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TOPOLOGICAL GAMES AND STRONG QUASI-CONTINUITY

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Abstract. Let X be a Baire space, Y be a W-space and Z be a regular topological space. We will show that every KC-function $f : X \times Y \to Z$ is strongly quasi-continuous at each point of $X \times Y$. In particular, when X is a Baire space and Y is Corson compact, every KC-function f from $X \times Y$ to a Moore space Z is jointly continuous on a dense subset of $X \times Y$. We also give a few applications of our results on continuity of group actions.

1. Introduction

Let X, Y and Z be topological spaces, following Kempisty [13], a function $\varphi : X \to Z$ is called *quasi-continuous at a point* $x \in X$ if for arbitrary neighborhoods V and W of x and $\varphi(x)$ respectively, one can find a nonempty open subset G of V such that $\varphi(G) \subset W$. The function $\varphi : X \to Y$ is called *quasi-continuous* if it is quasi-continuous at each point of X. By a Kempisty continuous function (KC-*function* for short), we mean a function $f : X \times Y \to Z$ which is quasi-continuous in the first variable and continuous in the second variable.

A mapping $f : X \times Y \to Z$ is called *strongly quasi-continuous* at $(x, y) \in X \times Y$ if for each neighborhood W of $f(x, y)$ in Z and for each product of open sets $U \times V \subset X \times Y$ containing (x, y), there is a nonvoid open set $U_1 \subset U$ and a neighborhood $V_1 \subset V$ of y such that $f(U_1 \times V_1) \subset W$.

The notion of quasicontinuity was used by R. Baire [2] in the study of points of continuity of separately continuous functions. There is a rich literature concerning the problem of determining points of continuity for two variable functions (see

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for example [6, 7, 17, 19, 20, 21, 22]). In particular, Piotrowski in [23] proved the following result:

Theorem 1.1. *Let X be a Baire space, Y ba first countable and Z be metric. If $f : X \times Y \to Z$ is a KC-function, then for all $y \in Y$, there is a G_δ subset D_y of X such that f is jointly continuous in $D_y \times \{y\}$.*

In 1976, by means of a topological game, G. Gruenhage [11] introduced a class of topological spaces, called W-spaces, which contains the class of all first countable spaces and is stable under Σ-products and open mappings.

In this paper, we use a topological game argument to show that every KC-function $f : X \times Y \to Z$ is strongly quasi-continuous, provided that X is a Baire space, Y is a W-space and Z is regular. In particular, when Z is a Moore space, it follows that for each $y \in Y$ the set of joint continuity of f is a dense G_δ subset of $X \times \{y\}$. Since, as the class of W-spaces strictly contains first countable spaces (see section 2), our result extends Theorem 1.1. It follows that every KC-action $\pi : G \times Y \to Y$ is jointly continuous if Y is a Moore W-space and G is a Baire left topological group.

2. Topological games

In this section, we will introduce two topological games which will be used in the sequel. Each topological game is described by two types of rules; the playing rules, that determine how to play the game, and the winning rule which determines the winner. The winning rule differs from game to game and, actually, identifies the game.

Let (X, τ) be a topological space. The Banach–Mazur game $\mathcal{BM}(X)$ [5] between two players α and β is done as follows.
Player β starts the game by selecting a nonempty open set U_1 of X; then player α chooses a non-empty open set $V_1 \subset U_1$. When (U_i, V_i) , $1 \leq i \leq n-1$, have been defined, player β picks a nonempty open set $U_n \subset V_{n-1}$ and α answers by selecting a nonempty open set $V_n \subset U_n$. In this way two players generate a sequence of nonempty open subsets of X

$$U_1 \supset V_1 \supset U_2 \cdots \supset U_n \supset V_n \dots$$

which is called a play.
The player α wins the play $(U_i, V_i)_{i \geq 1}$ if $(\bigcap_{n=1}^{\infty} V_n) \neq \emptyset$. Otherwise the player β is said to have won the play.
By a strategy for one of the players, we mean a rule that specifies each move of the player. We say that the player α has a winning strategy for the game $\mathcal{BM}(X)$ if there exists a strategy s, such that α wins all plays provided that he/she acts according to the strategy s. In this case, we say that X is an α-favorable space, otherwise X is said to be an α-unfavorable space for this game. Similarly, winning strategy for the player β and β-favorablity are defined.
It is known that X is a Baire space if and only if the player β does not have a winning strategy in the game $\mathcal{BM}(X)$ (see [25] Theorems 1 and 2). Therefore every α-favorable space X is a Baire space. However, the converse in not true in general (see for example [12]).

We need also to the following topological game which was introduced in [11]. Let Y be a topological space and $y_0 \in Y$. The topological game $\mathcal{G}(Y, y_0)$ is played by two players $\mathcal{O}$ and $\mathcal{P}$ as follows. Player $\mathcal{O}$ goes first by selecting an open neighborhood H_1 of y_0. $\mathcal{P}$ answers by choosing a point $y_1 \in H_1$.
In general, in step n, if selections $H_1, y_1, \ldots, H_n, y_n$ have already been specified, $\mathcal{O}$ selects an open set H_{n+1} with $y_0 \in H_{n+1}$ and then $\mathcal{P}$ answers by choosing a point $y_{n+1} \in H_{n+1}$. If

$$g_1 = (H_1, y_1),\ \cdots, g_n = (H_1, y_1, \cdots, H_n, y_n)$$

are the first "n" move of some play (of the game), we call g_n the n^{th} (*partial play*) of the game. We say that $\mathcal{O}$ wins the game $g = (H_n, y_n)_{n\geq 1}$ if $y_n \to y_0$.

A strategy s for one of the players is defined similar to that of Banach–Mazur game. We call $y \in Y$ a W-point (respectively w-point) in Y if $\mathcal{O}$ has (respectively $\mathcal{P}$ fails to have) a winning strategy in the game $\mathcal{G}(Y, y)$. A space Y in which each point of Y is a W-point (respectively w-point) is called a W-space (respectively w-space). It is known that every first countable space is a W-space [11, Theorem 3. 3]. However, the converse in not true in general [16, Example 2. 7].

There are w-spaces which are not W-space. For example [10] if Y is the one point compactification $T \cup \{\infty\}$ of an Aronszajn tree T with the interval topology, then neither $\mathcal{P}$ nor $\mathcal{O}$ has a winning strategy in $G(Y, \infty)$.

3. Strong quasi-continuity and joint continuity

Let X, Y and Z be topological spaces. In this section, we give a topological game argument to show under certain conditions on X, Y and Z every KC-function $f : X \times Y \to Z$ is strongly quasi-continuous. Our results can be considered as a partial extension of some results in [18] and [23].

Theorem 3.1. *Let Y be a topological space and Z be a regular space. If either*

(1) *X is a Baire space and the player $\mathcal{O}$ has a winning strategy in $\mathcal{G}(Y, y_0)$ or*
(2) *X is an α-favorable space and the player $\mathcal{P}$ does not have a winning strategy in $G(Y, y_0)$.*

Then every KC-function $f : X \times Y \to Z$ is strongly quasi-continuous on $X \times \{y_0\}$.

Proof. On the contrary, suppose that (1) or (2) holds but f is not strongly quasi-continuous at (x_0, y_0) for some point $x_0 \in X$. By the definition, there is an open set W containing $z_0 = f(x_0, y_0)$ and there is some product of open sets $U \times V \subset X \times Y$ containing (x_0, y_0) such that for each open set $U' \subset U$ and each neighborhood $H' \subset H$ of y_0, there is some $(x', y') \in U' \times H'$ such that $f(x', y') \notin W$.

Since Z is regular, there is an open subset G with $f(x_0, y_0) \in G$ and $\overline{G} \subset W$. By quasi-continuity of $f(\cdot, y_0)$, there is a non-empty open subset $U' \subset U$ such that $f(U' \times \{y_0\}) \subset G$. We define simultaneously a strategy s for β in $\mathcal{BM}(X)$ and a strategy t for $\mathcal{P}$ in $\mathcal{G}(Y, y_0)$ by induction as follows. Let $U_1 = U'$ be the first move of β-player and $V_1 \subset U_1$ be the answer of the player α to this movement. Suppose that H_1 is the first choice of $\mathcal{O}$-player. Then by our assumption, $f(V_1 \times H_1)$ is not a subset of $\overline{G}$. Therefore there is some $(x_1, y_1) \in V_1 \times H_1$ such that $f(x_1, y_1) \notin \overline{G}$.

Define $t(H_1) = y_1$. By quasi-continuity of $f(\cdot, y_1)$, we can find a non-empty open subset U_1 of V_1 such that $f(U_1 \times \{y_1\}) \cap \overline{G} = \emptyset$. Let $s(V_1) = U_1$.

Let for $n \geq 1$, the partial plays $p_n = (U_1, V_1, \ldots, U_n, V_n)$ in $\mathcal{BM}(X)$ and $g_n = (H_1, y_1, \ldots, H_n)$ in $\mathcal{G}(Y, y_0)$ are specified. Since by our assumption $f(V_n \times H_n)$ is not contained in $\overline{G}$, there is some $(x_n, y_n) \in V_n \times H_n$ such that $f(x_n, y_n) \notin \overline{G}$. By quasi-continuity of $x \mapsto f(x, y_n)$, there is a non-empty open subset U_{n+1} of V_n such that $f(U_{n+1} \times \{y_n\}) \cap \overline{G} = \emptyset$.

Define $s(U_1, V_1, \ldots, U_n, V_n) = U_{n+1}$ and $t(H_1, y_1, \ldots, H_n) = y_n$. In this way, by induction on n, a strategy for β in $\mathcal{BM}(X)$ and a strategy for $\mathcal{P}$ in $\mathcal{G}(Y, y_0)$ are defined.

If (1) or (2) holds, there are a s-play $p = (U_n, V_n)$ and t-play $g = (H_n, y_n)$ which are won by α and $\mathcal{O}$ respectively. Let $x^* \in \bigcap_{n \geq 1} U_n$. Then by continuity of $y \mapsto f(x^*, y)$ at y_0 and the fact that $f(x^*, y_0) \in G$, there is an open neighborhood H of y_0 such that $f(x^*, y) \in G$ for all $y \in H$. Since $\mathcal{O}$ wins the play $g = (H_n, y_n)$, there is some $n_0 \in \mathbb{N}$ such that $y_n \in H$ for all $n \geq n_0$. Hence $f(x^*, y_0) \in G$. However, our construction shows that $f(x, y_n) \notin \overline{G}$ for all $x \in U_n$. This contradiction proves the Theorem.

$\square$

The following result follows immediately from Theorem 3.1.

Corollary 3.2. *Let Y be a topological space and Z be a regular space. If either*

(1) *X is a Baire space and Y is a W-space or*
(2) *X is an α-favorable space and Y is a w-space.*

Then every KC-function $f : X \times Y \to Z$ is strongly quasi-continuous.

4. Applications

Let Z be a topological space $z \in Z$ and $\mathcal{U}$ be a collection of subsets of Z, then the *star of z with respect to $\mathcal{U}$* is defined by $st(z, \mathcal{U}) = \bigcup \{U \in \mathcal{U} : z \in U\}$. A sequence $\{\mathcal{G}_n\}$ of open covers of Z is said to be a *development* of Z if for each $z \in Z$, the set $\{st(z, \mathcal{G}_n) : n \in \mathbb{N}\}$ is a base at z.

A *developable space* is a space which has a development. A *Moore* space is a regular developable space.

Piotrowski [22, Theorem A] proved that if X is Baire space, Y is a topological space, Z is a developable space and $f : X \times Y \to Z$ is strongly quasi-continuous, then the points of joint continuity of f is a dense G_δ subset in $X \times \{y\}$ for all $y \in Y$ (see also [17, Theorem 2] for another proof of this result). Hence the following result follows from Theorem 3.1.

Corollary 4.1. *Let Z be a Moore space. If either*

(1) *X is a Baire space and Y is a W-space or*
(2) *X is an α-favorable space and Y is a w-space.*

Then for every KC-function $f : X \times Y \to Z$ and $y_0 \in Y$, there is a dense G_δ subset D_{y_0} of X such that f is jointly continuous at each point of $D_{y_0} \times \{y_0\}$.

Definition 4.2. A compact space Y is called *Corson compact* if for some κ, Y embeds in

$$\{\mathbf{x} \in \mathbb{R}^\kappa : x_\alpha = 0 \text{ for all but countably many } \alpha \in \kappa\}.$$

Corollary 4.3. *Let X be a Baire space, Y be a Corson compact space and Z be a regular space. Then every KC-function $f : X \times Y \to Z$ is strongly quasi-continuous. In particular, if Z is a Moore space, then f is jointly continuous on a dense subset of $X \times Y$.*

Proof. Since every Corson compact is a W-space [11, Theorem 4.6], the result follows from Theorem 3.1 and Corollary 4.1. □

In order to give another application of our result, we need to the following definition.

Definition 4.4. Let G be a group equipped with a topology. The group G is called left topological if for each $g \in G$, the left translation $h \in G \to gh \in G$ is continuous. By trivial change in the above definition, a right topological group can be defined. If G is both left and right topological, then G is called semitopological. A semitopological group is called paratopological if the product mapping is jointly continuous. If in addition the inverse function $x \mapsto x^{-1}$ is continuous, then G is said to be a topological group.

Let G be a left topological group and Y be a topological space. We say that G acts on X if there exists a function $\pi : G \times Y \to Y$ such that

$$\pi(gh, y) = \pi(g, \pi(h, y)) \quad (g, h \in G, y \in Y). \tag{4.1}$$

Ellis [9] proved that every separately continuous action $\pi : G \times Y \to Y$ is jointly continuous provided that G is a locally compact semitopological group and Y is a locally compact space. Theorem 3.1 enable us to give the following related result. The interested reader is referred to [1, 3, 4, 8, 14, 15] for further information in this direction.

Theorem 4.5. *Let Y be a Moore space and G be a left topological group. If either*

(1) *G is a Baire space and Y is a W-space or*
(2) *G is an α-favorable space and Y is a w-space.*

Then every KC-action $\pi : G \times Y \to Y$ jointly continuous.

Proof. Let $(g_0, y_0) \in G \times Y$. By Corollary 4.1, there is a dense G_δ subset D_{y_0} of G such that π is jointly continuous at each point of $D_{y_0} \times \{y_0\}$. Let $\{g_\alpha\}$ and $\{y_\alpha\}$ converge to g and y_0 respectively and take some arbitrary point $h \in D_{y_0}$. Since π is continuous at (h, y_0) and

$$\lim_\alpha hg^{-1}g_\alpha = h, \quad \lim_\alpha y_\alpha = y_0,$$

we see that $\lim_\alpha \pi(hg^{-1}g_\alpha, y_\alpha) = \pi(h, y_0)$. Therefore by using (4.1), we have

$$\begin{aligned} \lim_\alpha \pi(g_\alpha, y_\alpha) &= \lim_\alpha \pi(gh^{-1}, \pi(hg^{-1}g_\alpha, y_\alpha)) \\ &= \pi(gh^{-1}, \pi(h, y_0)) = \pi(g, y_0). \end{aligned}$$

This proves our result. □

Since every Moore space is first countable (hence is a W-space), the following result follows from Theorem 4.5.

Corollary 4.6. [24, Theorem 4]. *Let G be a Baire semitopological group which is also a Moore space. Then G is a paratopological group.*

The main result of [14] states that every strongly Baire semitopological group is a paratopological group. Since every Baire Moore space is strongly Baire, Corollary 4.6 is a special case of this result.

Remark 4.7. Cao et al in [4, Corollaries 2.4 and 2.11] have recently shown that every Baire and Moore paratopological group G is a topological group. In the view of Corollary 4.6, the paratopological group G is a topological group.

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