frac{1}{h}\Bigg[\frac{1}{p!}\int_0^1\int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_{p-1}} \partial_2^p(\Delta^p f)\big(u(x+h),0\big)(x+h)^p du dt_{p-1}\dots dt_1 \\
&- \frac{1}{p!}\int_0^1\int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_{p-1}} \partial_2^p(\Delta^p f)(ux,0)(x^p) du dt_{p-1}\dots dt_1\Bigg] \\
&= \frac{1}{h\cdot p!}\Bigg[\int_0^1\int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_{p-1}(x+h)} \partial_2^p(\Delta^p f)(s,0)(x+h)^{p-1} ds dt_{p-1}\dots dt_1
\end{aligned}$$

$$
\begin{aligned}
- &\int_0^1\int_0^{t_1}\int_0^{t_2}\dots\int_0^{t_{p-1}x}\partial_2^p(\Delta^p f)(s,0)\left(x^{p-1}\right)ds\,dt_{p-1}\dots dt_1\Bigg]\\
= &\frac{1}{h\cdot p!}\Bigg[\int_0^1\int_0^{t_1}\int_0^{t_2}\dots\int_0^{v_{p-2}(x+h)}\int_0^{v_{p-1}}\partial_2^p(\Delta^p f)(s,0)(x+h)^{p-2}ds\,dv_{p-1}\dots dt_1\\
- &\int_0^1\int_0^{t_1}\int_0^{t_2}\dots\int_0^{v_{p-2}x}\int_0^{v_{p-1}}\partial_2^p(\Delta^p f)(s,0)\left(x^{p-2}\right)ds\,dv_{p-1}\dots dt_1\Bigg]\\
= &\dots=\frac{1}{h\cdot p!}\Bigg[\int_0^1\int_0^{t_1(x+h)}\int_0^{v_2}\dots\int_0^{v_{p-1}}\partial_2^p(\Delta^p f)(s,0)(x+h)ds\,dv_{p-1}\dots dt_1\\
- &\int_0^1\int_0^{t_1x}\int_0^{v_2}\dots\int_0^{v_{p-1}}\partial_2^p(\Delta^p f)(s,0)(x)ds\,dv_{p-1}\dots dt_1\Bigg]\\
= &\frac{1}{h\cdot p!}\Bigg[\int_0^{x+h}\int_0^{v_1}\int_0^{v_2}\dots\int_0^{v_{p-1}}\partial_2^p(\Delta^p f)(s,0)ds\,dv_{p-1}\dots dv_1\\
- &\int_0^{x}\int_0^{v_1}\int_0^{v_2}\dots\int_0^{v_{p-1}}\partial_2^p(\Delta^p f)(s,0)ds\,dv_{p-1}\dots dv_1\Bigg]\\
= &\frac{1}{h\cdot p!}\int_x^{x+h}\int_0^{v_1}\int_0^{v_2}\dots\int_0^{v_{p-1}}\partial_2^p(\Delta^p f)(s,0)ds\,dv_{p-1}\dots dv_1\\
= &\frac{1}{h\cdot p!}\int_0^1\int_0^{x+th}\int_0^{v_2}\dots\int_0^{v_{p-1}}\partial_2^p(\Delta^p f)(s,0)(h)ds\,dv_{p-1}\dots dv_2\,dt\\
= &\frac{1}{p!}\int_0^1\int_0^{x+th}\int_0^{v_2}\dots\int_0^{v_{p-1}}\partial_2^p(\Delta^p f)(s,0)ds\,dv_{p-1}\dots dv_2\,dt\\
\longrightarrow &\frac{1}{p!}\int_0^1\int_0^{x}\int_0^{v_2}\dots\int_0^{v_{p-1}}\partial_2^p(\Delta^p f)(s,0)ds\,dv_{p-1}\dots dv_2\,dt\\
= &\frac{1}{p!}\int_0^x\int_0^{v_2}\int_0^{v_3}\dots\int_0^{v_{p-1}}\partial_2^p(\Delta^p f)(s,0)ds\,dv_{p-1}\dots dv_2
\end{aligned}
$$

for $h \to 0$. Hence the function $f - \Gamma_{p-1}$ is differentiable at every point $x \in \mathbb{R}$ and

$$D(f - \Gamma_{p-1})(x) = \frac{1}{p!} \int_0^x \int_0^{v_2} \int_0^{v_3} \dots \int_0^{v_{p-1}} \partial_2^p (\Delta^p f)(s,0) ds dv_{p-1} \dots dv_2, \quad x \in \mathbb{R}.$$

Similarly as above we can show that

$$D^2(f - \Gamma_{p-1})(x) = \frac{1}{p!} \int_0^x \int_0^{v_3} \int_0^{v_4} \dots \int_0^{v_{p-1}} \partial_2^p (\Delta^p f)(s,0) ds dv_{p-1} \dots dv_3, \quad x \in \mathbb{R}.$$

Proceeding step by step $(p-3)$-times we obtain

$$D^{p-1}(f - \Gamma_{p-1})(x) = \frac{1}{p!} \int_0^x \partial_2^p (\Delta^p f)(s,0) ds, \quad x \in \mathbb{R},$$

and consequently

$$D^p(f - \Gamma_{p-1})(x) = \frac{1}{p!} \partial_2^p (\Delta^p f)(x,0), \quad x \in \mathbb{R}.$$

Since $\partial_2^p(\Delta^p f) \in C^0(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, then $\partial_2^p(\Delta^p f)(x,0) \in C^0(\mathbb{R}, \mathbb{R})$ for all $x \in \mathbb{R}$. Therefore $D^p(f - \Gamma_{p-1}) \in C^0(\mathbb{R}, \mathbb{R})$, i.e. $f - \Gamma_{p-1} \in C^p(\mathbb{R}, \mathbb{R})$. Moreover, it is easily seen that we have

$$D^i(f - \Gamma_{p-1})(0) = 0, \quad 0 \le i \le p-1.$$

The proof is completed. □

Theorem 3.1 states, in particular, that the class of infinitely many times differentiable functions has the difference property of higher orders. We show that the class of analytic functions also has this property.

Theorem 3.2. *Let $p \in \mathbb{N}$ be fixed and let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that $\Delta^p f$ is analytic. Then there exists a polynomial function $\Gamma_{p-1}\colon \mathbb{R} \to \mathbb{R}$ of $(p-1)$-th order such that $f - \Gamma_{p-1}$ is analytic.*

Proof. By Theorem 3.1 there exists a polynomial function $\Gamma_{p-1}\colon \mathbb{R} \to \mathbb{R}$ of $(p-1)$-th order such that $g := f - \Gamma_{p-1} \in C^\infty(\mathbb{R}, \mathbb{R})$. Then obviously $\Delta^p g = \Delta^p f$ and hence $\Delta^p g$ is analytic. Making use of the equality

$$\frac{1}{p!} \partial_2^p (\Delta^p g)(x,0) = D^p g(x), \quad x \in \mathbb{R},$$

we obtain that $D^p g$ is analytic, and consequently that g is analytic. □

4. Concluding remarks

Remark 4.1. In connection with Theorem 3.1 there arises a natural question: whether the polynomial function Γ_{p-1} is unique?

Remark 4.2. Let $p \in \mathbb{N}$ be fixed and let

$$F(x,y) = \Delta^p f(x,y). \tag{4.1}$$

We can rewrite equations (2.3) and (2.4) in the following forms

$$\sum_{i=0}^{p}(-1)^i \binom{p}{i} F(x+is, y+it) = \sum_{i=0}^{p}(-1)^i \binom{p}{i} F(x+iy, s+it), \tag{4.2}$$

$$F(x,2y) = \sum_{i=0}^{p} \binom{p}{i} F(x+iy, y), \tag{4.3}$$

respectively, where F is an unknown function. Obviously, if a function F is equal to the difference operator $\Delta^p f$, then it satisfies (4.2) and (4.3). Conversely, it seems to be interesting to give some sets of conditions (under some reasonable assumptions on function F) which would be both necessary and sufficient for F to have the representation (4.1).

Similarly, equation (2.5) can be rewritten in the following form

$$G_k(x,y) = \frac{1}{p!}\sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i} i^k H(x+iy), \quad k \in \mathbb{N}_0,$$

where G_k and H are unknown functions. In this case we may also try to find the general solution of the above functional equation.

Remark 4.3. One can check that if a function f is p-times continuously differentiable then the polynomial function Γ_{p-1} is given by the following formula

$$\Gamma_{p-1}(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{p-1}x^{p-1},$$

where the coefficients of the above polynomial are given by

$$a_i = \frac{1}{i!}D^i f(0), \quad i = 0,1,\ldots,p-1.$$

Indeed, differentiating both sides of the equality (1.2) p-times with respect to y we obtain

$$\partial_2^p(\Delta^p f)(x,y) = \sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i} i^p D^p f(x+iy) = p! D^p f(x+iy), \quad x,y \in \mathbb{R},$$

due to the equality (2.2). Putting $y=0$ we have

$$\partial_2^p(\Delta^p f)(x,0) = p! D^p f(x), \quad x \in \mathbb{R}.$$

Now, applied this equality to the definition of the polynomial function Γ_{p-1} we are able to obtain the desired formula.

Remark 4.4. A possible application of Theorem 3.1 is towards the study of the Ulam–Hyers stability problem of the Fréchet functional equation in the class of differentiable functions (cf. [18]).

Acknowledgement. The author is grateful to the referee for valuable comments and suggestions which improve the previous version of the paper.

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