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BOUNDS FOR THE RATIO OF TWO GAMMA FUNCTIONS—FROM WENDEL'S AND RELATED INEQUALITIES TO LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS

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Abstract. In the survey paper, along one of several main lines of bounding the ratio of two gamma functions, the authors retrospect and analyse Wendel's double inequality, Kazarinoff's refinement of Wallis' formula, Watson's monotonicity, Gautschi's double inequality, Kershaw's first double inequality, and the (logarithmically) complete monotonicity results of functions involving ratios of two gamma or q-gamma functions obtained by Bustoz, Ismail, Lorch, Muldoon, and other mathematicians.

1. Preliminaries

In this paper, we need some definitions, notions, and notations below.

1.1. **The gamma and q-gamma functions.** It is well known that the classical Euler's gamma function may be defined for $\operatorname{Re} z > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

The logarithmic derivative of $\Gamma(z)$, denoted by $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, is called psi or digamma function, and $\psi^{(k)}(z)$ for $k \in \mathbb{N}$ are called polygamma functions. It is

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common knowledge that special functions $\Gamma(z)$, $\psi(z)$ and $\psi^{(k)}(z)$ for $k \in \mathbb{N}$ are fundamental and important and have much extensive applications in mathematical sciences.

The q-analogues of Γ and ψ are defined [4, pp. 493–496] for $x > 0$ by

$$\Gamma_q(x) = \begin{cases} (1-q)^{1-x} \displaystyle\prod_{i=0}^{\infty} \frac{1-q^{i+1}}{1-q^{i+x}}, & 0 < q < 1 \\ (q-1)^{1-x} q^{\binom{x}{2}} \displaystyle\prod_{i=0}^{\infty} \frac{1-q^{-(i+1)}}{1-q^{-(i+x)}}, & q > 1 \end{cases} \tag{1.1}$$

and

$$\psi_q(x) = \frac{\Gamma_q'(x)}{\Gamma_q(x)} = -\ln(1-q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k+x}}{1-q^{k+x}} \tag{1.2}$$

$$= -\ln(1-q) - \int_0^{\infty} \frac{e^{-xt}}{1-e^{-t}} d\mu_q(t) \tag{1.3}$$

for $0 < q < 1$, where $d\mu_q(t)$ is a discrete measure with positive masses $-\ln q$ at the positive points $-k \ln q$ for $k \in \mathbb{N}$, more accurately,

$$\mu_q(t) = \begin{cases} -\ln q \displaystyle\sum_{k=1}^{\infty} \delta(t + k \ln q), & 0 < q < 1, \\ t, & q = 1, \end{cases}$$

where δ denotes the Dirac delta function. See [22, p. 311]. The q-gamma function $\Gamma_q(z)$ meets

$$\lim_{q \to 1^+} \Gamma_q(z) = \lim_{q \to 1^-} \Gamma_q(z) = \Gamma(z)$$

and

$$\Gamma_q(x) = q^{\binom{x-1}{2}} \Gamma_{1/q}(x).$$

1.2. **Definition and properties of completely monotonic functions.** A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $0 \le (-1)^n f^{(n)}(x) < \infty$ for $x \in I$ and $n \ge 0$.

The class of completely monotonic functions has the following basic properties.

Theorem 1.1 ([52, p. 161]). *A necessary and sufficient condition that $f(x)$ should be completely monotonic for $0 < x < \infty$ is that*

$$f(x) = \int_0^{\infty} e^{-xt} d\alpha(t), \tag{1.4}$$

where $\alpha(t)$ is nondecreasing and the integral converges for $0 < x < \infty$.

Theorem 1.2 ([8, p. 83]). *If $f(x)$ is completely monotonic on I, $g(x) \in I$, and $g'(x)$ is completely monotonic on $(0, \infty)$, then $f(g(x))$ is completely monotonic on $(0, \infty)$.*

1.3. **The logarithmically completely monotonic functions.** A positive and k-times differentiable function $f(x)$ is said to be k-log-convex (or k-log-concave, respectively) with $k \ge 2$ on an interval I if and only if $[\ln f(x)]^{(k)}$ exists and $[\ln f(x)]^{(k)} \ge 0$ (or $[\ln f(x)]^{(k)} \le 0$, respectively) on I.

A positive function $f(x)$ is said to be logarithmically completely monotonic on an interval $I \subseteq \mathbb{R}$ if it has derivatives of all orders on I and its logarithm $\ln f(x)$ satisfies $0 \le (-1)^k[\ln f(x)]^{(k)} < \infty$ for $k \in \mathbb{N}$ on I.

The notion "logarithmically completely monotonic function" was first put forward in [5] without an explicit definition. This terminology was explicitly recovered in the preprints of [38, 41, 43]. It has been proved once and again in the papers [7, 17, 38] that any logarithmically completely monotonic function on an interval I must be completely monotonic on I. It was also pointed out in [7] that logarithmically completely monotonic functions are the same as those studied by Horn [20] under the name "infinitely divisible completely monotonic functions". For more information, please refer to [7], [45, pp. 2154–2155, Remark 8], [47, pp. 41–42, Remark 4.7] and plenty of references therein.

Recently a new concept and terminology "completely monotonic degree" was naturally introduced and initially studied in [16].

1.4. **Outline of this paper.** The history of bounding the ratio of two gamma functions is longer than six decades since the paper [51] was published in 1948.

The motivations to bound the ratio of two gamma functions are diverse, including, for example, establishment of asymptotic relation, refinement of Wallis' formula, approximation of π, and some needs in statistics and other mathematical sciences.

In this survey paper, along one of several main lines of bounding the ratio of two gamma functions, we review and analyse Wendel's double inequality, Kazarinoff's refinement of Wallis' formula, Watson's monotonicity, Gautschi's double inequality, Kershaw's first double inequality, and the complete monotonicity and logarithmically complete monotonicity results of functions involving ratios of two gamma or q-gamma functions by Bustoz, Ismail, Lorch, Muldoon, and other mathematicians.

2. Inequalities for bounding the ratio of two gamma functions

In this section, we look back and analyse some related inequalities for bounding the ratio of two gamma functions.

2.1. **Wendel's double inequality.** Our starting point is the paper [51], which is the earliest one we can search out to the best of our ability.

In order to establish the classical asymptotic relation

$$\lim_{x\to\infty} \frac{\Gamma(x+s)}{x^s\Gamma(x)} = 1 \tag{2.1}$$

for real numbers s and x, J. G. Wendel [51] elegantly proved the double inequality

$$\left(\frac{x}{x+s}\right)^{1-s} \le \frac{\Gamma(x+s)}{x^s\Gamma(x)} \le 1 \tag{2.2}$$

for $0 < s < 1$ and $x > 0$.

Proof of (2.1) *and* (2.2) *by Wendel.* Let $0 < s < 1$, $f(t) = e^{-(1-s)t}t^{(1-s)x+s-1}$, $g(t) = e^{-st}t^{sx}$, $p = \frac{p}{p-1} = \frac{1}{1-s}$, and $p = \frac{1}{s}$. Applying Hölder's inequality

$$\int_0^\infty |f(t)g(t)|dt \le \left[\int_0^\infty |f(t)|^p dt\right]^{1/p} \left[\int_0^\infty |g(t)|^q dt\right]^{1/q}$$

and

$$\Gamma(x+1) = x\Gamma(x) \tag{2.3}$$

leads to

$$\Gamma(x+s) = \int_0^\infty e^{-t}t^{x+s-1}dt \le \left(\int_0^\infty e^{-t}t^x dt\right)^s \left(\int_0^\infty e^{-t}t^{x-1}dt\right)^{1-s}$$
$$= [\Gamma(x+1)]^s[\Gamma(x)]^{1-s} = x^s\Gamma(x). \tag{2.4}$$

Replacing s by $1-s$ in (2.4) results in $\Gamma(x+1-s) \le x^{1-s}\Gamma(x)$, from which we obtain

$$\Gamma(x+1) \le (x+s)^{1-s}\Gamma(x+s), \tag{2.5}$$

by substituting $x+s$ for x. Combining (2.4) and (2.5) yields

$$\frac{x}{(x+s)^{1-s}}\Gamma(x) \le \Gamma(x+s) \le x^s\Gamma(x).$$

Therefore, the inequality (2.2) follows.

Letting x tend to infinity in (2.2) yields (2.1) for $0 < s < 1$. The extension to all real s is immediate on repeated application of (2.3). □

Remark 2.1. The inequality (2.2) can be rewritten for $0 < s < 1$ and $x > 0$ as

$$(x+s)^{s-1}\frac{\Gamma(x+1)}{\Gamma(x+s)} \le 1 \le x^{s-1}\frac{\Gamma(x+1)}{\Gamma(x+s)}.$$

Utilizing the relation (2.1) results in

$$\lim_{x\to\infty}(x+s)^{s-1}\frac{\Gamma(x+1)}{\Gamma(x+s)} = \lim_{x\to\infty}x^{s-1}\frac{\Gamma(x+1)}{\Gamma(x+s)} = 1 \tag{2.6}$$

which hints us that the functions

$$(x+s)^{s-1}\frac{\Gamma(x+1)}{\Gamma(x+s)} \quad \text{and} \quad x^{s-1}\frac{\Gamma(x+1)}{\Gamma(x+s)}, \tag{2.7}$$

or

$$(x+s)\left[\frac{\Gamma(x+1)}{\Gamma(x+s)}\right]^{1/(s-1)} \quad \text{and} \quad x\left[\frac{\Gamma(x+1)}{\Gamma(x+s)}\right]^{1/(s-1)},$$

are possibly increasing and decreasing in x respectively.

Remark 2.2. In [1, p. 257, 6.1.46], the limit

$$\lim_{x\to\infty}\left[x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)}\right] = 1 \tag{2.8}$$

for real numbers a and b was listed. Since

$$x^{t-s}\frac{\Gamma(x+s)}{\Gamma(x+t)}=\frac{\Gamma(x+s)}{x^s\Gamma(x)}\cdot\frac{x^t\Gamma(x)}{\Gamma(x+t)},$$

the limits (2.1) and (2.8) are equivalent to each other. Hence, the limit (2.8) should be presumedly called Wendel's limit in the literature.

Remark 2.3. Due to unknown reasons, Wendel's paper [51] was seemingly neglected by nearly all mathematicians for more than fifty years, until it was mentioned in [31], to the best of our knowledge.

2.2. **Kazarinoff's double inequality.** Stimulated by

$$\frac{1}{\sqrt{\pi(n+1/2)}}<\frac{(2n-1)!!}{(2n)!!}<\frac{1}{\sqrt{\pi n}},\quad n\in\mathbb{N},\tag{2.9}$$

one form of the celebrated formula of John Wallis, which had been quoted for more than a century before 1950s by writers of textbooks, D. K. Kazarinoff proved in [23] that the sequence $\theta(n)$ defined by

$$\frac{(2n-1)!!}{(2n)!!}=\frac{1}{\sqrt{\pi[n+\theta(n)]}}\tag{2.10}$$

satisfies $\frac14<\theta(n)<\frac12$ for $n\in\mathbb{N}$, that is,

$$\frac{1}{\sqrt{\pi(n+1/2)}}<\frac{(2n-1)!!}{(2n)!!}<\frac{1}{\sqrt{\pi(n+1/4)}},\quad n\in\mathbb{N}.\tag{2.11}$$

Remark 2.4. It was said in [23] that it is unquestionable that inequalities similar to (2.11) can be improved indefinitely but at a sacrifice of simplicity, which is why the inequality (2.9) had survived so long.

Remark 2.5. Kazarinoff's proof of (2.11) is based upon the property

$$[\ln\phi(t)]''-\{[\ln\phi(t)]'\}^2>0,\tag{2.12}$$

where

$$\phi(t)=\int_0^{\pi/2}\sin^t x\,dx=\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma((t+1)/2)}{\Gamma((t+2)/2)}$$

for $-1<t<\infty$. The inequality (2.12) was proved by making use of Legendre's formula

$$\psi(x)=-\gamma+\int_0^1\frac{t^{x-1}-1}{t-1}dt$$

for $x>0$ and by estimating the integrals

$$\int_0^1\frac{x^t}{1+x}dx\quad\text{and}\quad\int_0^1\frac{x^t\ln x}{1+x}dx,$$

where $\gamma=0.57721566\cdots$ is Euler-Mascheroni's constant. Since (2.12) is equivalent to the statement that the reciprocal of $\phi(t)$ has an everywhere negative second derivative, therefore $\phi(t)$ for any positive t is less than the harmonic mean of $\phi(t-1)$ and $\phi(t+1)$, this implies

$$\frac{\Gamma((t+1)/2)}{\Gamma((t+2)/2)} < \frac{2}{\sqrt{2t+1}}, \quad t > -\frac{1}{2}. \tag{2.13}$$

As a subcase of this result, the right-hand side inequality in (2.11) is established.

Remark 2.6. Replacing t by $2t$ in (2.13) and rearranging yield

$$\frac{\Gamma(t+1)}{\Gamma(t+1/2)} > \sqrt{t+\frac{1}{4}} \quad \Longleftrightarrow \quad \left(t+\frac{1}{4}\right)^{1/2-1} \frac{\Gamma(t+1)}{\Gamma(t+1/2)} > 1$$

for $t > -\frac{1}{4}$. From (2.1), it follows that

$$\lim_{x\to\infty}\left[\left(t+\frac{1}{4}\right)^{1/2-1} \frac{\Gamma(t+1)}{\Gamma(t+1/2)}\right] = 1.$$

This suggests that the function

$$\left(t+\frac{1}{4}\right)^{1/2-1} \frac{\Gamma(t+1)}{\Gamma(t+1/2)} \quad \text{or} \quad \left(t+\frac{1}{4}\right)\left[\frac{\Gamma(t+1)}{\Gamma(t+1/2)}\right]^{1/(1/2-1)}$$

is perhaps decreasing, more strongly, logarithmically completely monotonic.

Remark 2.7. The inequality (2.12) may be reorganized as

$$\psi'\left(\frac{t+1}{2}\right) - \psi'\left(\frac{t+2}{2}\right) > \left[\psi\left(\frac{t+1}{2}\right) - \psi\left(\frac{t+2}{2}\right)\right]^2$$

for $t > -1$. This inequality is a special case of the complete monotonicity of a function involving divided differences of the di- and tri-gamma functions, which may be recited as the following theorem.

Theorem 2.8 ([15, Theorem 1] and [39, Theorem 1.2])**.** *Let s and t be real numbers and $\alpha = \min\{s,t\}$. Then the function*

$$\Delta_{s,t}(x) = \begin{cases} \left[\dfrac{\psi(x+t)-\psi(x+s)}{t-s}\right]^2 + \dfrac{\psi'(x+t)-\psi'(x+s)}{t-s}, & s \ne t \\ [\psi'(x+s)]^2 + \psi''(x+s), & s = t \end{cases} \tag{2.14}$$

for $|t-s| < 1$ and $-\Delta_{s,t}(x)$ for $|t-s| > 1$ are completely monotonic in $x \in (-\alpha, \infty)$.

2.3. **Watson's monotonicity.** In 1959, motivated by the result in [23] mentioned in Section 2.2, G. N. Watson [49] observed that

$$\begin{aligned} \frac{1}{x}\cdot\frac{[\Gamma(x+1)]^2}{[\Gamma(x+1/2)]^2} &= {}_2F_1\left(-\frac{1}{2},-\frac{1}{2};x;1\right) \\ &= 1 + \frac{1}{4x} + \frac{1}{32x(x+1)} + \sum_{r=3}^{\infty} \frac{[(-1/2)\cdot(1/2)\cdot(3/2)\cdot(r-3/2)]^2}{r!x(x+1)\cdots(x+r-1)} \end{aligned} \tag{2.15}$$

for $x > -\frac{1}{2}$, which implies that the much general function

$$\theta(x) = \left[\frac{\Gamma(x+1)}{\Gamma(x+1/2)}\right]^2 - x \tag{2.16}$$

for $x > -\frac{1}{2}$, whose special case is the sequence $\theta(n)$ for $n \in \mathbb{N}$ defined in (2.10), is decreasing and satisfies

$$\lim_{x\to\infty} \theta(x) = \frac{1}{4} \quad \text{and} \quad \lim_{x\to(-1/2)^+} \theta(x) = \frac{1}{2}.$$

This implies apparently the sharp inequalities

$$\frac{1}{4} < \theta(x) < \frac{1}{2} \tag{2.17}$$

for $x > -\frac{1}{2}$,

$$\sqrt{x+\frac{1}{4}} < \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \le \sqrt{x+\frac{1}{4}+\left[\frac{\Gamma(3/4)}{\Gamma(1/4)}\right]^2} = \sqrt{x+0.36423\cdots} \tag{2.18}$$

for $x \ge -\frac{1}{4}$, and, by using Wallis cosine formula in [50],

$$\frac{1}{\sqrt{\pi(n+4/\pi-1)}} \le \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n+1/4)}}, \quad n \in \mathbb{N}. \tag{2.19}$$

Remark 2.9. In [49], an alternative proof of the double inequality (2.17) was also provided.

Remark 2.10. It is clear that the inequality (2.18) extends and improves (2.2) for $s = \frac{1}{2}$.

Remark 2.11. The left-hand side inequality in (2.19) is better than the corresponding one in (2.11).

Remark 2.12. The formula (2.15) implies complete monotonicity of the function $\theta(x)$ defined by (2.16) on $\left(-\frac{1}{2}, \infty\right)$.

2.4. **Gautschi's double inequalities.** The first result of the paper [13] was the double inequality

$$\frac{(x^p+2)^{1/p} - x}{2} < e^{x^p}\int_x^\infty e^{-t^p}\,dt \le c_p\left[\left(x^p + \frac{1}{c_p}\right)^{1/p} - x\right] \tag{2.20}$$

for $x \ge 0$ and $p > 1$, where

$$c_p = \left[\Gamma\left(1+\frac{1}{p}\right)\right]^{p/(p-1)}$$

or $c_p = 1$. By an easy transformation, the inequality (2.20) was written in terms of the complementary gamma function

$$\Gamma(a,x) = \int_x^\infty e^{-t}t^{a-1}\,dt$$

as

$$\frac{p[(x+2)^{1/p}-x^{1/p}]}{2}<e^x\Gamma\left(\frac{1}{p},x\right)\le pc_p\left[\left(x+\frac{1}{c_p}\right)^{1/p}-x^{1/p}\right] \tag{2.21}$$

for $x \ge 0$ and $p > 1$. In particular, if letting $p \to \infty$, the double inequality

$$\frac{1}{2}\ln\left(1+\frac{2}{x}\right)\le e^xE_1(x)\le\ln\left(1+\frac{1}{x}\right)$$

for the exponential integral $E_1(x)=\Gamma(0,x)$ with $x>0$ was derived from (2.21), in which the bounds exhibit the logarithmic singularity of $E_1(x)$ at $x=0$. As a direct consequence of the inequality (2.21) for $p=\frac{1}{s}$, $x=0$, and $c_p=1$, the following simple inequality for the gamma function was deduced:

$$2^{s-1}\le\Gamma(1+s)\le 1,\quad 0\le s\le 1. \tag{2.22}$$

The second result of the paper [13] was a sharper and more general inequality

$$e^{(s-1)\psi(n+1)}\le\frac{\Gamma(n+s)}{\Gamma(n+1)}\le n^{s-1} \tag{2.23}$$

for $0\le s\le 1$ and $n\in\mathbb{N}$ than (2.22). This inequality was obtained by proving that the function

$$f(s)=\frac{1}{1-s}\ln\frac{\Gamma(n+s)}{\Gamma(n+1)}$$

is monotonically decreasing for $0\le s<1$. Since $\psi(n)<\ln n$, it was derived from the inequality (2.23) that

$$\left(\frac{1}{n+1}\right)^{1-s}\le\frac{\Gamma(n+s)}{\Gamma(n+1)}\le\left(\frac{1}{n}\right)^{1-s},\quad 0\le s\le 1, \tag{2.24}$$

which was also rewritten as

$$\frac{n!(n+1)^{s-1}}{(s+1)(s+2)\cdots(s+n-1)}\le\Gamma(1+s)\le\frac{(n-1)!n^s}{(s+1)(s+2)\cdots(s+n-1)}, \tag{2.25}$$

and so a simple proof of Euler's product formula in the segment $0\le s\le 1$ was shown by letting $n\to\infty$ in (2.25).

Remark 2.13. The double inequalities (2.23) and (2.24) can be rearranged as

$$n^{1-s}\le\frac{\Gamma(n+1)}{\Gamma(n+s)}\le\exp((1-s)\psi(n+1)) \tag{2.26}$$

and

$$n^{1-s}\le\frac{\Gamma(n+1)}{\Gamma(n+s)}\le(n+1)^{1-s} \tag{2.27}$$

for $n\in\mathbb{N}$ and $0\le s\le 1$. Furthermore, the inequality (2.27) can be rewritten as

$$n^{1-s}\frac{\Gamma(n+s)}{\Gamma(n+1)}\le 1\le(n+1)^{1-s}\frac{\Gamma(n+s)}{\Gamma(n+1)} \tag{2.28}$$

or

$$n\left[\frac{\Gamma(n+s)}{\Gamma(n+1)}\right]^{1/(1-s)}\le 1\le(n+1)\left[\frac{\Gamma(n+s)}{\Gamma(n+1)}\right]^{1/(1-s)}. \tag{2.29}$$

This supplies us some possible clues to see that the sequences at the very ends of the inequalities (2.28) and (2.29) are monotonic.

Remark 2.14. The left-hand side inequality in (2.2) and the upper bound in (2.26) have the following relations:

(1) For $0 \le s \le \frac{1}{2}$ and $n \in \mathbb{N}$,
$$(n+s)^{1-s} \le \exp((1-s)\psi(n+1)); \tag{2.30}$$
(2) For $s > e^{1-\gamma} - 1 = 0.52620\cdots$, the inequality (2.30) reverses.

These relations can be derived from the decreasing monotonicity of the function
$$Q(x) = e^{\psi(x+1)} - x$$
on $(-1, \infty)$ and the limit
$$\lim_{x\to\infty} Q(x) = \frac{1}{2}.$$
Hence, Wendel's double inequality (2.2) and Gautschi's first double inequality (2.26) are not included each other, but both of them contain Gautschi's second double inequality (2.27).

Remark 2.15. In reviews on the paper [13] by Mathematical Reviews and Zentralblatt MATH, there is no a word to comment on inequalities in (2.26) and (2.27). However, these two double inequalities later became a major source and origin of a large amount of study on bounding the ratio of two gamma functions.

2.5. **Kershaw's first double inequality.** Inspired by the inequality (2.24), among other things, D. Kershaw presented in [24] the following double inequality
$$\left(x + \frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left[x - \frac{1}{2} + \left(s + \frac{1}{4}\right)^{1/2}\right]^{1-s} \tag{2.31}$$
for $0 < s < 1$ and $x > 0$. In the literature, it is called as Kershaw's first double inequality for the ratio of two gamma functions.

Kershaw's proof of (2.31). Define the function g_β by
$$g_\beta(x) = \frac{\Gamma(x+1)}{\Gamma(x+s)}(x+\beta)^{s-1}$$
for $x > 0$ and $0 < s < 1$, where the parameter β is to be determined. It is not difficult to show, with the aid of (2.1), that $\lim_{x\to\infty} g_\beta(x) = 1$. Define
$$G(x) = \frac{g_\beta(x)}{g_\beta(x+1)} = \frac{x+s}{x+1}\left(\frac{x+\beta+1}{x+\beta}\right)^{1-s}.$$
Then
$$\frac{G'(x)}{G(x)} = \frac{(1-s)[(\beta^2+\beta-s)+(2\beta-s)x]}{(x+1)(x+s)(x+\beta)(x+\beta+1)}.$$
This will leads to

(1) if $\beta = \frac{s}{2}$, then $G'(x) < 0$ for $x > 0$;

(2) if $\beta = -\frac{1}{2} + \left(s + \frac{1}{4}\right)^{1/2}$, then $G'(x) > 0$ for $x > 0$.

Further by standard arguments, the double inequality (2.31) follows. □

Remark 2.16. It is easy to see that the inequality (2.31) refines and extends inequalities (2.2) and (2.27).

Remark 2.17. The inequality (2.31) may be rearranged as

$$\left[x-\frac{1}{2}+\left(s+\frac{1}{4}\right)^{1/2}\right]^{s-1}\frac{\Gamma(x+1)}{\Gamma(x+s)}<1<\left(x+\frac{s}{2}\right)^{s-1}\frac{\Gamma(x+1)}{\Gamma(x+s)}$$

for $x>0$ and $0<s<1$. By virtue of (2.1) or (2.8), it is easy to see that

$$\lim_{x\to\infty}\left[x-\frac{1}{2}+\left(s+\frac{1}{4}\right)^{1/2}\right]^{s-1}\frac{\Gamma(x+1)}{\Gamma(x+s)}=\lim_{x\to\infty}\left(x+\frac{s}{2}\right)^{s-1}\frac{\Gamma(x+1)}{\Gamma(x+s)}=1.$$

This suggests us the monotonicity, more strongly, the logarithmically complete monotonicity, of the functions

$$\left[x-\frac{1}{2}+\left(s+\frac{1}{4}\right)^{1/2}\right]^{s-1}\frac{\Gamma(x+1)}{\Gamma(x+s)}\quad\text{and}\quad\left(x+\frac{s}{2}\right)^{s-1}\frac{\Gamma(x+1)}{\Gamma(x+s)}$$

or

$$\left[x-\frac{1}{2}+\left(s+\frac{1}{4}\right)^{1/2}\right]\left[\frac{\Gamma(x+1)}{\Gamma(x+s)}\right]^{1/(s-1)}\quad\text{and}\quad\left(x+\frac{s}{2}\right)\left[\frac{\Gamma(x+1)}{\Gamma(x+s)}\right]^{1/(s-1)}.$$

Remark 2.18. Some more inequalities with the type of (2.31) were constructed and applied in [25, 28, 53] and related references therein.

3. Some completely monotonic functions involving ratios of two gamma or q-gamma functions

In this section, we review and analyse complete monotonicity of functions involving ratios of two gamma or q-gamma functions.

3.1. **Ismail-Lorch-Muldoon's monotonicity results.** Motivated by work on inequalities for the ratio of two gamma functions in [24, 25, 28] and [53, p. 155], M. E. H. Ismail, L. Lorch and M. E. Muldoon pointed out at the beginning of [21] that simple monotonicity of the ratio of two gamma functions are useful.

In [33, pp. 118–119], the asymptotic formula

$$z^{b-a}\frac{\Gamma(z+a)}{\Gamma(z+b)}\sim 1+\frac{(a-b)(a+b-1)}{2z}+\cdots \tag{3.1}$$

as $z\to\infty$ along any curve joining $z=0$ and $z=\infty$ is listed, where $z\ne -a,-a-1,\dots$ and $z\ne -b,-b-1,\dots$. Suggested by (3.1), the following complete monotonicity was proved in [21, Theorem 2.4]: Let $a>b\ge 0$, $a+b\ge 1$ and

$$h(x)=\ln\left[x^{a-b}\frac{\Gamma(x+b)}{\Gamma(x+a)}\right].$$

Then both $h'(x)$ and

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} \tag{3.2}$$

are completely monotonic on $(0,\infty)$; The results fail when $a+b<1$ replaces $a+b\ge 1$ in the hypotheses.

Meanwhile, the following q-analogue of [21, Theorem 2.4] was also provided in [21, Theorem 2.5]: Let $a > b \ge 0$, $a + b \ge 1$, $q > 0$, $q \ne 1$ and

$$h_q(x) = \ln\left[|1 - q^x|^{a-b} \frac{\Gamma_q(x + b)}{\Gamma_q(x + a)}\right].$$

Then $h_q'(x)$ is completely monotonic on $(0, \infty)$; So is the function

$$|1 - q^x|^{b-a} \frac{\Gamma_q(x + a)}{\Gamma_q(x + b)}; \tag{3.3}$$

The result fails if $a + b < 1$.

Remark 3.1. The proof of [21, Theorem 2.4] can be outlined as follows: Using the integral representation

$$\psi(z) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-tz}}{1 - e^{-t}} dt \tag{3.4}$$

for $\mathrm{Re} z > 0$ yields

$$h'(x) = \int_0^\infty \left[\frac{e^{-at} - e^{-bt}}{1 - e^{-t}} + a - b\right] e^{-xt} dt. \tag{3.5}$$

It was obtained in [21, Lemma 4.1] that if $0 \le b < a$, $a+b \ge 1$, and $b^2+(a-1)^2 \ne 0$, then

$$\frac{w^b - w^a}{1 - w} < a - b, \quad 0 < w < 1; \tag{3.6}$$

the inequality (3.6) fails if the condition $a + b \ge 1$ is replaced by $a + b < 1$. Combining (3.5) and (3.6) with Theorem 1.2 on page 133 results in [21, Theorem 2.4].

The proof of [21, Theorem 2.5] was fulfilled by using the formula (1.2), the inequality (3.6), Theorem 1.1 on page 133, and Theorem 1.2 on page 133.

Remark 3.2. We remark that [21, Theorem 2.4 and Theorem 2.5] can be restated using the terminology "logarithmically completely monotonic function" as follows: The functions defined by (3.2) and (3.3) are logarithmically completely monotonic on $(0, \infty)$ if and only if $a + b \ge 1$ for $a > b \ge 0$, $q > 0$ and $q \ne 1$.

Remark 3.3. The function

$$\frac{e^{-at} - e^{-bt}}{1 - e^{-t}} \tag{3.7}$$

for $t \in (-\infty, \infty)$ in (3.5) has been researched and applied in [19] and closely related references therein. See also Section 4.1.

3.2. **Bustoz-Ismail's monotonicity results.** In [9], it was noticed that inequalities like (2.17) are "immediate consequences of the complete monotonicity of certain functions. Indeed, one should investigate monotonicity properties of functions involving quotients of gamma functions and as a by-product derive inequalities of the aforementioned type. This approach is simpler and yields more general results."

In [9], it was revealed that

(1) the function

$$\frac{1}{(x+c)^{1/2}} \cdot \frac{\Gamma(x+1)}{\Gamma(x+1/2)}, \quad x > \max\left\{-\frac{1}{2}, -c\right\} \tag{3.8}$$

is completely monotonic on $(-c,\infty)$ if $c \le \frac{1}{4}$, so is the reciprocal of (3.8) on $\left[-\frac{1}{2},\infty\right)$ if $c \ge \frac{1}{2}$; see [9, Theorem 1];

(2) the function

$$(x+c)^{a-b}\frac{\Gamma(x+b)}{\Gamma(x+a)} \tag{3.9}$$

for $1 \ge b-a > 0$ is completely monotonic on $(\max\{-a,-c\},\infty)$ if $c \le \frac{a+b-1}{2}$, so is the reciprocal of (3.9) on $(\max\{-a,-c\},\infty)$ if $c \ge a$; see [9, Theorem 3];

(3) the function

$$\frac{\Gamma(x+1)}{\Gamma(x+s)}\left(x+\frac{s}{2}\right)^{s-1} \tag{3.10}$$

for $0 \le s \le 1$ is completely monotonic on $(0,\infty)$; when $0 < s < 1$, it satisfies $(-1)^n f^{(n)}(x) > 0$ for $x > 0$; see [9, Theorem 7];

(4) the function

$$\left(x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\right)^{1-s}\frac{\Gamma(x+s)}{\Gamma(x+1)} \tag{3.11}$$

for $0 < s < 1$ is strictly decreasing on $(0,\infty)$; see [9, Theorem 8].

Remark 3.4. A special case of Theorem 1.2 says that the function $\exp(-h(x))$ is completely monotonic on an interval I if $h'(x)$ is completely monotonic on I. This was iterated as [9, Lemma 2.1].

In [12, p. 15 and p. 20], the following integral representation was listed: For $\operatorname{Re} z > 0$,

$$\psi\left(\frac{1}{2}+\frac{z}{2}\right)-\psi\left(\frac{z}{2}\right) = 2\int_0^\infty \frac{e^{-zt}}{1+e^{-t}}dt. \tag{3.12}$$

The formula (3.12) and [9, Lemma 2.1] are basic tools to prove [9, Theorem 1].

Remark 3.5. The basic tools to prove [9, Theorem 3] also include the formula (3.4) and the non-negativeness of the functions

$$\omega(t) = 2(b-a)\sinh\frac{t}{2} - 2\sinh\frac{(b-a)t}{2} \tag{3.13}$$

for $b > a$ and $t \ge 0$ and

$$(a-b)(1-e^{-t}) + e^{(c-a)t} - e^{(c-b)t} \tag{3.14}$$

for $b > a$, $c \ge a$ and $t \ge 0$. As mentioned in Remark 3.3, the positivity of the functions (3.13) and (3.14) for $t > 0$ can be deduced from monotonic properties of the function (3.7) or (4.1), studied in [19, Theorems 2.1–2.3], and closely related references therein. See also Section 4.1.

Remark 3.6. The proof of the complete monotonicity of the function (3.10) in [9, Theorem 7] relies on the series representation

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{x+n}\right) \tag{3.15}$$

in [12, p. 15], the positivity of the function

$$(1-s)\sinh t - \sinh[(1-s)t] \tag{3.16}$$

on $(0,\infty)$ for $0 < s < 1$, and Theorem 1.2 applied to $f(x) = e^{-x}$, as mentioned in Remark 3.4.

The proof of the decreasing monotonicity of the function (3.11) just used the formula (3.15) and [9, Lemma 2.1] stated in Remark 3.4.

Remark 3.7. It is clear that the function (3.16) is a special case of (3.13).

Remark 3.8. In fact, the logarithmically complete monotonicity of the functions (3.8), (3.9), (3.10) and their reciprocals was proved in [9].

3.3. **Ismail-Muldoon's monotonicity results.** It was claimed in [22, p. 310] that "Many inequalities for special functions follow from monotonicity properties. Often such inequalities are special cases of the complete monotonicity of related special functions. For example, an inequality of the form $f(x) \ge g(x)$ for $x \in [a,\infty)$ with equality if and only if $x = a$ may be a disguised form of the complete monotonicity of $\frac{g(\varphi(x))}{f(\varphi(x))}$ where $\phi(x)$ is a nondecreasing function on (a,∞) and $\frac{g(\varphi(a))}{f(\varphi(a))} = 1$".

Among other things, suggested by [9, Theorem 3] mentioned on page 143, the following complete monotonicity was presented in [22, Theorem 2.5]: Let $a < b \le a+1$ and

$$g(x) = \left(\frac{1-q^{x+c}}{1-q}\right)^{a-b}\frac{\Gamma_q(x+b)}{\Gamma_q(x+a)}.$$

Then $-[\ln g(x)]'$ is completely monotonic on $(-c,\infty)$ if $0 \le c \le \frac{a+b-1}{2}$ and $[\ln g(x)]'$ is completely monotonic on $(-a,\infty)$ if $c \ge a \ge 0$; Neither is completely monotonic for $\frac{a+b-1}{2} < c < a$.

As a supplement of [22, Theorem 2.5], it was proved separately in [22, Theorem 2.6] that the first derivative of the function

$$\ln\left[\left(\frac{1-q^x}{1-q}\right)^a\frac{\Gamma_q(x)}{\Gamma_q(x+a)}\right], \quad 0<q<1$$

is completely monotonic on $(0,\infty)$ for $a \ge 1$.

Remark 3.9. The proof of [22, Theorem 2.5] depends on deriving

$$\frac{d}{dx}\ln g(x) = -\int_0^\infty e^{-xt}\left[\frac{e^{-bt}-e^{-at}}{1-e^{-t}} + (b-a)e^{-ct}\right]d\mu_q(t)$$

and on [22, Lemma 1.2]: If $0 < \alpha < 1$, then

$$\alpha e^{(\alpha-1)t} < \frac{\sinh(\alpha t)}{\sinh t} < \alpha, \quad t > 0. \tag{3.17}$$

The inequalities become equalities when $\alpha = 1$ and they are reversed when $\alpha > 1$.

The proof of [22, Theorem 2.6] is similar to that of [22, Theorem 2.5].

Remark 3.10. It is clear that Theorem 2.5 and Theorem 2.6 in [22] can be rewritten using the phrase "logarithmically completely monotonic function".

Remark 3.11. The inequality (3.17) is also a consequence of the monotonicity of the function (4.1) obtained in [19, Theorems 2.1–2.3] and related references listed therein.

Remark 3.12. From [22, Theorem 2.5], the following inequality was derived in [22, Theorem 3.3]: For $0 < q \le 1$, the inequality

$$\frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} > \left(\frac{1-q^{x+s/2}}{1-q}\right)^{1-s} \tag{3.18}$$

holds for $0 < s < 1$ and $x > -\frac{s}{2}$. In [3], it was pointed out that the inequality

$$\frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} < \left(\frac{1-q^{x+s}}{1-q}\right)^{1-s}, \quad s \in (0,1) \tag{3.19}$$

is also valid for $x > -s$. As refinements of (3.18) and (3.19), the following double inequality was presented in [3, Theorem 3.1]: For real numbers $0 < q \ne 1$ and $s \in (0,1)$, the double inequality

$$\left[\frac{1-q^{x+\alpha(q,s)}}{1-q}\right]^{1-s} < \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} < \left[\frac{1-q^{x+\beta(q,s)}}{1-q}\right]^{1-s}, \quad x > 0$$

holds with the best possible values

$$\alpha(q,s) = \begin{cases} \dfrac{\ln[(q^s-q)/(1-s)(1-q)]}{\ln q}, & 0 < q < 1 \\ \dfrac{s}{2}, & q > 1 \end{cases}$$

and

$$\beta(q,s) = \frac{\ln\{1-(1-q)[\Gamma_q(s)]^{1/(s-1)}\}}{\ln q}.$$

As a direct consequence, it was derived in [3, Corollary 3.2] that the inequality

$$[x+a(s)]^{1-s} \le \frac{\Gamma(x+1)}{\Gamma(x+s)} \le [x+b(s)]^{1-s} \tag{3.20}$$

holds for $s \in (0,1)$ and $x \ge 0$ with the best possible values $a(s) = \frac{s}{2}$ and $b(s) = [\Gamma(s)]^{1/(s-1)}$.

The inequality (3.20) was ever claimed in [26, p. 248], but with a wrong proof. It was also generalized and extended in [11, Theorem 3].

4. Some logarithmically completely monotonic functions involving ratios of two gamma or q-gamma functions

In this section, we look back and analyse necessary and sufficient conditions for functions involving ratios of two gamma or q-gamma functions to be logarithmically completely monotonic.

4.1. **Some properties of a function involving exponential functions.** For real numbers α and β with $\alpha \ne \beta$ and $(\alpha,\beta) \notin \{(0,1),(1,0)\}$, let

$$q_{\alpha,\beta}(t) = \begin{cases} \dfrac{e^{-\alpha t} - e^{-\beta t}}{1-e^{-t}}, & t \ne 0; \\ \beta - \alpha, & t = 0. \end{cases} \tag{4.1}$$

As seen in Section 3, it is easy to have an idea that the function $q_{\alpha,\beta}(t)$ or its variations play indispensable roles in the proofs of [9, Theorem 3], [9, Theorem 7], [21, Theorem 2.4], [21, Theorem 2.5], [22, Theorem 2.5] and [22, Theorem 2.6].

In order to bound ratios of two gamma or q-gamma functions, necessary and sufficient conditions for $q_{\alpha,\beta}(t)$ to be either monotonic or logarithmically convex were established in [18, 19, 37, 44] little by little.

Proposition 4.1 ([19, Theorems 2.1–2.3])**.** *Let t, α and β with $\alpha \ne \beta$ and $(\alpha,\beta) \notin \{(0,1),(1,0)\}$ be real numbers. Then*

(1) *the function $q_{\alpha,\beta}(t)$ increases on $(0,\infty)$ if and only if*

$$\begin{aligned}(\alpha,\beta) &\in D_1(\alpha,\beta) \\ &\triangleq \{(\alpha,\beta) : (\beta-\alpha)(1-\alpha-\beta) \ge 0, (\beta-\alpha)(|\alpha-\beta|-\alpha-\beta) \ge 0\};\end{aligned} \tag{4.2}$$

(2) *the function $q_{\alpha,\beta}(t)$ decreases on $(0,\infty)$ if and only if*

$$\begin{aligned}(\alpha,\beta) &\in D_2(\alpha,\beta) \\ &\triangleq \{(\alpha,\beta) : (\beta-\alpha)(1-\alpha-\beta) \le 0, (\beta-\alpha)(|\alpha-\beta|-\alpha-\beta) \le 0\};\end{aligned} \tag{4.3}$$

(3) *the function $q_{\alpha,\beta}(t)$ increases on $(-\infty,0)$ if and only if $(\beta-\alpha)(1-\alpha-\beta) \ge 0$ and $(\beta-\alpha)(2-|\alpha-\beta|-\alpha-\beta) \ge 0$;*

(4) *the function $q_{\alpha,\beta}(t)$ decreases on $(-\infty,0)$ if and only if $(\beta-\alpha)(1-\alpha-\beta) \le 0$ and $(\beta-\alpha)(2-|\alpha-\beta|-\alpha-\beta) \le 0$;*

(5) *the function $q_{\alpha,\beta}(t)$ increases on $(-\infty,\infty)$ if and only if $(\beta-\alpha)(|\alpha-\beta|-\alpha-\beta) \ge 0$ and $(\beta-\alpha)(2-|\alpha-\beta|-\alpha-\beta) \ge 0$;*

(6) *the function $q_{\alpha,\beta}(t)$ decreases on $(-\infty,\infty)$ if and only if $(\beta-\alpha)(|\alpha-\beta|-\alpha-\beta) \le 0$ and $(\beta-\alpha)(2-|\alpha-\beta|-\alpha-\beta) \le 0$.*

Proposition 4.2 ([19, Theorem 3.1] and [44, Lemma 1])**.** *The function $q_{\alpha,\beta}(t)$ on $(-\infty,\infty)$ is logarithmically convex if $\beta-\alpha>1$ and logarithmically concave if $0<\beta-\alpha<1$.*

Proposition 4.3 ([37, Theorem 1.1])**.** *If $1>\beta-\alpha>0$, then $q_{\alpha,\beta}(u)$ is 3-log-convex on $(0,\infty)$ and 3-log-concave on $(-\infty,0)$; If $\beta-\alpha>1$, then $q_{\alpha,\beta}(u)$ is 3-log-concave on $(0,\infty)$ and 3-log-convex on $(-\infty,0)$.*

Proposition 4.4 ([18])**.** *Let $\lambda \in \mathbb{R}$. If $\beta-\alpha>1$, then the function $q_{\alpha,\beta}(t)q_{\alpha,\beta}(\lambda-t)$ is increasing on $\left(\frac{\lambda}{2},\infty\right)$ and decreasing on $\left(-\infty,\frac{\lambda}{2}\right)$; if $0<\beta-\alpha<1$, it is decreasing on $\left(\frac{\lambda}{2},\infty\right)$ and increasing on $\left(-\infty,\frac{\lambda}{2}\right)$.*

Remark 4.5. By noticing that the function $q_{\alpha,\beta}(t)$ can be rewritten as

$$q_{\alpha,\beta}(t) = \frac{\sinh[(\beta-\alpha)t/2]}{\sinh(t/2)} \exp\frac{(1-\alpha-\beta)t}{2},$$

it is easy to see that the inequality (3.6), the non-negativeness of the functions (3.13) and (3.14), the positivity of the function (3.16) and the inequality (3.17) are at all special cases of the monotonicity of the function $q_{\alpha,\beta}(t)$ on $(0,\infty)$.

4.2. **Necessary and sufficient conditions related to the ratio of two gamma functions.** In this section, we survey necessary and sufficient conditions for some functions involving the ratio of two gamma functions to be logarithmically completely monotonic.

4.2.1. The logarithmically complete monotonicity of the function

$$h_a(x) = \frac{(x+a)^{1-a}\Gamma(x+a)}{x\Gamma(x)} = \frac{(x+a)^{1-a}\Gamma(x+a)}{\Gamma(x+1)}$$

for $x > 0$ and $a > 0$ and the reciprocal of the first function in (2.7) were considered in [46].

Theorem 4.6 ([46, Theorem 1.2])**.** *The function $h_a(x)$ has the following properties:*

(1) *The function $h_a(x)$ is logarithmically completely monotonic on $(0,\infty)$ if $0 < a < 1$;*
(2) *The function $[h_a(x)]^{-1}$ is logarithmically completely monotonic on $(0,\infty)$ if $a > 1$;*
(3) *For any $a > 0$,*

$$\lim_{x\to 0^+} h_a(x) = \frac{\Gamma(a+1)}{a^a} \quad \text{and} \quad \lim_{x\to\infty} h_a(x) = 1.$$

In order to obtain a refined upper bound in (2.2), the logarithmically complete monotonicity of the function $f_a(x) = \frac{\Gamma(x+a)}{x^a\Gamma(x)}$ for $x \in (0,\infty)$ and $a \in (0,\infty)$, the middle term in (2.2) or the reciprocal of the second function in (2.7), were considered in [46, Theorem 1.3].

Theorem 4.7 ([46, Theorem 1.3])**.** *The function $f_a(x)$ has the following properties:*

(1) *The function $f_a(x)$ is logarithmically completely monotonic on $(0,\infty)$ and $\lim_{x\to 0+} f_a(x) = \infty$ if $a > 1$;*
(2) *The function $[f_a(x)]^{-1}$ is logarithmically completely monotonic on $(0,\infty)$ and $\lim_{x\to 0+} f_a(x) = 0$ if $0 < a < 1$;*
(3) *$\lim_{x\to\infty} f_a(x) = 1$ for any $a \in (0,\infty)$.*

As a straightforward consequence of combining Theorem 4.6 and Theorem 4.7, the following refinement of the upper bound in the inequality (2.2) was established.

Theorem 4.8 ([46, Theorem 1.4])**.** *Let $x\in(0,\infty)$. If $0<a<1$, then*

$$\begin{aligned}\left(\frac{x}{x+a}\right)^{1-a}&<\frac{\Gamma(x+a)}{x^a\Gamma(x)}\\&<\begin{cases}\dfrac{\Gamma(a+1)}{a^a}\left(\dfrac{x}{x+a}\right)^{1-a}\le 1, & 0<x\le\dfrac{ap(a)}{1-p(a)},\\ 1, & \dfrac{ap(a)}{1-p(a)}<x<\infty,\end{cases}\end{aligned}\tag{4.4}$$

where

$$p(x)=\begin{cases}\left[\dfrac{x^x}{\Gamma(x+1)}\right]^{1/(1-x)}, & x\ne 1,\\ e^{-\gamma}, & x=1.\end{cases}\tag{4.5}$$

If $a>1$, the reversed inequality of (4.4) *holds.*

Remark 4.9. The logarithmically complete monotonicity of the function (4.5) and its generalized form were researched in [36, Theorem 1.4], and [46, Theorem 1.5] respectively.

4.2.2. In [27, Theorem 1], the functions

$$\frac{\Gamma(x+t)}{\Gamma(x+s)}\left(x+\frac{s+t-1}{2}\right)^{s-t}\quad\text{and}\quad\frac{\Gamma(x+s)}{\Gamma(x+t)}(x+s)^{t-s}\tag{4.6}$$

for $0<s<t<s+1$ are proved to be logarithmically completely monotonic with respect to x on $(-s,\infty)$.

Remark 4.10. We cannot understand why the authors of the paper [27] chose so special functions in (4.6). More accurately, we have no idea why the constants $\frac{s+t-1}{2}$ and s were chosen in the polynomial factors of the functions listed in (4.6). Perhaps this can be interpreted by Theorems 4.12 and 4.16 below.

4.2.3. For real numbers a, b and c, denote $\rho=\min\{a,b,c\}$ and let

$$H_{a,b;c}(x)=(x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)}\tag{4.7}$$

for $x\in(-\rho,\infty)$.

By a recourse to the incomplete monotonicity of the function $q_{\alpha,\beta}(t)$ defined by (4.1), the following incomplete conclusions about the logarithmically complete monotonicity of the function $H_{a,b;c}(x)$ were procured in [34].

Theorem 4.11 ([34, Theorem 1])**.** *Let a, b and c be real numbers and $\rho=\min\{a,b,c\}$. Then*

(1) *the function $H_{a,b;c}(x)$ is logarithmically completely monotonic on $(-\rho,\infty)$ if*

$$\begin{aligned}(a,b;c)\in\left\{(a,b;c):a+b\ge 1,c\le b<c+\frac12\right\}&\cup\left\{(a,b;c):a>b\ge c+\frac12\right\}\\ \cup\{(a,b;c):2a+1\le a+b\le 1,a<c\}&\cup\{(a,b;c):b-1\le a<b\le c\}\\ \setminus\{(a,b;c):a=c+1,b=c\},&\end{aligned}$$

(2) *so is the function* $[H_{a,b;c}(x)]^{-1}$ *if*

$$\begin{aligned}(a,b;c)\in&\left\{(a,b;c):a+b\ge1,c\le a<c+\frac{1}{2}\right\}\cup\left\{(a,b;c):b>a\ge c+\frac{1}{2}\right\}\\&\cup\{(a,b;c):b<a\le c\}\cup\{(a,b;c):b+1\le a,c\le a\le c+1\}\\&\cup\{(a,b;c):b+c+1\le a+b\le1\}\\&\setminus\{(a,b;c):a=c+1,b=c\}\setminus\{(a,b;c):b=c+1,a=c\}.\end{aligned}$$

4.2.4. In [35, Theorem 1] and [39, Theorem 1.1], the function

$$\delta_{s,t}(x)=\begin{cases}\dfrac{\psi(x+t)-\psi(x+s)}{t-s}-\dfrac{2x+s+t+1}{2(x+s)(x+t)}, & s\ne t\\ \psi'(x+s)-\dfrac{1}{x+s}-\dfrac{1}{2(x+s)^2}, & s=t\end{cases}\tag{4.8}$$

for $|t-s|<1$ and $-\delta_{s,t}(x)$ for $|t-s|>1$ were proved to be completely monotonic on the interval $(-\min\{s,t\},\infty)$. By employing the formula (3.4), the monotonicity of $q_{\alpha,\beta}(t)$ on $(0,\infty)$ and the complete monotonicity of $\delta_{s,t}(x)$, necessary and sufficient conditions are presented for the function $H_{a,b;c}(x)$ to be logarithmically completely monotonic on $(-\rho,\infty)$ in [42] as follows.

Theorem 4.12 ([42, Theorem 1])**.** *Let* a*,* b *and* c *be real numbers and* $\rho=\min\{a,b,c\}$*. Then*

(1) *the function* $H_{a,b;c}(x)$ *is logarithmically completely monotonic on* $(-\rho,\infty)$ *if and only if*

$$\begin{aligned}(a,b;c)\in D_1(a,b;c)\triangleq&\{(a,b;c):(b-a)(1-a-b+2c)\ge0\}\\&\cap\{(a,b;c):(b-a)(|a-b|-a-b+2c)\ge0\}\\&\setminus\{(a,b;c):a=c+1=b+1\}\setminus\{(a,b;c):b=c+1=a+1\};\end{aligned}\tag{4.9}$$

(2) *so is the function* $H_{b,a;c}(x)$ *on* $(-\rho,\infty)$ *if and only if*

$$\begin{aligned}(a,b;c)\in D_2(a,b;c)\triangleq&\{(a,b;c):(b-a)(1-a-b+2c)\le0\}\\&\cap\{(a,b;c):(b-a)(|a-b|-a-b+2c)\le0\}\\&\setminus\{(a,b;c):b=c+1=a+1\}\setminus\{(a,b;c):a=c+1=b+1\}.\end{aligned}\tag{4.10}$$

Proof. In [1, p. 255, 6.1.1], it was listed that, for $x>0$ and $\omega>0$,

$$\frac{1}{x^\omega}=\frac{1}{\Gamma(\omega)}\int_0^\infty t^{\omega-1}e^{-xt}dt.\tag{4.11}$$

By virtue of (4.11) and (3.4), a straightforward calculation gives

$$[\ln H_{a,b,c}(x)]'=-\int_0^\infty[q_{a-c,b-c}(t)+(a-b)]e^{-(x+c)t}dt,$$

and, for $k\in\mathbb{N}$,

$$(-1)^k[\ln H_{a,b,c}(x)]^{(k)}=\int_0^\infty[q_{a-c,b-c}(t)+(a-b)]t^{k-1}e^{-(x+c)t}dt,$$

where $q_{a-c,b-c}(t)$ is the function defined by (4.1). From $q_{a-c,b-c}(0) = b-a$, it is revealed that if $q_{a-c,b-c}(t)$ is increasing (or decreasing respectively) on $(0,\infty)$ then $q_{a-c,b-c}(t)+(a-b) \gtreqless 0$ in $t\in(0,\infty)$ and $(-1)^k[\ln H_{a,b,c}(x)]^{(k)} \gtreqless 0$ in $x\in(-\rho,\infty)$ for $k\in\mathbb{N}$. Combining this with Proposition 4.1 demonstrates that the function $H_{a,b,c}(x)$ is logarithmically completely monotonic on $(-\rho,\infty)$ if $(a-c,b-c)\in D_1(a-c,b-c)$ and that the function $[H_{a,b,c}(x)]^{-1}$ is logarithmically completely monotonic on $(-\rho,\infty)$ if $(a-c,b-c)\in D_2(a-c,b-c)$, where $D_1(a-c,b-c)$ and $D_2(a-c,b-c)$ are defined by (4.2) and (4.3). The sufficiency of Theorem 4.12 is proved.

If the function $H_{a,b,c}(x)$ is logarithmically completely monotonic on $(-\rho,\infty)$, then $[\ln H_{a,b,c}(x)]'\le 0$, which is equivalent to

$$c \ge \frac{b-a}{\psi(x+b)-\psi(x+a)} - x \triangleq \chi_{a,b}(x) \tag{4.12}$$

for $b>a$ on $(-\rho,\infty)$. Since $\lim_{x\to0^+}\psi(x)=-\infty$, then $\lim_{x\to(-a)^+}\chi_{a,b}(x)=a\le c$ for $b>a$. From the complete monotonicity of the function (4.8) for $|t-s|<1$ and $-\delta_{s,t}(x)$ for $|t-s|>1$ on the interval $(-\alpha,\infty)$, where s and t are two real numbers and $\alpha=\min\{s,t\}$, and $\lim_{x\to\infty}\delta_{s,t}(x)=0$, it is deduced that

$$c\ge\chi_{a,b}(x)\ge\frac{2(x+a)(x+b)}{2x+a+b+1}-x\to\frac{a+b-1}{2}>a \tag{4.13}$$

for $b-a>1$ and that

$$\chi_{a,b}(x)\le\frac{2(x+a)(x+b)}{2x+a+b+1}-x\to\frac{a+b-1}{2}<a \tag{4.14}$$

for $b-a<1$ as x tends to ∞. The necessity of $H_{a,b,c}(x)$ being logarithmically completely monotonic on $(-\rho,\infty)$ follows.

The proof of necessity of $H_{b,a,c}(x)$ being logarithmically completely monotonic on $(-\rho,\infty)$ is same as above. The necessity of Theorem 4.12 is proved. □

Remark 4.13. The limit (2.1) implies that $\lim_{x\to\infty}H_{a,b;c}(x)=1$ is valid for all defined numbers a,b,c. Combining this with the logarithmically complete monotonicity of $H_{a,b;c}(x)$ yields that the inequality $H_{a,b;c}(x)>1$ holds if $(a,b;c)\in D_1(a,b;c)$ and reverses if $(a,b;c)\in D_2(a,b;c)$, that is, the inequality

$$x+\lambda<\left[\frac{\Gamma(x+a)}{\Gamma(x+b)}\right]^{1/(a-b)}<x+\mu,\quad b>a \tag{4.15}$$

holds for $x\in(-a,\infty)$ if $\lambda\le\min\bigl\{a,\frac{a+b-1}{2}\bigr\}$ and $\mu\ge\max\bigl\{a,\frac{a+b-1}{2}\bigr\}$, which is equivalent to

$$\min\left\{a,\frac{a+b-1}{2}\right\}<\left[\frac{\Gamma(a)}{\Gamma(b)}\right]^{1/(a-b)}<\max\left\{a,\frac{a+b-1}{2}\right\},\quad b>a>0.$$

It is noted that a special case $0<a<b<1$ of the inequality (4.15) was derived in [10] from [11, Theorem 1] (see also [18, 44]). Moreover, by available of the inequality (2.2) and others, the double inequalities

$$\frac{x+a}{x+b}(x+b)^{b-a}\le\frac{\Gamma(x+b)}{\Gamma(x+a)}\le(x+a)^{b-a},\quad x>0$$

and

$$(x+a)e^{-\gamma/(x+a)} < \left[\frac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{1/(b-a)} < (x+b)e^{-1/2(x+b)}, \quad x \ge 1$$

were proved in [48] to be valid for $0 < a < b < 1$.

Maybe two references [6, 32] are also useful and worth being mentioned.

Remark 4.14. Since the complete monotonicity of the function (4.8) was not established and the main result about the monotonicity of the function $q_{\alpha,\beta}(t)$ is incomplete at that time, necessary conditions for the function (4.7) to be logarithmically completely monotonic was not discovered in [34, Theorem 1] and the sufficient conditions in [34, Theorem 1] are imperfect.

Remark 4.15. It is not difficult to see that all (complete) monotonicity on functions involving the ratio of two gamma functions showed in [9, 21] and related results in [34, 46] are special cases of the above Theorem 4.12.

4.2.5. From Theorem 4.12 above, the following double inequalities for divided differences of the psi and polygamma functions may be deduced immediately.

Theorem 4.16 ([42, Theorem 3])**.** *Let $b > a \ge 0$ and $k \in \mathbb{N}$. Then the double inequality*

$$\frac{(k-1)!}{(x+\alpha)^k} < \frac{(-1)^{k-1}\left[\psi^{(k-1)}(x+b)-\psi^{(k-1)}(x+a)\right]}{b-a} < \frac{(k-1)!}{(x+\beta)^k} \tag{4.16}$$

for $x \in (-\rho,\infty)$ holds if $\alpha \ge \max\left\{a, \frac{a+b-1}{2}\right\}$ and $0 \le \beta \le \min\left\{a, \frac{a+b-1}{2}\right\}$.

Remark 4.17. It is amazing that taking $b - a = 1$ in (4.16) leads to

$$\psi^{(k-1)}(x+a+1) - \psi^{(k-1)}(x+a) = (-1)^{k-1}\frac{(k-1)!}{(x+a)^k}$$

for $a \ge 0$, $x > 0$ and $k \in \mathbb{N}$, which is equivalent to the recurrence formula

$$\psi^{(n)}(z+1) - \psi^{(n)}(z) = (-1)^n n! z^{-n-1}, \quad z > 0, \quad n \ge 0 \tag{4.17}$$

listed in [1, p. 260, 6.4.6]. For related information, see [19, Remark 4.2].

Remark 4.18. For more information on results of divided differences for the psi and polygamma functions, please refer to [15, 35, 39, 40] and related references therein.

Remark 4.19. It is worthwhile to note that some errors and defects appeared in [34] have been corrected and consummated in [42].

4.3. **Necessary and sufficient conditions related to the ratio of two q-gamma functions.** The known results obtained by many mathematicians show that most of properties relating to the ratio of two gamma functions may be replanted to cases of the ratio of two q-gamma functions, as done in [21, Theorem 2.5] and [22, Theorems 2.5 and 2.6].

Let a, b and c be real numbers, $\rho = \min\{a, b, c\}$, and define

$$H_{q;a,b;c}(x) = \left(\frac{1-q^{x+c}}{1-q}\right)^{a-b}\frac{\Gamma_q(x+b)}{\Gamma_q(x+a)} \tag{4.18}$$

for $x \in (-\rho, \infty)$, where $\Gamma_q(x)$ is the q-gamma function defined by (1.1).

It is clear that the function (4.18) is a q-analogue of the function (4.7).

In virtue of the monotonicity of $q_{\alpha,\beta}(t)$ on $(0, \infty)$ and the formula (1.3), a q-analogue of Theorem 4.12 was procured.

Theorem 4.20 ([19, Theorem 4.3])**.** *Let a, b and c be real numbers and $\rho = \min\{a, b, c\}$. Then the function $H_{q;a,b;c}(x)$ is logarithmically completely monotonic on $(-\rho, \infty)$ if and only if $(a, b; c) \in D_2(a, b; c)$, so is the function $H_{q;b,a;c}(x)$ if and only if $(a, b; c) \in D_1(a, b; c)$, where $D_1(a, b; c)$ and $D_2(a, b; c)$ are defined by (4.9) and (4.10) respectively.*

Remark 4.21. All complete monotonicity obtained in [21, Theorem 2.5] and [22, Theorems 2.5 and 2.6] are special cases of Theorem 4.20.

Similar to Theorem 4.16, the following double inequality of divided differences of the q-psi function $\psi_q(x)$ for $0 < q < 1$ may be derived from Theorem 4.20.

Theorem 4.22 ([19, Theorem 4.4])**.** *Let $b > a \ge 0$, $k \in \mathbb{N}$ and $0 < q < 1$. Then, for $x \in (-\rho, \infty)$, the inequality*

$$\frac{(-1)^{k-1}\big[\psi_q^{(k-1)}(x+b) - \psi_q^{(k-1)}(x+a)\big]}{b-a} < (-1)^{k-1}[\ln(1-q^{x+c})]^{(k)} \tag{4.19}$$

holds if $0 \le c \le \min\big\{a, \frac{a+b-1}{2}\big\}$ and reverses if $c \ge \max\big\{a, \frac{a+b-1}{2}\big\}$. Consequently, the identity

$$\psi_q^{(k-1)}(x+1) - \psi_q^{(k-1)}(x) = [\ln(1-q^x)]^{(k)} \tag{4.20}$$

holds for $x \in (0, \infty)$ and $k \in \mathbb{N}$.

Remark 4.23. Since identities (4.17) and (4.20) may be derived from inequalities (4.16) and (4.19), we can regard inequalities (4.16) and (4.19) as generalizations of identities (4.17) and (4.20).

5. Logarithmically complete monotonicity for ratios of products of the gamma and q-gamma functions

In this section, we would like to look back and analyse some (logarithmically) complete monotonicity of ratios of products of the gamma and q-gamma functions.

Let a_i and b_i for $1 \le i \le n$ be real numbers and $\rho_n = \min_{1 \le i \le n}\{a_i, b_i\}$. For $x \in (-\rho_n, \infty)$, define

$$h_{\boldsymbol{a},\boldsymbol{b};n}(x) = \prod_{i=1}^{n} \frac{\Gamma(x+a_i)}{\Gamma(x+b_i)}, \tag{5.1}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ denote $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ respectively.

5.1. Complete monotonicity.

In [9, Theorem 6], by virtue of the formula (3.4) and a special case of Theorem 1.2 mentioned in Remark 3.4 above, the function

$$x \mapsto \frac{\Gamma(x)\Gamma(x+a+b)}{\Gamma(x+a)\Gamma(x+b)} \tag{5.2}$$

for $a, b \ge 0$, a special cases of $h_{\boldsymbol{a},\boldsymbol{b};n}(x)$ for $n = 2$, was proved to be completely monotonic on $(0, \infty)$.

In [2, Theorem 10], the function $h_{\boldsymbol{a},\boldsymbol{b};n}(x)$ was proved to be completely monotonic on $(0, \infty)$ provided that $0 \le a_1 \le \cdots \le a_n$, $0 \le b_1 \le \cdots \le b_n$ and $\sum_{i=1}^{k} a_i \le \sum_{i=1}^{k} b_i$ for $1 \le k \le n$. Its proof used the formula (3.4), a special case of Theorem 1.2 applied to $f(x) = e^{-x}$, and the following conclusion cited from [30, p. 10]: Let a_i and b_i for $i = 1, \dots, n$ be real numbers such that $a_1 \le \cdots \le a_n$, $b_1 \le \cdots \le b_n$, and $\sum_{i=1}^{k} a_i \le \sum_{i=1}^{k} b_i$ for $k = 1, \dots, n$. If the function f is decreasing and convex on $\mathbb{R}$, then

$$\sum_{i=1}^{n} f(b_i) \le \sum_{i=1}^{n} f(a_i).$$

In [22, Theorem 4.1], the functions

$$-\frac{d}{dx} \ln \frac{\Gamma_q(x+a_1)\Gamma_q(x+a_2)\cdots\Gamma_q(x+a_n)}{[\Gamma(x+\bar{a})]^n}$$

and

$$\frac{d}{dx} \ln \frac{\Gamma_q(x+a_1)\Gamma_q(x+a_2)\cdots\Gamma_q(x+a_n)}{[\Gamma_q(x)]^{n-1}\Gamma_q(x+a_1+a_2+\cdots+a_n)} \tag{5.3}$$

were proved to be completely monotonic on $(0, \infty)$, where $a_1, \dots, a_n$ are positive numbers, $n\bar{a} = a_1 + \cdots + a_n$, and $0 < q \le 1$.

In [29], the function

$$x \mapsto \frac{[\Gamma(x)]^{n-1}\Gamma\bigl(x + \sum_{i=1}^{n} a_i\bigr)}{\prod_{i=1}^{n} \Gamma(x+a_i)} \tag{5.4}$$

for $a_i > 0$ and $i = 1, \dots, n$ was found to be decreasing on $(0, \infty)$.

Motivated by the decreasing monotonic property of the function (5.4), H. Alzer proved in [2, Theorem 11] that the function

$$x \mapsto \frac{[\Gamma(x)]^{\alpha}\Gamma\bigl(x + \sum_{i=1}^{n} a_i\bigr)}{\prod_{i=1}^{n} \Gamma(x+a_i)}$$

is completely monotonic on $(0, \infty)$ if and only if $\alpha = n - 1$.

Remark 5.1. It is clear that the decreasingly monotonic property of the function (5.4) is just the special case $q \to 1^-$ of the complete monotonicity of the function (5.3). Therefore, it seems that the authors of the papers [2, 29] were not aware of the results in [22, Theorem 4.1].

Remark 5.2. The complete monotonicity mentioned just now are indeed logarithmically completely monotonic ones.

5.2. **Logarithmically complete monotonicity.** Let S_n be the symmetric group over n symbols, $a_1, a_2, \dots, a_n$. Let O_n and E_n be the sets of odd and even permutations over n symbols, respectively. For $a_1 > a_2 > \cdots > a_n > 0$, define

$$F(x) = \frac{\prod_{\sigma \in E_n} \Gamma\bigl(x + a_{\sigma(2)} + 2a_{\sigma(3)} + \cdots + (n-1)a_{\sigma(n)}\bigr)}{\prod_{\sigma \in O_n} \Gamma\bigl(x + a_{\sigma(2)} + 2a_{\sigma(3)} + \cdots + (n-1)a_{\sigma(n)}\bigr)}.$$

It was proved in [14, Theorem 1.1] that the function $F(x-a_2-2a_3-\cdots-(n-1)a_n)$ is logarithmically completely monotonic on $(0,\infty)$.

In [14, Theorem 1.2], it was presented that the functions

$$F_n(x)=\frac{\Gamma(x)\prod_{k=1}^{[n/2]}\Bigl[\prod_{m\in P_{n,2k}}\Gamma\Bigl(x+\sum_{j=1}^{2k}a_{m_j}\Bigr)\Bigr]}{\prod_{k=1}^{[(n+1)/2]}\Bigl[\prod_{m\in P_{n,2k-1}}\Gamma\Bigl(x+\sum_{j=1}^{2k-1}a_{m_j}\Bigr)\Bigr]} \tag{5.5}$$

for any $a_k>0$ and $k\in\mathbb{N}$ are logarithmically completely monotonic on $(0,\infty)$ and that any product of functions of the type (5.5) with different parameters a_k is logarithmically completely monotonic on $(0,\infty)$ as well, where $P_{n,k}$ for $1\le k\le n$ is the set of all vectors $\boldsymbol{m}=(m_1,\dots,m_k)$ whose components are natural numbers such that $1\le m_\nu<m_\mu\le n$ for $1\le\nu<\mu\le k$ and $P_{n,0}$ is the empty set.

Remark 5.3. The above Theorem 1.2 is more general than Theorem 1.1. The case $n=2$ in Theorem 1.2 corresponds to the complete monotonicity of the function (5.2) obtained in [9, Theorem 6].

In [14, Theorem 3.2], it was showed that if

$$F_q(x)=\frac{\prod_{\sigma\in E_n}\Gamma_q\bigl(x+a_{\sigma(2)}+2a_{\sigma(3)}+\cdots+(n-1)a_{\sigma(n)}\bigr)}{\prod_{\sigma\in O_n}\Gamma_q\bigl(x+a_{\sigma(2)}+2a_{\sigma(3)}+\cdots+(n-1)a_{\sigma(n)}\bigr)}$$

for $a_1>a_2>\cdots>a_n>0$, then $F_q(x-a_2-2a_3-\cdots-(n-1)a_n)$ is a logarithmically completely monotonic function of x on $(0,\infty)$.

In [14, Theorem 3.3], it was stated that the functions

$$F_{n,q}(x)=\frac{\Gamma_q(x)\prod_{k=1}^{[n/2]}\Bigl[\prod_{m\in P_{n,2k}}\Gamma_q\Bigl(x+\sum_{j=1}^{2k}a_{m_j}\Bigr)\Bigr]}{\prod_{k=1}^{[(n+1)/2]}\Bigl[\prod_{m\in P_{n,2k-1}}\Gamma_q\Bigl(x+\sum_{j=1}^{2k-1}a_{m_j}\Bigr)\Bigr]} \tag{5.6}$$

for any $a_k>0$ with $k=1,\dots,n$ are logarithmically completely monotonic on $(0,\infty)$, so is any product of functions (5.6) with different parameters a_k.

Remark 5.4. It is obvious that [14, Theorem 3.2 and Theorem 3.3] are q-analogues of [14, Theorem 1.1 and Theorem 1.2].

5.3. **Some recent conclusions.** By a recourse to the monotonicity of $q_{\alpha,\beta}(t)$ on $(0,\infty)$, the following sufficient conditions for the function $h_{\boldsymbol{a},\boldsymbol{b};n}(x)$ to be logarithmically completely monotonic on $(0,\infty)$ are devised.

Theorem 5.5 ([19, Theorem 4.5])**.** *If*

$$(b_i-a_i)(1-a_i-b_i)\ge0 \quad \text{and} \quad (b_i-a_i)(|a_i-b_i|-a_i-b_i)\ge0 \tag{5.7}$$

hold for $1\le i\le n$ *and*

$$\sum_{i=1}^n b_i\ge\sum_{i=1}^n a_i, \tag{5.8}$$

then the function $h_{\boldsymbol{a},\boldsymbol{b};n}(x)$ *is logarithmically completely monotonic on* $(-\rho_n,\infty)$. *If inequalities in* (5.7) *and* (5.8) *are reversed, then the function* $h_{\boldsymbol{b},\boldsymbol{a};n}(x)$ *is logarithmically completely monotonic on* $(-\rho_n,\infty)$.

Proof. Taking the logarithm of $h_{\boldsymbol{a},\boldsymbol{b};n}(x)$ defined by (5.1) and differentiating consecutively gives

$$[\ln h_{\boldsymbol{a},\boldsymbol{b};n}(x)]^{(k)} = \sum_{i=1}^{n}\bigl[\psi^{(k-1)}(x+a_i) - \psi^{(k-1)}(x+b_i)\bigr]$$

for $k \in \mathbb{N}$. Using the integral representation (3.4) and its derivative

$$\psi^{(m)}(x) = (-1)^{m+1}\int_0^\infty \frac{e^{-tx}t^m}{1-e^{-t}}dt$$

for $x > 0$ and $m \in \mathbb{N}$ yields

$$(-1)^k[\ln h_{\boldsymbol{a},\boldsymbol{b};n}(x)]^{(k)} = \int_0^\infty t^{k-1}e^{-xt}\sum_{i=1}^{n} q_{a_i,b_i}(t)dt,$$

where $q_{a_i,b_i}(t)$ is defined by (4.1). Since

$$\sum_{i=1}^{n} q_{a_i,b_i}(0) = \sum_{i=1}^{n}(b_i - a_i) = \sum_{i=1}^{n} b_i - \sum_{i=1}^{n} a_i$$

and, by Proposition 4.1, the function $q_{a_i,b_i}(t)$ increases on $(0,\infty)$ if and only if inequalities in (5.7) hold, when the inequality (5.8) and the inequalities in (5.7) are valid for $1 \le i \le n$, it follows that $(-1)^k[\ln h_{\boldsymbol{a},\boldsymbol{b};n}(x)]^{(k)} \ge 0$ for $k \in \mathbb{N}$, and so the function $h_{\boldsymbol{a},\boldsymbol{b};n}(x)$ is logarithmically completely monotonic on $(-\rho_n,\infty)$.

Similarly, when the inequalities in (5.7) and (5.8) are reversed, the function $h_{\boldsymbol{b},\boldsymbol{a};n}(x)$, the reciprocal of $h_{\boldsymbol{a},\boldsymbol{b};n}(x)$, is logarithmically completely monotonic on $(-\rho_n,\infty)$. The proof of Theorem 5.5 is completed. □

The q-analogue of Theorem 5.5 is as follows.

Theorem 5.6 ([19, Theorem 4.6])**.** *Let a_i and b_i for $1 \le i \le n$ be real and $\rho_n = \min_{1\le i\le n}\{a_i, b_i\}$. For $x \in (-\rho_n,\infty)$, define*

$$h_{q;\boldsymbol{a},\boldsymbol{b};n}(x) = \prod_{i=1}^{n}\frac{\Gamma_q(x+a_i)}{\Gamma_q(x+b_i)}$$

for $0 < q < 1$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ stand for $(a_1,a_2,\dots,a_n)$ and $(b_1,b_2,\dots,b_n)$ respectively. If inequalities in (5.7) and (5.8) hold, then the function $h_{q;\boldsymbol{a},\boldsymbol{b};n}(x)$ is logarithmically completely monotonic on $(-\rho_n,\infty)$. If inequalities in (5.7) and (5.8) are reversed, then the function $h_{q;\boldsymbol{b},\boldsymbol{a};n}(x)$ is logarithmically completely monotonic on $(-\rho_n,\infty)$.

Proof. Taking the logarithm of $h_{q;\boldsymbol{a},\boldsymbol{b};n}(x)$ and differentiating successively reveal

$$[\ln h_{q;\boldsymbol{a},\boldsymbol{b};n}(x)]^{(k)} = \sum_{i=1}^{n}\bigl[\psi_q^{(k-1)}(x+a_i) - \psi_q^{(k-1)}(x+b_i)\bigr]$$

for $k \in \mathbb{N}$. Using (1.3) and its derivative

$$\psi_q^{(m)}(x) = (-1)^{m+1}\int_0^\infty \frac{t^m e^{-tx}}{1-e^{-t}}d\gamma_q(t)$$

for $x > 0$ and $m \in \mathbb{N}$ results in

$$(-1)^k [\ln h_{q;\boldsymbol{a},\boldsymbol{b};n}(x)]^{(k)} = \int_0^\infty t^{k-1} e^{-xt} \sum_{i=1}^n q_{a_i,b_i}(t) dt.$$

The rest is the same as that in the proof of Theorem 5.5. The proof of Theorem 4.20 is thus completed. □

6. An open problem

From Theorem 1.1, we can see that a completely monotonic function $f(x)$ on the interval $(0, \infty)$ must be a Laplace transform of the measure $\alpha(t)$ in (1.4). We now naturally pose a problem: Can on find the concrete measures $\alpha(t)$ for the (logarithmically) completely monotonic functions mentioned in this paper?

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