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# COSINE FUNCTIONS REVISITED

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Abstract. In this short note, a new approach is provided to prove that every nonzero continuous cosine function on a compact group $G$ is the normalized character of a representation of $G$ into the special unitary group $SU(2)$.

## 1. Introduction

Let $G$ be a group. A *cosine function on* $G$ is a solution to the cosine equation (also called d'Alembert equation)

$$f(xy) + f(xy^{-1}) = 2f(x)f(y), \tag{1.1}$$

where $f : G \to \mathbb{C}$ is the function to determine. Eq. (1.1) has been attracting a great deal of attention recently. See, e.g., [1]-[7], [10]-[16], and the references therein. This is mainly because there are close relations between the structure of cosine functions and harmonic analysis. In particular, it has been in [1, 14, 15, 16] shown that any nonzero continuous cosine function on a compact group $G$ is the normalized character of a 2-dimensional (continuous) representation of $G$ into the 2-dimensional special unitary group $SU(2)$.

It should be mentioned that in [1] we deal with a much more general equation by using (of course, not surprisingly) much more complex approaches. The solutions of Eq. (1.1) are obtained as a consequence of the main results there. In [16], the structure of cosine functions mentioned above is derived from a so called Small Dimension Lemma, which states that an irreducible representation of a compact group under certain assumptions must have dimension $\leq 2$. Its proof is not

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long, but not quite elementary, since it uses the equivalence between the given representation, say $\pi$, and the restriction of the right regular representation onto each row space $\mathcal{E}_i = \text{span}\{\pi_{ij} : j = 1, ..., d_\pi\}$ of $\pi$ ($i = 1, ..., d_\pi$), and Schur's orthogonality relations.

The main purpose of this short note is to give a very elementary proof to the corresponding lemma (i.e., Lemma 2.1 below) in our current context. We are going to provide three proofs. The first one is the most elementary, in which only some basics from Linear Algebra are needed. The second and third ones are still elementary, but use Burnside's Theorem, which states that the only irreducible algebra of the linear transformations on the finite dimensional vector space $\mathcal{V}$ of dimension $> 1$ is the algebra of all linear transformations mapping $\mathcal{V}$ into $\mathcal{V}$ (cf., e.g., [9, Chapter 1]).

## 2. Main Result

In this section, let us begin with recalling that a collection of linear transformations is said to be *irreducible* if it has only trivial invariant subspaces.

**Lemma 2.1.** *Let $K$ be a closed irreducible subgroup of the unitary group $U(n)$ ($n \geq 1$). Suppose for every $k \in K$ there is $c_k \in \mathbb{C}$ such that*

$$k + k^{-1} = c_k I_n. \tag{2.1}$$

*Then either $n = 1$, or $n = 2$ and $K \leq SU(2)$.*

*Proof.* Suppose $n > 1$. Clearly, from (2.1) we have $k^2 - c_k k + I_n = 0$ for all $k \in K$. So the polynomial $t^2 - c_k t + 1$ is divisible by the minimal polynomial of $k$. Hence, we have either (i) $k$ is a scalar matrix, or (ii) $k$ has exactly two distinct eigenvalues $\lambda_k$ and $\bar{\lambda}_k$ satisfying $|\lambda_k| = 1$ and $\lambda_k \neq \pm 1$.

Since $K$ is irreducible, there exists an element $a \in K$ satisfying the properties of (ii). For brevity, let $\lambda$ and $\bar{\lambda}$ be the exactly two distinct eigenvalues of $a$. After a similarity, we may assume that $a = \text{diag}(\lambda I_r, \bar{\lambda} I_s)$ with $r \geq 1, s \geq 1$. WLOG, one can further assume $r \geq s$.

*Proof 1.* For $k \in K$, with respect to the decomposition $\mathbb{C}^n = \mathbb{C}^r \oplus \mathbb{C}^s$, we write $k$ as the $2 \times 2$ block matrix $k = \begin{pmatrix} \mathbf{k}_{11} & \mathbf{k}_{12} \\ \mathbf{k}_{21} & \mathbf{k}_{22} \end{pmatrix}$. Then $k + k^* = c_k I_n$ (as $k^{-1} = k^*$ in (2.1)) implies that

$$\mathbf{k}_{21} = -\mathbf{k}_{12}^*, \tag{2.2}$$

$$\mathbf{k}_{11} + \mathbf{k}_{11}^* = c_k I_r, \quad \mathbf{k}_{22} + \mathbf{k}_{22}^* = c_k I_s. \tag{2.3}$$

Applying (2.3) to $ak = \begin{pmatrix} \lambda \mathbf{k}_{11} & \lambda \mathbf{k}_{12} \\ -\bar{\lambda} \mathbf{k}_{12}^* & \bar{\lambda} \mathbf{k}_{22} \end{pmatrix}$, we deduce

$$\lambda \mathbf{k}_{11} + \bar{\lambda} \mathbf{k}_{11}^* = c_{ak} I_r, \quad \bar{\lambda} \mathbf{k}_{22} + \lambda \mathbf{k}_{22}^* = c_{ak} I_s. \tag{2.4}$$

Since $\lambda \neq \pm 1$, a simple calculation using (2.3) and (2.4) yields

$$\mathbf{k}_{11} = \alpha_k I_r, \quad \mathbf{k}_{22} = \bar{\alpha}_k I_s$$

where $\alpha_k = \frac{c_{ak} - \bar{\lambda} c_k}{\lambda - \bar{\lambda}} \in \mathbb{C}$. In other words, every $k \in K$ is of the form

$$k = \begin{pmatrix} \alpha_k I_r & \mathbf{k}_{12} \\ -\mathbf{k}_{12}^* & \bar{\alpha}_k I_s \end{pmatrix}. \tag{2.5}$$

For $k, k' \in K$, considering the (1,1)-block entry of $kk'$, we get from (2.5) that

$$\mathbf{k}_{12}\mathbf{k}_{12}'^* = (\alpha_k \alpha_{k'} - \alpha_{kk'}) I_r. \tag{2.6}$$

Since $K$ is irreducible, there is $b \in K$ with $b_{12} \neq 0$. Letting $k = k' = b$ in (2.6) gives rise to $\mathbf{b}_{12}\mathbf{b}_{12}'^* = \gamma I_r$ for some $\gamma \in \mathbb{C}$ with $\gamma \neq 0$. Since

$$r = \operatorname{rank}(\gamma I_r) = \operatorname{rank}(\mathbf{b}_{12}\mathbf{b}_{12}^*) \leq \operatorname{rank}(\mathbf{b}_{12}) \leq s,$$

we have $r = s$. Hence there exist $U \in SU(r)$ and a nonzero $\beta \in \mathbb{C}$ such that $b_{12} = \beta U$. Letting $k' = b$ in (2.6), we conclude that for every $k \in K$, $\mathbf{k}_{12} = \beta_k U$ for some $\beta_k \in \mathbb{C}$. So

$$k = \begin{pmatrix} \alpha_k I_r & \beta_k U \\ -\overline{\beta_k} U^{-1} & \overline{\alpha_k} I_r \end{pmatrix}.$$

Let $v \in \mathbb{C}^r$ be an eigenvector of $U$. Then the 2-dimensional subspace of $\mathbb{C}^n$ spanned by $\begin{pmatrix} v \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ v \end{pmatrix}$ is $K$-invariant. Since $K$ is irreducible, we obtain that $n = 2$ and $k = \begin{pmatrix} \alpha_k & \beta_k \\ -\overline{\beta_k} & \overline{\alpha_k} \end{pmatrix} \in SU(2)$. □

*Proof 2.* Once having the general form (2.5) of the elements in $K$, we can conclude from Burnside's Theorem that $r = s (= 1)$ and then finish the proof from (2.5). Indeed, consider the algebra $\mathcal{A} \subseteq M_n(\mathbb{C})$ generated by $K$. Since $K$ is irreducible, it is easy to see so is $\mathcal{A}$. By Burnside's Theorem (which, in our context, says that the only irreducible subalgebra of $M_n(\mathbb{C})$ $(n > 1)$ is $M_n(\mathbb{C})$ itself), we obtain $\mathcal{A} = M_n(\mathbb{C})$. On the other hand, let $x$ be an arbitrary element in $\mathcal{A}$. As usual, we write $x = (x_{ij})_n$ (instead of being a $2 \times 2$ block matrix as in (2.5)). If $r > 1$, it follows from (2.5) that $x_{12} = 0$. This clearly gives a contradiction. Therefore $r = 1$, and we are done. □

*Proof 3.* This time, we also apply Burnside's Theorem, but immediately after obtaining (2.2). That is, we do not need to know the general form of elements in $K$ given in (2.5). Suppose on the contrary that $r > 1$. Let $k = (k_{ij})_n$ be an arbitrary element in $K$. It follows from (2.2) that

$$k_{21} + \overline{k_{12}} = 0.$$

(Notice that $k_{12}$ above is a complex number and $\mathbf{k}_{12}$ in (2.2) is an $r \times s$ submatrix.) Applying this to $ak$, and noticing $(ak)_{12} = \lambda k_{12}$ and $(ak)_{21} = \lambda k_{21}$, we obtain

$$\lambda k_{21} + \bar{\lambda}\overline{k_{12}} = 0.$$

Clearly, the above two identities imply that $k_{12} = 0$ as $\lambda \neq \bar{\lambda}$. This contradicts Burnside's Theorem by the arbitrariness of $k$. So $r = 1$.

From the above discussion, we have that $K \leq U(2)$, and that for every $k \in K$, either $k = \pm I_2$, or $k$ is conjugate to $\mathrm{diag}(\lambda_k, \overline{\lambda_k})$ with $\lambda_k \neq \overline{\lambda_k}$. Therefore, $K \leq SU(2)$. $\square$

Any one of the above proofs proves the lemma. $\square$

As a consequence of Lemma 2.1, we reproduce the structure of continuous cosine functions on a compact group $G$ (cf. [14]-[16]).

**Theorem 2.2.** *Let $G$ be a compact group. Then $f$ is a nonzero continuous cosine function on $G$ if and only if there is a representation $\rho : G \to SU(2)$ such that*

$$f = \frac{\chi_\rho}{2}, \tag{2.7}$$

*where $\chi_\rho$ is the character of $\rho$.*

*Proof.* Apparently, it suffices to prove "only if" directon: any nonzero continuous cosine function $f$ on $G$ is of the form given in (2.7).

Same as [1, 2, 16] and keeping the same notation there, we first transfer Eq. (1.1) to an operator equation. Clearly, Eq. (1.1) is equivalent to

$$R_y f + R_{y^{-1}} f = 2f(y)f \quad \text{for all} \quad y \in G.$$

Then applying the Fourier transform, we deduce

$$(\pi(y) + \pi(y)^{-1})\hat{f}(\pi) = 2f(y)\hat{f}(\pi). \tag{2.8}$$

It is very well-known that cosine functions are central. So for every $[\pi] \in \hat{G}$, we have that $\hat{f}(\pi)$ is a scalar matrix (cf. [8]). That is, $\hat{f}(\pi) = \lambda_\pi I_{d_\pi}$ for some $\lambda_\pi \in \mathbb{C}$. Then (2.8) becomes

$$\lambda_\pi(\pi(y) + \pi(y)^{-1}) = 2\lambda_\pi f(y) I_{d_\pi} \quad \text{for all} \quad y \in G.$$

Since $f \neq 0$, obviously $\mathrm{supp}(\hat{f}) = \{[\pi] \in \hat{G} \mid \lambda_\pi \neq 0\} \neq \emptyset$. Let $[\pi] \in \mathrm{supp}(\hat{f})$. Then we have

$$\pi(y) + \pi(y)^{-1} = 2f(y) I_{d_\pi}. \tag{2.9}$$

Since $\pi$ is irreducible, applying Lemma 2.1 to $K = \pi(G)$, one obtains either $d_\pi = 1$, or $d_\pi = 2$ and $\pi(G) \leq SU(2)$.

If $d_\pi = 1$, then $f = \frac{1}{2}(\pi + \bar{\pi})$. Let $\rho : G \to SU(2)$ be the representation defined by $\rho = \pi \oplus \bar{\pi}$. Then $f = \frac{1}{2}\chi_\rho$.

If $d_\pi = 2$ and $\pi(G) \leq SU(2)$, taking trace in (2.9) and noticing that $\mathrm{tr}(U^{-1}) = \mathrm{tr}(U)$ for any $U \in SU(2)$, we get $f = \frac{1}{2}\chi_\rho$. So the representation $\rho = \pi : G \to SU(2)$ gives the desired one. $\square$

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