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ASYMPTOTIC INTERTWINING BY THE IDENTITY OPERATOR AND PERMANENCE OF SPECTRAL PROPERTIES

B. P. DUGGAL

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ABSTRACT. We consider permanence of spectral properties of Banach space operators asymptotically intertwined by the identity operator.

1. INTRODUCTION AND PRELIMINARIES

If A, B are operators in $B(\mathcal{X})$, and $\triangle_{AB}(X) \in B(B(\mathcal{X}))$ is the generalized derivation $\triangle_{AB}(X) = AX - XB$, then B is said to be asymptotically intertwined to A by $X \in B(\mathcal{X})$, denoted $(A, B) \in (AX)$, if

$$\lim_{n\longrightarrow\infty} ||\triangle_{AB}^n(X)||^{\frac{1}{n}} = \lim_{n\longrightarrow\infty} ||\triangle_{AB}(\triangle_{AB}^{n-1}(X))||^{\frac{1}{n}} = 0.$$

Asymptotically intertwined operators intertwined by the identity operator $I \in B(\mathcal{X})$ share a number of properties; see [8, Lemmas, 3.4.7, 3.4.8 and Proposition 3.7.11]. In particular, if A has property (β) and $\lim_{n\to\infty} ||\triangle_{AB}^n(I)||^{\frac{1}{n}} = 0$, then B has property (β).

Intertwining by the identity operator preserves the single-valued extension property (in one direction), but fails to preserve the polaroid property. Here we say that an operator $A \in B(\mathcal{X})$ is polar at a point $\lambda \in \text{iso}\sigma(A)$ if $A - \lambda$ $(= A - \lambda I)$ has finite ascent and descent, and A is polaroid if it is polar at every $\lambda \in \text{iso}\sigma(A)$. Let us say that the operator $B \in B(\mathcal{X})$ is finitely intertwined to $A \in B(\mathcal{X})$ by the identity operator if there exists an integer $k \geq 1$ such that

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$\triangle_{AB}^k(I) = 0$. (Such an intertwining of A and B by the identity has been called a Helton class of order k in [7].) We prove that if B is finitely intertwined to A by the identity operator and $\text{iso}\sigma(B) \subseteq \text{iso}\sigma(A)$, then B inherits the polaroid property from A; again, if $(A, B) \in (AI)$, $\lambda \in \text{iso}\sigma(B) \cap \text{iso}\sigma(A)$ and λ is a finite rank pole (of the resolvent) of A, then λ is a finite rank pole of B. $(A, B) \in (AI)$ does not, in general, imply the equality $\sigma(A) = \sigma(B)$, or that the decomposition property (δ) transfers from A to B. We prove that if A and B^* have the single-valued extension property and $(A, B) \in (AI)$, then A and B have the same spectrum, the same Browder spectrum and the same Weyl spectrum. If the local spectra $\sigma_A(x)$ and $\sigma_B(x)$ satisfy the inclusion $\sigma_B(x) \subseteq \sigma_A(x)$ for all $0 \neq x \in \mathcal{X}$ and $(A, B) \in (AI)$, then A satisfies (Dunford's) condition (C) if and only if B satisfies condition (C); if also A has the single-valued extension property, then A satisfies property (δ) if and only if B satisfies property (δ). If, instead, $(A, B) \in (AI)$, B^* has the single-valued extension property and $\sigma_A(x) = \sigma(A)$ for all $0 \neq x \in \mathcal{X}$, then either A, B are quasi-nilpotent or A, B satisfy the abstract shift condition.

An operator $T \in B(\mathcal{X})$ is *upper semi Fredholm*, $T \in \Phi_+(\mathcal{X})$, if $T(\mathcal{X})$ is closed and $\alpha(T) = \dim(T^{-1}(0)) < \infty$, T is *lower semi Fredholm*, $T \in \Phi_-(\mathcal{X})$, if (the deficiency index) $\beta(T) = \dim(\mathcal{X}/T(\mathcal{X})) < \infty$, and T is Fredholm if T is both upper and lower semi Fredholm. The *semi-Fredholm index* of T, $\text{ind}(T)$, is the (finite or infinite) integer $\text{ind}(T) = \alpha(T) - \beta(T)$. The operator T is *Weyl* if it is Fredholm of zero index, and T is said to be *Browder* if it is Fredholm of finite ascent and descent. The upper essential spectrum, the lower essential spectrum, the essential spectrum, the Browder spectrum and the Weyl spectrum of T are, respectively, the sets $\sigma_{le}(T) = \{\lambda \in \sigma(T) : T - \lambda \notin \Phi_+(\mathcal{X})\}$, $\sigma_{ue}(T) = \{\lambda \in \sigma(T) : T - \lambda \notin \Phi_-(\mathcal{X})\}$, $\sigma_e(T) = \sigma_{le}(T) \cup \sigma_{ue}(T)$, $\sigma_b(T) = \{\lambda \in \sigma(T) : T - \lambda \text{ is not Browder}\}$ and $\sigma_w(T) = \{\lambda \in \sigma(T) : T - \lambda \text{ is not Weyl}\}$. We say that an operator $T \in B(\mathcal{X})$ satisfies *Browder's theorem* if the complement of $\sigma_w(T)$ in $\sigma(T)$ is the set $\pi_0(T)$ of finite rank poles of (the resolvent of) T (equivalently, if $\sigma_b(T) = \sigma_w(T)$ [4, Theorem 3.1]).

Let $\mathtt{C}$ denote the set of complex numbers. A Banach space operator T, $T \in B(\mathcal{X})$, has *the single-valued extension property* at $\lambda_0 \in \mathtt{C}$, SVEP at λ_0 for short, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \to \mathcal{X}$ which satisfies

$$(T - \lambda)f(\lambda) = 0 \quad \text{for all} \quad \lambda \in \mathcal{D}_{\lambda_0}$$

is the function $f \equiv 0$. *T has SVEP if it has SVEP at every* $\lambda \in \mathbf{C}$. The single valued extension property plays an important role in local spectral theory and Fredholm theory (see [8] and [1]; also see [6]). Evidently, every T has SVEP at points in the resolvent $\rho(T) = \mathtt{C} \setminus \sigma(T)$ and the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. It is easily verified that SVEP is inherited by restrictions, and that if T has SVEP and $\triangle_{TY}^n(I) = 0$ for some $Y \in B(\mathcal{X})$ and finite positive integer n, then Y has SVEP.

The *local resolvent set* $\rho_T(x)$ of $T \in B(\mathcal{X})$ at $x \in \mathcal{X}$ is the union of all open subsets $\mathcal{U}$ of $\mathtt{C}$ for which there is an analytic function $f : \mathcal{U} \to \mathcal{X}$ which satisfies $(T - \lambda)f(\lambda) = x$ for all $\lambda \in \mathcal{U}$; the *local spectrum* $\sigma_T(x)$ of T at x is then the set $\sigma_T(x) = \mathtt{C} \setminus \rho_T(x)$, and the *local spectral subspace* $X_T(F)$, $F \subseteq \mathtt{C}$, of T is the (not necessarily closed) subspace $X_T(F) = \{x \in \mathcal{X} : \sigma_T(x) \subseteq F\}$. For an arbitrary closed subset F of $\mathbf{C}$ and $T \in B(\mathcal{X})$, let $\chi_T(F) = \{x \in \mathcal{X} : (T-\lambda)f_x(\lambda) \equiv x$ for some analytic function $f_x : \mathbf{C} \setminus F \longrightarrow \mathcal{X}\}$. The *glocal spectral subspace* $\chi_T(F)$ is a hyper-invariant linear manifold of T such that $\chi_T(F) \subseteq X_T(F)$, with equality whenever T has SVEP (and, of course, F is closed) [8, p. 220]. Any further notation or terminology will be introduced progressively in the sequel, on an as and when required basis.

2. Main results

Recall, [8, page 253], that the operators $A, B \in B(\mathcal{X})$ are said to be *quasi-nilpotent equivalent* if both (A, B) and $(B, A) \in (AI)$. Quasi-nilpotent equivalence preserves a number of spectral properties amongst them (Bishop's) property (β), (decomposition) property (δ), (Dunford's) condition (C), SVEP, spectrum and local spectrum [8, Proposition 3.4.11]. Recall, [8], $T \in B(\mathcal{X})$ satisfies: condition (C) if $X_T(F)$ is closed for every closed set $F \subseteq \mathtt{C}$; property (δ) if $\mathcal{X} = \chi_T(\overline{U}) + \chi(\overline{V})$ for every open cover $\{U, V\}$ of $\mathtt{C}$; and T satisfies property (β) if and only if T^* satisfies property (δ). The following lemma generalizes a result known to hold for finitely intertwined by identity operators (see [5] and [7]).

Lemma 2.1. *If $(A, B) \in (AI)$ for some $A, B \in B(\mathcal{X})$, then A has SVEP implies B has SVEP.*

Proof. The hypothesis $(A, B) \in (AI)$ implies the inclusion $X_B(F) \subseteq X_A(F)$ for every closed subset F of $\mathtt{C}$ [8, Corollary 3.4.5]. Recall, [1, Theorem 2.8], that A has SVEP if and only if $X_A(\emptyset) = \{0\}$. Hence, if A has SVEP, then $X_B(\emptyset) \subseteq X_A(\emptyset) = \{0\}$ implies B has SVEP. □

Remark 2.2. (i). Observe that $(A, B) \in (AI) \Longleftrightarrow (B^*, A^*) \in (AI)$; hence Lemma 2.1 implies that if B^* has SVEP, then A^* has SVEP.

(ii). More can said in the case in which $\triangle^k_{AB}(I) = 0$ for some integer $k \geq 1$: If $\triangle^k_{AB}(I) = 0$ for some integer $k \geq 1$, then A has SVEP at a point μ implies B has SVEP at μ. This is seen as follows. If $\mu \notin \sigma(B)$, then B has SVEP at μ. Hence assume $\mu \in \sigma(B)$. Assume further that B does not have SVEP at μ. Then there exists a non-trivial analytic function f such that $(B-\lambda)f(\lambda) = 0$ for every λ in an ϵ-neighbourhood of μ. Since $\triangle^k_{AB}(I) = 0 \Longleftrightarrow \triangle^k_{(A-\lambda)(B-\lambda)}(I) = 0$, $(A-\lambda)^k f(\lambda) = 0$. But then (since A has SVEP at μ) $f(\lambda) = 0$ – a contradiction.

We start by considering the preservation of *the polaroid property*. For this we introduce some notation and terminology relevant to our considerations.

The *quasinilpotent part* $H_0(T)$ and the *analytic core* $K(T)$ of $T \in B(\mathcal{X})$ are defined by

$$H_0(T) = \{x \in \mathcal{X} : \lim_{n \longrightarrow \infty} ||T^n x||^{\frac{1}{n}} = 0\}$$

and

$$K(T) = \{x \in \mathcal{X} : \text{there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \text{ for which } x = x_0, T(x_{n+1}) = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, ...\}.$$

We note that $H_0(T)$ and $K(T-\lambda)$ are (generally) non-closed hyperinvariant subspaces of T such that $(T)^{-q}(0) \subseteq H_0(T)$ for all $q = 0, 1, 2, \ldots$ and $TK(T) = K(T)$ [9]. An operator $T \in B(\mathcal{X})$ has a (*generalized Kato*) *decomposition* at every isolated point λ of $\sigma(T)$, $\lambda \in \text{iso}\sigma(T)$, namely $\mathcal{X} = H_0(T-\lambda) \oplus K(T-\lambda)$ [9]. Observe that $H_0(T-\lambda) = \chi_T(\lambda)$. The ascent of T, asc(T), is the least non-negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ and the descent of T, dsc(T), is the least non-negative integer n such that $T^n(\mathcal{X}) = T^{n+1}(\mathcal{X})$; if no such integer n exists, then T is said to have infinite ascent/descent.

An operator $T \in B(\mathcal{X})$ is polar at $\lambda \in \text{iso}\sigma(T)$ if $\text{asc}(T-\lambda) = \text{dsc}(T-\lambda) < \infty$; T is said to be polaroid if T is polar at every $\lambda \in \text{iso}\sigma(T)$. The polaroid property is not preserved under asymptotic intertwining by I, even quasinilpotent equivalence. Thus, if $A = 0$ and B is the weighted forward unilateral shift

$$B(x_1, x_2, \ldots) = (0, \frac{x_1}{2}, \frac{x_2}{3}, \ldots), \ \ (x_n) \in \ell^2(\mathbb{N}),$$

then A is polaroid, the operator B (being non-nilpotent quasinilpotent) is not polaroid and $\lim_{n\to\infty} ||\Delta^n_{AB}(I)||^{\frac{1}{n}} = \lim_{n\to\infty} ||\Delta^n_{BA}(I)||^{\frac{1}{n}} = 0$. (Also see [3, Example 3.17].) However, if $\Delta^n_{AB}(I) = 0$ for some finite n and $\text{iso}\sigma(B) \subseteq \text{iso}\sigma(A)$, then the polaroid property transfers from A to B. Let $\sigma_a(T) = \{\lambda \in \mathbb{C} : T-\lambda$ is not bounded below$\}$ denote the approximate point spectrum of T. Recall that T is said to be left polar at $\lambda \in \text{iso}\sigma_a(T)$ if $\text{asc}(T-\lambda) = d < \infty$ and $(T-\lambda)^{d+1}\mathcal{X}$ is closed; T is finitely left polar at λ if T is left polar at λ and $\alpha(T-\lambda) < \infty$. T is finitely left polaroid (finitely polaroid) if it is finitely left polar (resp., finitely polar) at every $\lambda \in \text{iso}\sigma_a(T)$ (resp., $\lambda \in \text{iso}\sigma(T)$).

Theorem 2.3. *Let $A, B \in B(\mathcal{X})$.*
(a). If $\Delta^n_{AB}(I) = 0$ for some integer $n \geq 1$ and iso$\sigma(B) \subseteq$ iso$\sigma(A)$, then A polaroid implies B polaroid.
(b). If $\Delta^n_{AB}(I) = 0$ for some integer $n \geq 1$ and iso$\sigma_a(B) \subseteq$ iso$\sigma_a(A)$, then A finitely left polaroid implies B finitely polaroid.

Proof. We start by proving that if $\Delta^n_{AB}(I) = 0$ and $H_0(A-\lambda) = (A-\lambda)^{-p}(0)$ for some integer $p \geq 1$ at a point $\lambda \in \sigma(A) \cap \sigma(B)$, then $H_0(B-\lambda) = (B-\lambda)^{-q}(0)$ for some integer $q \geq 1$. Since $\Delta^n_{AB}(I) = 0 \Longleftrightarrow \Delta^n_{(A-\lambda)(B-\lambda)}(I) = 0$ for every λ, we may assume that $\lambda = 0$. The hypothesis $\Delta^n_{AB}(I) = 0$ implies the inclusion $H_0(B) \subseteq H_0(A)$ [8, Corollary 3.4.5]. Let $H_0(A) = A^{-p}(0)$ for some integer $p \geq 1$. Assume without loss of generality that $n = p+s$, for some integer $s \geq 1$. Observe that if $x \in H_0(B)$, then $B^t x \in H_0(B)$ for all integers $t \geq 1$. Let $x \in H_0(B)$. Then $\Delta^n_{AB}(I) = 0$ implies

$$\sum_{i=s+1}^{n} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} A^{n-i} B^i x = 0.$$

Multiplying by A^{n-1} on the left, we have $A^{p-1}B^nx = 0$, and hence upon multiplying $\Delta^n_{AB}(I) = 0$ by A^{p-2} on the left and by B on the right

$$\begin{aligned} & \sum_{i=0}^{n} (-1)^i \binom{n}{i} A^{n+p-2-i}B^{i+1}x = 0 \\ \Longrightarrow \quad & \sum_{i=s+1}^{n} (-1)^i \binom{n}{i} A^{n+p-2-i}B^{i+1}x = 0 \\ \Longrightarrow \quad & \{(-1)^{n-1}nA^{p-1}B^n + (-1)^n A^{p-2}B^{n+1}\}x = 0 \\ \Longrightarrow \quad & A^{p-2}B^{n+1}x = 0 \end{aligned}$$

for all $x \in H_0(B)$. Continuing in this fashion we have eventually that $B^{n+p-1}x = 0$ for every $x \in H_0(B)$. Since the inclusion $B^{-t}(0) \subseteq H_0(B)$ holds for every operator B and integer $t \geq 0$, we conclude that $H_0(B) = (B^{n+p-1})^{-1}(0)$, and so B has finite ascent at 0.

(a). The hypothesis iso$\sigma(B) \subseteq$ iso$\sigma(A)$ implies that $A - \lambda$ is polar at every $\lambda \in$ iso$\sigma(B)$. In particular $H_0(A-\lambda) = (A-\lambda)^{-p}(0)$ for some integer $p \geq 1$, and hence (asc$(B-\lambda) < \infty$ and) there exists an integer $q \geq 1$ such that $H_0(B-\lambda) = (B-\lambda)^{-q}(0)$. The point λ being isolated in $\sigma(B)$,

$$\begin{aligned} & \mathcal{X} = H_0(B-\lambda) \oplus K(B-\lambda) = (B-\lambda)^{-q}(0) \oplus K(B-\lambda) \\ \Longrightarrow \quad & (B-\lambda)^q\mathcal{X} = 0 \oplus (B-\lambda)^q K(B-\lambda) = K(B-\lambda) \\ \Longrightarrow \quad & \mathcal{X} = (B-\lambda)^{-q}(0) \oplus (B-\lambda)^q)\mathcal{X}. \end{aligned}$$

Thus B is polar at every $\lambda \in$ iso$\sigma(B)$.

(b). If A is left polar at 0 with asc$(A) = q < \infty$, then $0 \in$ iso$\sigma_a(A)$, ($A^{q+1}\mathcal{X}$ closed implies) $A^q\mathcal{X}$ is closed, $A_{(q)} = A|_{A^q\mathcal{X}}$ is upper semi Fredholm and asc$(A_{(q)}) = q < \infty$. Hence there exists an integer $p \geq 1$ such that $H_0(A) = A^{-p}(0)$ [2, Theorem 2.3]. Since $\lambda \in$ iso$\sigma(B) \Longrightarrow \lambda \in$ iso$\sigma_a(B) \Longrightarrow \lambda \in$ iso$\sigma_a(A)$, asc$(B-\lambda) < \infty$ and $H_0(B-\lambda) = (B-\lambda)^{-t}(0)$, for some integer $t \geq 1$, at every $\lambda \in$ iso$\sigma(B)$. This, as in part (a), implies B is polar at λ. Suppose now that A is finitely left polaroid. Then, since $H_0(B-\lambda) \subseteq H_0(A-\lambda)$, we have also that $\dim \mathrm{H_0(B-\lambda)} < \infty$. Hence B is finitely polaroid. □

It is evident from the example of the operators $A = 0$ and $B = Q$ a non-nilpotent quasinilpotent that asymptotic intertwining by the identity operator does not preserve finite ascent. The preservation of the finite ascent property under finite intertwining by the identity is proved in [7], Lemma 2.2 and Theorem 2.3, for Hilbert space operators: the argument of [7] works just as well for Banach space operators A and B. Apparently, $\triangle^n_{AB}(I) = 0 \Longrightarrow \triangle^n_{B^*A^*}(I^*) = 0$, and so if asc$(B^* - \lambda I^*) < \infty$ and $\lambda \in \sigma(A)$ then asc$(A^* - \lambda I^*) < \infty$. Since B is polar at $\lambda \in$ iso$\sigma(B)$ if and only if B^* is polar at λ, it follows that if $\triangle^n_{AB}(I) = 0$ and $\sigma(A) = \sigma(B)$ then A is polaroid if and only if B is polaroid.

The *Drazin spectrum* $\sigma_D(T)$ of $T \in B(\mathcal{X})$ is the set $\{\lambda \in \sigma(T) : \text{asc}(T-\lambda)$ or $\text{dsc}(T-\lambda) \not< \infty\}$.

Corollary 2.4. *Let $A, B \in B(\mathcal{X})$. If $\Delta_{AB}^n(I) = 0$ for some integer $n \geq 1$ and $\sigma(A) = \sigma(B)$, then $\sigma_D(A) = \sigma_D(B)$.*

Proof. Theorem 2.3 implies that A and B have the same poles. Since the Drazin spectrum is the complement of the set of poles in the spectrum, the proof follows. □

The Drazin spectrum $\sigma_D(T)$ is a regularity [10, page 50] and so satisfies the spectral mapping theorem for every $f \in \mathcal{H}_{nc}(\sigma(T))$, where $\mathcal{H}_{nc}(\sigma(T))$ is the set of functions which are holomorphic on a neighbourhood of $\sigma(T)$ and non-constant on each component of their domain of definition. It is straightforward to see that if $f \in H_{nc}(\sigma(T))$ and $\lambda \in \text{iso}\sigma(f(T)) = \text{iso}f(\sigma(T))$, then $\lambda \in f(\text{iso}\sigma(T))$. Hence, if T is polaroid, then $f(T)$ is polaroid for every $f \in H_{nc}(\sigma(T))$. Recall, [4], that T is said to satisfy Weyl's theorem if the complement of the Weyl spectrum $\sigma_w(T)$ of T in $\sigma(T)$ is set of finite multiplicity isolated eigenvalues of T: a necessary and sufficient condition for T to satisfy Weyl's theorem is that T has SVEP at points $\lambda \notin \sigma_w(T)$ and is polaroid at points $\lambda \in \text{iso}\sigma(T)$ such that $0 < \alpha(T - \lambda) < \infty$ [4, Theorem 4.3]. Hence:

Corollary 2.5. *If A has SVEP and either of the hypotheses (a) and (b) of Theorem 2.3 is satisfied, then $f(B)$ satisfies Weyl's theorem for every $f \in \mathcal{H}_{nc}(\sigma(B))$.*

Proof. $f(B)$ has SVEP (since B has SVEP [1, Theorem 2.39]) and is polaroid (as seen above). □

The following theorem provides a sufficient condition for the permanence of the finitely polaroid property under asymptotic intertwining by the identity operator.

Theorem 2.6. *Let $A, B \in B(\mathcal{X})$. If $(A, B) \in (AI)$, $\lambda \in iso\sigma(A) \cap iso\sigma(B)$ and $\lambda \in \pi_0(A)$, then $\lambda \in \pi_0(B)$.*

Proof. The hypothesis $(A, B) \in (AI)$ implies $H_0(B - \lambda) \subseteq H_0(A - \lambda)$, and if $\lambda \in \pi_0(A)$ then $\mathrm{dimH_0(A - \lambda)} < \infty$. Thus $\mathrm{dimH_0(B - \lambda)} < \infty$, and this since $(B - \lambda)^{-t}(0) \subseteq H_0(B - \lambda)$ for every integer $t \geq 0$ implies that $\alpha(B - \lambda) < \infty$. The hypothesis $\lambda \in \text{iso}\sigma(B)$ implies $\mathcal{X} = H_0(B - \lambda) \oplus K(B - \lambda)$, where both $H_0(B - \lambda)$ and $K(B - \lambda)$ are closed. Obviously,

$$(B - \lambda)\mathcal{X} = (B - \lambda)H_0(B - \lambda) \oplus (B - \lambda)K(B - \lambda)$$

being the sum of a closed subspace with a finite dimensional subspace is closed; hence $B - \lambda \in \Phi_+(\mathcal{X})$. Observe that $\lambda \in \text{iso}\sigma(B)$ implies both B and B^* have SVEP at λ; hence B is polar at λ [1, Theorem 3.77]. Since $\mathrm{dimH_0(B - \lambda)} < \infty$, $\lambda \in \pi_0(B)$. □

An operator $T \in B(\mathcal{X})$ is said to satisfy the "abstract shift condition" if the hyper-range $T^\infty\mathcal{X} = \bigcap_{n\in\mathbb{N}} T^n\mathcal{X} = \{0\}$. If we let

$$\kappa(T) := \inf\{||\mathrm{Tx}|| : \mathrm{x} \in \mathcal{X}, ||\mathrm{x}|| = 1\}, \;\; \iota(\mathrm{T}) = \mathrm{lim_{n\to\infty}}\kappa(\mathrm{T^n})^{\frac{1}{n}} = \mathrm{sup_{n\in\mathbb{N}}}\kappa(\mathrm{T^n})^{\frac{1}{n}}$$

and $\bigtriangledown(0, r) = \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$, then T satisfies the abstract shift condition implies $\bigtriangledown(0, \iota(T)) \subseteq \sigma_T(x)$ for all $0 \neq x \in \mathcal{X}$ [8, Theorem 1.6.3].

Let $\sigma_{su}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not onto}\}$ denote the surjectivity spectrum of T. The following propositions give sufficient conditions for the equality of the spectrum, certain distinguished parts thereof, and the preservation of property (δ) and condition (C) for operators $(A, B) \in (AI)$.

Proposition 2.7. *Let $(A, B) \in (AI)$; A and $B \in B(\mathcal{X})$.*

(a) Suppose that $\sigma_B(x) \subseteq \sigma_A(x)$ for every non-zero $x \in \mathcal{X}$.

(i) If A has SVEP, then $\sigma(A) = \sigma(B)$ and A satisfies property (δ) if and only if B satisfies property (δ).

(ii) A satisfies condition (C) if and only if B satisfies condition (C).

(b) If B^* has SVEP and $\sigma_A(x) = \sigma(A)$ for every non-zero $x \in \mathcal{X}$, then either A, B are quasinilpotent operators or else A, B satisfy the abstract shift condition.

Proof. (a) The hypothesis $(A, B) \in (AI)$ implies $\sigma_A(x) \subseteq \sigma_B(x)$ for every $x \in \mathcal{X}$; hence if $\sigma_B(x) \subseteq \sigma_A(x)$ for every non-zero $x \in \mathcal{X}$, then $\sigma_A(x) = \sigma_B(x)$ for every non-zero $x \in \mathcal{X}$.

(i) The hypothesis A has SVEP implies B has SVEP. Since $\sigma(T) = \sigma_{su}(T)$ whenever $T \in B(\mathcal{X})$ has SVEP, it follows from $\sigma_{su}(A) = \bigcup_{x\in\mathcal{X}} \sigma_A(x) = \bigcup_{x\in\mathcal{X}} \sigma_B(x) = \sigma_{su}(B)$ that $\sigma(A) = \sigma(B)$. Recall from [8, Lemma 3.4.7] that if $(A, B) \in (AI)$ and B satisfies property (δ), then A satisfies property (δ). (Indeed, if $(A, B) \in (AI)$, A has SVEP and B satisfies property (δ), then A and B are quasi-nilpotent equivalent [8, Corollary 3.4.5].) If, instead, A satisfies property (δ), then $\mathcal{X} = \chi_A(\overline{U}) + \chi_A(\overline{V})$ for every open cover $\{U, V\}$ of $\mathbb{C}$. The conclusion $\sigma_A(x) = \sigma_B(x)$ for every $0 \neq x \in \mathcal{X}$ implies $X_A(F) = X_B(F)$ for every closed subset $F \subseteq \mathbb{C}$. Since both A and B have SVEP, $X_A(F) = \chi_A(F) = \chi_B(F) = X_B(F)$ for every closed subset $F \subseteq \mathbb{C}$. Hence $\mathcal{X} = \chi_B(\overline{U}) + \chi_B(\overline{V})$ for every open cover $\{U, V\}$ of $\mathbb{C}$, implies B satisfies property (δ).

(ii) The conclusion $\sigma_A(x) = \sigma_B(x)$ for every $x \in \mathcal{X}$ implies $X_A(F) = X_B(F)$ for every closed subset $F \in \mathbb{C}$ [8, Corollary 3.6.4]. Evidently, $X_A(F)$ is closed if and only if $X_B(F)$ is closed; equivalently, A satisfies condition (C) if and only if B satisfies condition (C).

(b) The hypotheses $(A, B) \in (AI)$ and $\sigma_A(x) = \sigma(A)$ for all $0 \neq x \in \mathcal{X}$ imply $\sigma(A) = \sigma_A(x) \subseteq \sigma_B(x) \subseteq \sigma(B)$; again, since $(A, B) \in (AI) \Longrightarrow (B^*, A^*) \in (AI)$ and since B^* has SVEP (implies A^* has SVEP and so $\sigma(T) = \sigma(T^*) = \sigma_{su}(T^*)$ for $T = A$ or B), $\sigma(B) = \bigcup_x \sigma_{B^*}(x) \subseteq \bigcup_x \sigma_{A^*}(x) = \sigma(A)$. Thus $\sigma(A) = \sigma_A(x) = \sigma_B(x) = \sigma(B)$ for all $0 \neq x \in \mathcal{X}$, and hence $\iota(A) = r(A) = r(B) = \iota(B)$. We have two possibilities: Either $r(A) = r(B) = 0$ or $r(A) = r(B) > 0$. If $r(A) = r(B) = 0$, then A and B are quasi-nilpotent; if, instead, $r(A) = r(B) > 0$, then A and B satisfy the abstract shift condition [8, Proposition 1.6.4]. $\square$

Remark 2.8. (i) Recall from [8, Lemma 3.4.8] that property (β) transfers from A to B whenever $(A, B) \in (AI)$. Hence it follows from the argument of the proof of Proposition 2.7(a)(i) that if $(A, B) \in (AI)$ and $\sigma_B(x) \subseteq \sigma_A(x)$ for all $0 \neq x \in \mathcal{X}$, then A is decomposable if and only if B is decomposable (*cf.* [7, Theorem 3.4]). Proposition 2.7(a)(ii) generalizes [7, Theorem 3.7].

(ii) Under the hypotheses of Proposition 2.7(b), if A satisfies property (δ), then A and B are quasi-nilpotent. Reason: If $r(A) > 0$, then A satisfies $A^\infty\mathcal{X} = \{0\}$, and hence can not satisfy property (δ) [8, Theorem 1.6.3]. Observe that $(A, B) \in (AI)$

and B^* has SVEP ensures A quasi-nilpotent implies B quasi-nilpotent (*cf.* [7, Theorem 3.4]).

Although some of the hypotheses of Proposition 2.7 imply $\sigma(A) = \sigma(B)$, these hypotheses are in no way the best possible. We shall prove in the following that for operators $A, B \in B(\mathcal{X})$ such that $(A, B) \in (AI)$, the hypothesis A and B^* have SVEP is sufficient for $\sigma_\times(A) = \sigma_\times(B)$ for a variety of choices $\sigma_\times$ of some of the more distinguished parts of the spectrum σ. We shall require the following construction, known in the literature as the Sadovskii/Buoni, Harte, Wickstead construction [10, Page 159], in the proof of our next result. The construction leads to a representation of the Calkin algebra $B(\mathcal{X})/\mathcal{K}(\mathcal{X})$ as an algebra of operators on a suitable Banach space. Let $\ell^\infty(\mathcal{X})$ denote the Banach space of all bounded sequences $x = (x_n)_{n=1}^\infty$ of elements of $\mathcal{X}$ endowed with the norm $||x||_\infty := \sup_{n\in\mathbb{N}} ||x_n||$, and write T_∞, $T_\infty x := (Tx_n)_{n=1}^\infty$ for all $x = (x_n)_{n=1}^\infty$, for the operator induced by T on $\ell^\infty(\mathcal{X})$. The set $m(\mathcal{X})$ of all precompact sequences of elements of $\mathcal{X}$ is a closed subspace of $\ell^\infty(\mathcal{X})$ which is invariant for T_∞. Let $\mathcal{X}_q := \ell^\infty\mathcal{X})/m(\mathcal{X})$, and denote by T_q the operator T_∞ on $\mathcal{X}_q$. The mapping $T \mapsto T_q$ is then a unital homomorphism from $B(\mathcal{X}) \to B(\mathcal{X}_q)$ with kernel $\mathcal{K}(\mathcal{X})$ which induces a norm decreasing monomorphism from $B(\mathcal{X})/\mathcal{K}(\mathcal{X})$ to $B(\mathcal{X}_q)$ with the property that T is lower semi-Fredholm, $T \in \Phi_+(\mathcal{X})$, if and only if T_q is injective, if and only if T_q is bounded below (see [10, Section 17] for details). A part of the following theorem (for Hilbert space operators) is proved in [7, Theorem 3.1]. Let $\pi_0(T) = \{\lambda \in \text{iso}\sigma(T) : \lambda \text{ is a finite rank pole of the resolvent of } T\}$.

Proposition 2.9. *Let $A, B \in B(\mathcal{X})$ be such that $(A, B) \in (AI)$.*
(i) If A has SVEP, then

$$\sigma_{le}(B) \subseteq \sigma_{le}(A) \subseteq \sigma_e(A) = \sigma_{ue}(A) \subseteq \sigma(A) \subseteq \sigma(B).$$

(ii) If both A and B^* have SVEP, then

$$\sigma_\times(A) = \sigma_\times(B), \quad \textit{where } \sigma_\times = \sigma \ \textit{ or } \sigma_b \ \textit{ or } \sigma_w \ \textit{ or } \sigma_e \ \textit{ or } \sigma_{le} \ \textit{ or } \sigma_{re}.$$

Furthermore, $\sigma_b(X) = \sigma_w(X) = \sigma_e(X) = \sigma_{le}(X) = \sigma_{re}(X)$, where $X = A$ or B.

Proof. (i). The hypothesis $(A, B) \in (AI)$ implies $\sigma_A(x) \subseteq \sigma_B(x)$ for every $x \in \mathcal{X}$, and hence since A and B have SVEP (recall from Lemma 2.1 that $(A, B) \in (AI)$ and A has SVEP implies B has SVEP),

$$\sigma(A) = \sigma_a(A^*) = \sigma_{su}(A) = \bigcup_x \sigma_A(x) \subseteq \bigcup_x \sigma_B(x) = \sigma_{su}(B) = \sigma_a(B^*) = \sigma(B).$$

Since $(A, B) \in (AI)$ implies $((A_q, B_q) \in (AI)$ implies) $(B_q^*, A_q^*) \in (AI)$,

$$\sigma_a(B_q) = \sigma_{su}(B_q^*) \subseteq \sigma_{su}(A_q^*) = \sigma_a(A_q).$$

Now let $\lambda \notin \sigma_{le}(A)$. Then $(A-\lambda)_q = A_q - \lambda I_q$ is bounded below, and the following implications hold:

$$\lambda \notin \sigma_a(A_q) \Longrightarrow \lambda \notin \sigma_a(B_q) \Longleftrightarrow B - \lambda \in \Phi_+(\mathcal{X}) \Longleftrightarrow \lambda \notin \sigma_{le}(B).$$

Thus

$$\sigma_{le}(B) \subseteq \sigma_{le}(A) \subseteq \sigma_e(A).$$

Evidently, $\sigma_{ue}(A) \subseteq \sigma_e(A)$. Let $\lambda \notin \sigma_{ue}(A)$ ($\Longleftrightarrow A - \lambda \in \Phi_-(\mathcal{X})$). Then A has SVEP implies $\alpha(A-\lambda) \leq \beta(A-\lambda) < \infty$ [1, Corollary 3.19], and hence $\lambda \notin \sigma_e(A)$. Thus, if A has SVEP, then $\sigma_{ue}(A) = \sigma_e(A)$. This proves (i).

(ii). If A and B^* have SVEP, then $(A, B) \in (AI)$ implies A, A^*, B and B^* all have SVEP. Thus, since $(B^*, A^*) \in (AI)$ implies, $\sigma(B) = \sigma(B^*) = \sigma_{su}(B^*) = \bigcup_y \sigma_{B^*}(y) \subseteq \bigcup \sigma_{A^*}(y) = \sigma_{su}(A^*) = \sigma(A^*) = \sigma(A)$ (for every $y \in \mathcal{X}^*$), we have from $\sigma(A) \subseteq \sigma(B)$ (see (i)) that $\sigma(A) = \sigma(B)$. It is not difficult to verify that if an operator $T \in B(\mathcal{X})$ is such that both T and T^* have SVEP, then $\sigma_e(T) = \sigma_{le}(T) = \sigma_{ue}(T) = \sigma_b(T) = \sigma_w(T)$. (For example, if $\lambda \notin \sigma_{le}(T) \Longleftrightarrow T - \lambda \in \Phi_+(\mathcal{X})$, and both T and T^* have SVEP, then $\text{ind}(T-\lambda) = 0$ and $\text{asc}(T-\lambda) = \text{dsc}(T-\lambda) < \infty$, $\Longrightarrow T - \lambda$ is both Browder and Weyl; see also [1, pp. 141 - 142].) We prove that $\sigma_w(A) = \sigma_w(B)$: this would then prove the equality $\sigma_\times(A) = \sigma_\times(B)$ of (ii). The property that A and B have SVEP implies A and B satisfy Browder's theorem [4, Corollary 3.5], i.e. $\sigma(A) \setminus \sigma_w(A) = \pi_0(A)$ and $\sigma(B) \setminus \sigma_w(B) = \pi_0(B)$. Let $\lambda \notin \sigma_w(A)$; then $\lambda \in \text{iso}\sigma(A) = \text{iso}\sigma(B)$ is a finite rank pole (of the resolvent) of A. Hence, see Theorem 2.6, λ is a finite rank pole of B, implies $\lambda \notin \sigma_w(B)$. Since the same argument works with $(B^*, A^*) \in (AI)$, we have $\lambda \notin \sigma_w(B^*) \Longrightarrow \lambda \notin \sigma_w(A^*)$. Hence $\sigma_w(A) = \sigma_w(B)$. (We remark here that an operator T satisfies Browder's theorem if and only if T^* satisfies Browder's theorem [4, Remark 3.2]; since $\sigma(T) = \sigma(T^*)$ and $\sigma_w(T) = \sigma_w(T^*)$, we then have $\pi_0(T) = \pi_0(T^*)$.) □

Corollary 2.10. *Let $(A, B) \in (AI)$, where $A, B \in B(\mathcal{X})$. If $\sigma(B)$ is totally disconnected, then a necessary and sufficient condition for $\sigma(A)$ to be totally disconnected is that A has SVEP.*

Proof. If $\sigma(A)$ is totally disconnected, then A is super-decomposable [8, Proposition 1.4.5] and so both A and A^* have SVEP. Conversely, if $\sigma(B)$ is totally disconnected and A has SVEP, then $\sigma(A) = \sigma(B)$ (by Proposition 2.9). □

Corollary 2.10 implies that if A has SVEP and B is algebraic (i.e., there exists a non-trivial polynomial $p(.)$ such that $p(B) = 0$), then $\sigma(A) = \sigma(B)$ is a finite set. Observe that B algebraic implies that the points $\lambda \in \sigma(B)$ are poles of the resolvent of B. In particular, $\sigma_b(B) = \emptyset$; hence, see Theorem 2.9(ii), $\sigma_b(A) = \emptyset$ and A is algebraic (*cf.* [7, Proposition 3.6]).

Remark 2.11. Let $(A, B) \in (AI)$. If $\sigma_A(x) = \sigma(A)$ for every non-zero $x \in \mathcal{X}$, then A satisfies Dunford's condition (C) [8, page 83], and so has SVEP. Hence, if also B^* has SVEP, then $\sigma(A) = \sigma(B)$ (by Proposition 2.9(ii)) and $\sigma_A(x) = \sigma_B(x) = \sigma(B)$ for every non-zero $x \in \mathcal{X}$. Consequently, B also satisfies condition (C).

References

1. P. Aiena, *Fredholm and Local Spectral Theory with Applications to Multipliers*, Kluwer, 2004.
2. P. Aiena, *Quasi-Fredholm operators and localized SVEP*, Acta Sci. Math. (Szeged) **73** (2007), 251–263.
3. P. Aiena, M. Chō and M. González, *Polaroid type operators under quasi-affinities*, J. Math. Anal. Appl. **371** (2010), 485–495.

4. B.P. Duggal, *SVEP, Browder and Weyl Theorems*, Tópicas de Theoría de la Approximación III, Editors: M.A. Jiménez P., J. Bustamante G. and S.V. Djordjević, Textos Cientifficos BUAP Puebla (2009), 107–146. (Available at http : //www.fcfm.buap.mx/CA/analysis − mat/pdf/LIBRO_TOP_T_APPROX.pdf)
5. B.P. Duggal, *Finite intertwinings and subscalarity*, Oper. Matrices **4** (2010), 257–271.
6. H.G. Heuser, *Functional Analysis*, John Wiley and Sons, 1982.
7. I. Kim, Y. Kim, E. Ko and J.E. Lee, *Inherited properties through the Helton class of an operator*, Bull. Korean Math. Soc. **48** (2011), 183–195.
8. K.B. Laursen and M.N. Neumann, *Introduction to Local Spectral Theory*, Clarendon Press, Oxford 2000.
9. M. Mbekhta, *Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux*, Glasgow Math. J. **29** (1987), 159–175.
10. V. Müller, *Spectral Theory of Linear Operators, Operator Theory Advances and Applications*, 139, Birkhäuser Verlag , 2003.

Redwood Grove, Northfield Avenue, London W5 4SZ, England, U.K.
E-mail address: bpduggal@yahoo.co.uk