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1-PARAMETER SUBGROUPS OF THE CIRCLE-EXPONENT FUNCTION IN A-CONVEX ALGEBRAS

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ABSTRACT. We solve classical differential equations, linear and affine, in a more general setting than usual; no use is made of manifolds. We work within the context of general A-convex algebras.

1. Introduction

In differential geometry one speaks of integrable curves (solutions) of differential equations defined by (differentiable) vector fields on smooth manifolds. Yet, under the action of a Lie group on the particular manifold at issue ("*Klein geometry*"), one considers *invariant vector fields*, whose the Lie algebra is the "*tangent space*" of the Lie (action) group at the neutral element. We solve in the sequel analogous differential equations, by considering *A-convex* (topological) *algebras*, in connection with the underlying locally convex spaces, and the action of the (additive) Lie group $\mathbb{R}$ on the group of invertible and/or or quasi-invertible elements of the algebra. Precisely, the corresponding action of $\mathbb{R}$ on the latter group(s) is achieved, via the *exponent*/c-*exponent function*. So we arive at the *"Klein geometry" in A-convex algebras* (A. Mallios [3, 4]), through the latter functions.

Now, given a topological algebra $(\mathbb{A}, \tau)$ we consider, the family of *linear differential equations*,

$$\dot{\alpha} = T \circ \alpha, \tag{1.1}$$

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for suitable $T \in L(\mathbb{A}) := \{S : \mathbb{A} \to \mathbb{A} \text{ linear}\}$, where $\alpha : \mathbb{R} \to \mathbb{A}$ are appropriate (differentiable) curves (: solutions of (1.1)). Within the same spirit, we also consider the *"affine" equation*,

$$\dot{\alpha} = T + T \circ \alpha \tag{1.2}$$

(see also [8, 10]), where T is also considered, as the constant curve $c_T \equiv c : \mathbb{R} \to L_\tau(\mathbb{A}) : c_T(t) := T$, $t \in \mathbb{R}$. The case of a σ-complete unital left A-convex algebra $(\mathbb{A}, \tau)$ (in particular of an lmc-algebra) is analogous to the Banach case (see [2]). Equations (1.1) and (1.2) are considered for $T \in \mathcal{L}_\Gamma(\mathbb{A}) \supseteq l(\mathbb{A}) := \{l_u : u \in \mathbb{A}\}$, where for a convenient family Γ of left A-convex seminorms on $\mathbb{A}$ we have $\tau = \tau_\Gamma$, and also

$$l_u(x) := ux, \ \textit{with } u, x \textit{ in } \mathbb{A}.$$

Here we look for curves $\alpha : [-\varepsilon, \varepsilon] \subseteq \mathbb{R} \to \mathbb{A}$. It is possible to obtain solutions using 1-parameter subgroups,

$$\alpha_T : \mathbb{R} \to \Gamma(\mathbb{A})^\bullet : \alpha_T(t) := \exp(tT) \equiv 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} T^n, \ t \in \mathbb{R} \tag{1.3}$$

$$\beta_T : \mathbb{R} \to \Gamma(\mathbb{A})^\circ : \beta_T(t) := c\exp(tT) \equiv \sum_{n=1}^{\infty} \frac{t^n}{n!} T^n, \ t \in \mathbb{R} \tag{1.4}$$

where we put,

$$\mathcal{L}_\Gamma(\mathbb{A})^\bullet := \{T \in \mathcal{L}_\Gamma(\mathbb{A}) : T \text{ is invertible}\}$$
$$\mathcal{L}_\Gamma(\mathbb{A})^\circ := \{T \in \mathcal{L}_\Gamma(\mathbb{A}) : T \text{ is quasi-invertible}\}$$

for the groups of invertible and quasi-invertible elements of the algebra $\mathcal{L}_\Gamma(\mathbb{A})$, respectively.

Now, for a given locally convex space (V, τ) and family $\Gamma \subseteq Sem(V)$ of seminorms of V such that $\tau = \tau_\Gamma$ (with τ_Γ the topology on V having a subbase of neighborhoods of $0 \in V$ the family $\{\overline{S_p}(\varepsilon) := \{w \in V : p(w) < \varepsilon\}, \ \varepsilon > 0\} \equiv \mathfrak{A}_0(\tau_\Gamma)$) we have put, $\mathcal{L}_\Gamma(V) :=$

$$\{T \in L(V) : \tilde{p}(T) := \inf\{\varepsilon > 0 : p(Tx) \leq \varepsilon p(x), x \in V\} < +\infty, p \in \Gamma\} \tag{1.5}$$

and we call $\mathcal{L}_\Gamma(V)$ the *"algebra of Γ-uniformly continuous operators on V"* (see also [10, 11]). In [11] we prove that $\mathcal{L}_\Gamma(V)$ *depends on the particular family* Γ with $\tau_\Gamma = \tau$. (In other words, for families $\Gamma, \Delta \subseteq Sem(V) := \{p : V \to \mathbb{R}_+$, a vector space seminorm$\}$, with $\tau_\Gamma = \tau_\Delta = \tau$, it is possible that $\mathcal{L}_\Gamma(V) \neq \mathcal{L}_\Delta(V)$). Thus, it is convenient to consider the family $\mathcal{L}_{\tau(V)}$ of operators given by

$$\mathcal{L}_\tau(V) = \bigcup \{\mathcal{L}_\Gamma(V) : \Gamma \subseteq Sem(V) : \tau = \tau_\Gamma\}. \tag{1.6}$$

In [12, 13] we have seen that in a σ-complete left A-convex algebra $(\mathbb{A}, \tau)$ (in particular lmc-algebra, unital or not, yet, see e.g. [5, 6]) one can define (together with its exponent) the so called *"circle exponent function"* with values in the group $\mathbb{A}^\circ$ of quasi-invertible elements of $\mathbb{A}$, by the relation (see also [12]),

$$c\exp : \mathbb{A} \to \mathbb{A}^\circ : c\exp(a) := \sum_{n=1}^{\infty} \frac{t^n}{n!} a^n \in \mathbb{A}^\circ, \quad a \in \mathbb{A}. \tag{1.7}$$

This function satisfies the relation

$$c\exp(a+b) := c\exp(a)\circ c\exp(b), \quad a,b\in\mathbb{A} : ab=ba$$

while the *"circle operation"* in $\mathbb{A}$ is defined by

$$a\circ b := a+b+ab, \quad a,b\in\mathbb{A}. \tag{1.8}$$

Thus (in the case the topological algebra $(\mathbb{A},\tau)$ is σ-complete), we can define the family of curves:

$$\beta_u : \mathbb{R}\to\mathbb{A}^\circ : \beta_u(t)\equiv c\exp(tu) := \sum_{n=1}^{\infty}\frac{t^n}{n!}u^n\in\mathbb{A}^\circ,\ u\in\mathbb{A},\ t\in\mathbb{R}. \tag{1.9}$$

In case $\mathbb{A}$ is unital we can also define the curves:

$$\alpha_u : \mathbb{R}\to\mathbb{A}^\bullet : \alpha_u(t) := \sum_{n=0}^{\infty}\frac{t^n}{n!}u^n = 1+\beta_u(t),\ t\in\mathbb{R},\ u\in\mathbb{A}, \tag{1.10}$$

where we also put $\mathbb{A}^\bullet$ for *the group of invertible elements of* $\mathbb{A}$. These curves satisfy the (initial) conditions;

$$\beta_u(0)=0,\ \alpha_u(0)=1,\ u\in\mathbb{A}, \tag{1.11}$$

and we note that,

$$u\beta_u(t)=\beta_u(t)u,\ u\alpha_u(t)=\alpha_u(t)u,\ u\in\mathbb{A},\ t\in\mathbb{R}.$$

We also note that the families $\{\alpha_u : u\in\mathbb{A}\}$, $\{\beta_u : u\in\mathbb{A}\}$ realize the so-called *"differentiable* 1*-parameter subgroups"* of the groups $\mathbb{A}^\bullet$ and $\mathbb{A}^\circ$, respectively (see also [2, Chapters IX and X]):

$$\alpha_u\in\mathrm{Hom}^\infty(\mathbb{R},\mathbb{A}^\bullet),\ \beta_u\in\mathrm{Hom}^\infty(\mathbb{R},\mathbb{A}^\circ),\ u\in\mathbb{A}.$$

The applied notation above hints at the *"differentiability"* (smoothness) of the curves involved. At this point we also remark the following. Let

$$\phi : (\mathbb{R},t)\to(\mathbb{A},\cdot) : \phi(s+t) := \phi(s)\cdot\phi(t),\ s,t\in\mathbb{R}$$

be a semi-group morphism of the group $(\mathbb{R},+)$ (we consider it here, as a semi-group) into the semigroup $(\mathbb{A},\cdot)$ (under the ring-multiplication of $\mathbb{A}$). Then the image $\phi(\mathbb{R}) := \{\phi(t) : t\in\mathbb{R}\}\subseteq(\mathbb{A},\cdot)$ forms an abelian group $(\phi(\mathbb{R}),\cdot)$ (having as operation the ring-multiplication of $\mathbb{A}$). We also put $\phi(0)\equiv e$ the neutral element of $\phi(\mathbb{R})$.

In the topological case this group is contained in the group $(e\mathbb{A}e)^\bullet$ of invertible elements of the closed subalgebra

$$e\mathbb{A}e := \{eae : a\in\mathbb{A}\}\subseteq\mathbb{A}.$$

(In fact, for any $\beta\in\overline{e\mathbb{A}e}$ and net $\beta_\lambda = ea_\alpha e\in e\mathbb{A}e$, $\lambda\in\Lambda$, such that $\beta_\lambda \underset{\tau}{\longrightarrow} \beta$, we have,

$$\beta=\lim_\lambda\beta_\lambda=\lim_\lambda ea_\lambda e=\lim_\lambda(e(ea_\lambda e)e)=e\lim_\lambda(ea_\lambda e)e=e\beta e\in e\mathbb{A}e.$$

(because $(\mathbb{A},\tau)$ is supposed to have a separately continuous multiplication, so that left and right translations $x\mapsto ax$, $x\mapsto xa$ are continuous (see also [3]). Thus,

$e\mathbb{A}e$ is σ-complete if $\mathbb{A}$ is ($e\mathbb{A}e$ is the biggest subalgebra of $\mathbb{A}$ having $e \equiv \phi(0)$ as unit). Therefore we can define the family of 1-parameter subgroups:

$$\phi_u : \mathbb{R} \to (e\mathbb{A}e)^{\bullet} : \phi_u(t) := \exp(tu) \equiv e + \sum_{n=1}^{\infty} \frac{t^n}{n!} u^n,\ t \in \mathbb{R},\ u \in e\mathbb{A}e.$$

In other words, we obtain the corresponding to $\{a_u : u \in \mathbb{A}\}$ family $\{\phi_u : u \in e\mathbb{A}e\}$ in the (unital) algebra $(e\mathbb{A}e, e)$. This situation appears, in general, whenever we have an idempotent $0 \neq e = e^2 \in \mathbb{A}$. In each case we put

$$\mathrm{Hom}(\mathbb{R}, (e\mathbb{A}e)^{\bullet}) := \{\phi : \mathbb{R} \to (e\mathbb{A}e)^{\bullet} : \phi(s+t) = \phi(s)\phi(t),\ s, t \in \mathbb{R}\}$$

Moreover, for each $\psi \in \mathrm{Hom}^{\infty}(\mathbb{R}, (e\mathbb{A}e)^{\bullet})$, we easily get $\psi(0) = e$. But in the case we take $h \in \mathrm{Hom}^{\infty}(\mathbb{R}, e\mathbb{A}e) := \{\psi : \mathbb{R} \to e\mathbb{A}e : \psi(s+t) = \psi(s)\psi(t), s, t \in \mathbb{R}\}$ and we put $e' = \psi(0) \in \psi(\mathbb{R}) \subseteq e\mathbb{A}e$, we get $e' = ebe$ and so

$$e'\mathbb{A}e' = ebe\mathbb{A}ebe \subseteq e\mathbb{A}e$$

(such idempotents $e' \in \mathbb{A} : e'\mathbb{A}e' \subseteq e\mathbb{A}e$ are defined to be "*smaller than* e" (defining thus in the set of all idempotents $id(\mathbb{A})$ of $\mathbb{A}$ a partial order):

$$0 \neq e' \leq e \underset{\text{def}}{\Longleftrightarrow} e'\mathbb{A}e' \subseteq e\mathbb{A}e \Leftrightarrow e' \in e\mathbb{A}e, \quad 0 \neq e' = e'^2 \in \mathbb{A}.$$

Then, it is easy to see that $e' = e \Leftrightarrow e'\mathbb{A}e' = e\mathbb{A}e$ (because then e, e' are two units of $e\mathbb{A}e : e' = e'e = e$). Thus, in the case $e' \in e\mathbb{A}e$ with $0 \neq e'^2 \in e\mathbb{A}e$ and $e'\mathbb{A}e' \neq e\mathbb{A}e$, we get $e' \neq e$).

2. On left A-convex algebras

A-convex algebras have been introduced in [1]. Here by saying *left A-convex seminorm* p of $\mathbb{A}$ we mean that p satisfies (by definition) the relation

$$\forall a \in \mathbb{A}\ \exists \lambda \equiv \lambda_a > 0 : p(ax) \leq \lambda p(x),\ x \in \mathbb{A}. \tag{2.1}$$

A topological algebra $(\mathbb{A}, \tau)$ is said to be a *left A-convex algebra* if there exists a family $\Gamma \subseteq Sem(\mathbb{A})$ of left A-convex seminorms of $\mathbb{A}$ such that $\tau = \tau_{\Gamma}$ (see also comments in (1.5) for τ_{Γ}). In [3, proof of Lemma I.5.4] we see that each p-ball $S_p(\varepsilon) = \{w \in \mathbb{A} : p(w) \leq \varepsilon\}$ satisfies the condition,

$$\text{for each } a \in \mathbb{A}, \text{ there exists } \lambda > 0 : aS_p(\varepsilon) \subseteq \lambda S_p(\varepsilon).$$

(such a seminorm is called *absorbing*). But we see below that if a balanced absorbing convex set $U \subseteq \mathbb{A}$ satisfies moreover the relation,

$$\text{for each } a \in \mathbb{A}, \text{ there exists } \lambda > 0 : aU \subseteq \lambda U,$$

then its gauge $P = P_U$ is an *absorbing seminorm*. Thus, we can define *left A-convex* algebras as follows: A topological algebra $(\mathbb{A}, \tau)$ is said to be *left A-convex*, if there exists a subbase $\mathfrak{A}_0(\tau)$ of neighborhoods of $0 \in \mathbb{A}$, satisfying (2.1). We claim that the above two definitions are equivalent.

First, we have the following.

Proposition 2.1. *Let p be a seminorm of a vector space V and*

$$p(V) := \{T \in L(V) : \exists\ \varepsilon > 0 : p(Tx) \leq \varepsilon p(x), \quad x \in V\}.$$

Then for each $T \in L(V)$ the following are equivalent:
(a) $T \in p(V)$.
(b) $\exists \lambda > 0 : \forall \varepsilon > 0$ we have $T S_p(\varepsilon) \subseteq \lambda \overline{S_p}(\varepsilon)$.

Proof. $(a) \Rightarrow (b)$. Let $w \in T\overline{S_p}(\varepsilon)$. Then $w = Tx : p(x) < \varepsilon$. By (a) $p(w) = p(Tx) \leq \tilde{p}(T)p(x) < \tilde{p}(T)\varepsilon$. Thus $p\big(\frac{1}{\tilde{p}(T)}w\big) < \varepsilon \Leftrightarrow \frac{1}{\tilde{p}(T)}w \in S_p(\varepsilon) \Leftrightarrow w \in \tilde{p}(T)S_p(\varepsilon)$. With $\lambda = \tilde{p}(T)$, we have the relation (b). $(b) \Rightarrow (a)$. We firstly note that, if $T\overline{S_p}(\varepsilon) \subseteq \lambda\overline{S_p}(\varepsilon)$ for some $\varepsilon > 0$, then this holds also for any $\varepsilon > 0$. Thus let $T\overline{S_p}(\varepsilon) \subseteq \lambda\overline{S_p}(\varepsilon)$ for some $\lambda, \varepsilon > 0$. Then for $x \in V : p(x) < 1$ we get: $p(\varepsilon x) < \varepsilon$, $\varepsilon x \in \overline{S_p}(\varepsilon)$. $T\varepsilon x \in T\overline{S_p}(\varepsilon) \subseteq \lambda\overline{S_p}(\varepsilon) = \overline{S_p}(\lambda\varepsilon)$. Thus, $p(T\varepsilon x) \leq \lambda\varepsilon \Leftrightarrow p(Tx) \leq \lambda \Leftrightarrow \left(\frac{1}{\lambda}p \circ T\right)(x) \leq 1$. Then by [3, Lemma I.1.2] we obtain $\frac{1}{\lambda}p \circ T \leq p \Leftrightarrow p(Tx) \leq \lambda p(x)$, $x \in V$, $T \in p(V)$.

Remark 2.2. – By using the above proposition, we get the equivalence of the definition of p-uniformly bounded operators on V, given in [7, 14] (see also [10]). Moreover, we see that the lmc-algebra defined on (the algebra of) uniformly continuous operators in a locally convex space (V, τ) (corresponding to a given subbase $\mathfrak{A}_0(\tau)$ of neighborhoods of $0 \in V$), coincides with the algebra $\mathcal{L}_\Gamma(V)$ defined in [14] (or also [10]) for a particular family $\Gamma \subseteq Sem(V)$, defined by the family $\mathfrak{A}_0(\tau)$, as above. See also comments following (1.5).

We can now move on the following.

Proposition 2.3. *Let U be a neighborhood of zero of an algebra $\mathbb{A}$, and $p \equiv p_U$ be its Minkowski functional (: gauge). Then the following are equivalent:*
(a) For each $a \in \mathbb{A}$ there exists $\lambda \equiv \lambda_a > 0 : aU \subseteq \lambda U$.
(b) For each $a \in \mathbb{A}$ there exists $\lambda \equiv \lambda_a > 0 : p(ax) \leq \lambda p(x)$, $x \in V$.
Thus, the above definitions for left A-convex algebras are equivalent.

Proof. See [3].

Remark 2.4. – It is easy to see that if (V, τ) is σ-complete, then $\mathcal{L}_\Gamma(V)$ is σ-complete for every family $\Gamma \subseteq Sem(V) : \tau = \tau_\Gamma$. By $\tilde{p}(l_a) = \overline{p}(a)$, as above, we see that if $(\mathbb{A}, \tau)$ is a σ-complete left A-convex algebra, then $\mathcal{L}_\Gamma(\mathbb{A})$ is also σ-complete so that the corresponding lmc algebra $(\mathbb{A}, \overline{\Gamma})$ to $(\mathbb{A}, \Gamma)$ is also σ-complete. Therefore, we can define the exponent and circle-exponent functions in $(\mathbb{A}, \overline{\Gamma})$ and so in $(\mathbb{A}, \Gamma)$, by $p(u^{n+1}) \leq \overline{p}(u^n) \cdot p(u) \leq \overline{p}(u)^n \cdot p(u)$, with $u \in \mathbb{A}$ and $n \in \mathbb{N}$.

3. Solutions of basic linear and affine differential equations in $\mathcal{L}_\tau(V)$ and (V, τ)

In [2, Chapters IX and X] we see that, by considering a Banach algebra B, the curves $\alpha : (0, +\infty) \to B$ for which $\alpha(s + t) = \alpha(s)\alpha(t)$ and also $\lim_{s \to 0^+} \alpha(s) = e$ (unit of B) exists, one can find $u \in B$, such that (see also (1.10))

$$\alpha(t) = \exp(tu).$$

Thus, *the curve α can be extended to all the real line* $\mathbb{R}$. We assume here the framework of σ-complete left A-convex algebras $(\mathbb{A}, \Gamma)$ (instead of Banach algebras, as above), to obtain the same results. So we consider:

(i) A locally convex space (V, τ) in order to solve (1.1) for every $T \in \mathcal{L}_\tau(V)$ (see below and also (1.6)).

(ii) The family $\mathcal{L}_\tau(V) \subseteq L(V)$ of τ-*uniformly bounded* operators on (V, τ) (see (1.6)), which contains all algebras $\mathcal{L}_\Gamma(V) : \tau = \tau_\Gamma$, where $\Gamma \subseteq sem(V)$. In $\mathcal{L}_\tau(V)$ we can put and at the same time solve both equations (1.1) and (1.2) for all $T \in \mathcal{L}_\tau(V)$. Here the solutions are curves $\alpha : \mathbb{R} \to \mathcal{L}_\tau(V)$; more precisely, we get:

(ii′), 1).– The curve (1.3) satisfies the equation

$$\dot{a}_T = Ta_T, \quad T \in \mathcal{L}_\tau(V).$$

In fact, for appropriate $\Gamma \subseteq sem(V) : \tau = \tau_\Gamma : T \in \mathcal{L}_\Gamma(V)$, we easily compute (cf. [10] for details, in particular, (1.0) therein).

2).– Also for $S \in \mathcal{L}_T(V)^\bullet$, with

$$\mathcal{L}_T(V, \tau) \equiv \mathcal{L}_T(V) \underset{\text{def}}{:=} \{S \in \mathcal{L}_\tau(V) : \exists \Gamma \subseteq Sem(V),\ T, S \in \mathcal{L}_\Gamma(V)^\bullet\}),$$

we consider the curve

$$\alpha_{TS} : \mathbb{R} \to \mathcal{L}_\Gamma(V)^\bullet \subseteq \mathcal{L}_T(V) \subseteq \mathcal{L}_\tau(V) : \alpha_{TS}(t) := S\alpha_T(t),\ t \in \mathbb{R},$$

where we easily compute that $\alpha_{TS}(t) = S + \sum_{n=1}^{\infty} \frac{t^n}{n!} ST^n$.

Moreover, we obtain $\dot{\alpha}_{TS}(t) = S\dot{\alpha}_T(t) = ST\alpha_T(t) = S\alpha_T(t)T = \alpha_{TS}(t)T$, where more precisely we have to write $\lambda_T \circ a_T$ instead of $a_T(t)T$, $t \in \mathbb{R}$, with $\lambda_T(T') = T'T$ the right translation of the corresponding $\mathcal{L}_\Gamma(V)$. Therefore, we get

$$\dot{\alpha}_{TS} = \lambda_T \alpha_{TS} \equiv \alpha_{TS} T,\ T \in \mathcal{L}_\tau(V),\ S \in \mathcal{L}_T(V)^\bullet.$$

In other words, α_{TS} is a solution of (1.1) in $\mathcal{L}_\tau(V)^\bullet$ satisfying the initial condition

$$\alpha_{TS}(0) = SI = S, \quad S \in \mathcal{L}_T(V)^\bullet,$$

where, obviously, $\alpha_T = \alpha_{TI}$.

3).– Now, we note that the curve (1.4) satisfies the equation,

$$\dot{\beta}_T = T + T\beta_T,\ \beta_T(0) = 0,\ T \in \mathcal{L}_\tau(V).$$

In fact (see also (1.8)), we get

$$\begin{aligned}
\dot{\beta}_T(t) &:= \lim_{s \to 0} \frac{1}{s}\big(\beta_T(t+s) - \beta_T(t)\big) = \lim_{s \to 0} \frac{1}{s}\big(\beta_T(t) + \beta_T(s) + \beta_T(t)\beta_T(s) - \beta_T(t)\big) \\
&= \lim_{s \to 0} \big[\beta_T(s)(I + \beta_T(t))\big] = (I + \beta_T(t)) \lim_{s \to 0} \left(T + \frac{sT^2}{2!} + \frac{s^2T^3}{3!} + \cdots\right) \\
&= (I + \beta_T(t))T = T(I + \beta_T(t)) = T + T\beta_T(t) = T + \beta_T(t)T.
\end{aligned}$$

Therefore, β_T is a solution of (1.2) satisfying the initial condition $\beta_T(0) = 0$.

4).– Now, for any $S \in \mathcal{L}_T(V)^\circ$ we define, by analogy, the curve

$$\beta_{TS} := S \cdot \beta_T \equiv S + \beta_T + S\beta_T,\ T \in \mathcal{L}_\tau(V),\ S \in \mathcal{L}_T(V)^\bullet.$$

Hence, we can compute

$$\begin{aligned}\dot{\beta}_{TS} &= \dot{S} + \dot{\beta}_T + S\dot{\beta}_T = \dot{\beta}_T + S\dot{\beta}_T = T + T\beta_T + ST + ST\beta_T \\ &= T + (S + \beta_T + S\beta_T)T = T + \beta_{TS}T,\ T \in \mathcal{L}_\tau(V),\ S \in \mathcal{L}_T(V)^\circ.\end{aligned}$$

Thus, β_{TS} satisfies also (1.2) and the initial condition $\beta_{TS}(0) = S$. Following [9], we can say that every $\mathcal{L}_\Gamma(V)$ is the tangent space of the group $\mathcal{L}_\Gamma(V)^\bullet$ at I, and also the tangent space of the group $\mathcal{L}_\Gamma(V)^\circ$ at 0, (because $\beta_T(\mathbb{R}) \subseteq \mathcal{L}_\Gamma(V)^\circ$ and also $\dot{\beta}_T(0) = T$, $T \in \mathcal{L}_\Gamma(V)$). But given the σ-complete lc space (V, τ) we possibly have several equivalent families $\Gamma \subseteq Sem(V) : \tau = \tau_\Gamma$ in such a way that the family $\mathcal{L}_\tau(V)$ does not coincide with $\mathcal{L}_\Gamma(V)$, for all such Γ. On the other hand, it is a question whether $\mathcal{L}_\tau(V)$ has the structure of an algebra (or even of a vector space). Thus, we can put $\mathcal{L}_\tau(V)^\bullet$, $\mathcal{L}_T(V)^\bullet$, $\mathcal{L}_\tau(V)^\circ$, $\mathcal{L}_T(V)^\circ$ for the intersection of $L(V)^\bullet$ and $L(V)^\circ$ with the families $\mathcal{L}_\tau(V)$, $\mathcal{L}_T(V)$ respectively. In this respect, we can say that $\mathcal{L}_\tau(V)$ is the tangent family of the family $\mathcal{L}_\tau(V)^\bullet$ at $I \in L(V)$, but also of the family $\mathcal{L}_\tau(V)^\circ$ at $0 \in L(V)$. We can also define $\mathcal{L}_T(V)$ as the tangent family of the family $\mathcal{L}_T(V)^\bullet$ (of invertible operators) at $I \in L(V)$, but also of the family $\mathcal{L}_T(V)^\circ$ (of quasi invertible operators) at $0 \in L(V)$. Now, we try to solve (1.1) and (1.2) in (V, τ) itself using the 1-parameter subgroups a_T, β_T, as above. In this connection, consider the curve, $a_{xT} : \mathbb{R} \to (V, \tau)$,

$$a_{xT}(t) \underset{\text{def}}{:=} a_T(t)(x) \equiv x + \sum_{n=1}^{\infty} \frac{t^n}{n!} T^n(x),\ T \in \mathcal{L}_T(V), x \in V.$$

By an easy computation (see [9] for details), we find

$$\dot{a}_{xT}(t) = Ta_T(t)(x) \equiv Ta_{xT}(t),\ t \in \mathbb{R},\ x \in V,\ T \in \mathcal{L}_\tau(V)$$

$$a_{xT}(0) = a_T(0)(x) = I(x) = x,\ \ x \in V,\ \ T \in \mathcal{L}_\tau(V).$$

In other words a_{xT} satisfies (1.1) and the initial condition $a_{xT}(0) = x$, $x \in V$, $\alpha_{xT}(\mathbb{R}) \subseteq V$. Therefore, we obtain a solution a_{xT}, through every $x \in V$ for any differential equation $\dot{a} = Ta$ in V, with $T \in \mathcal{L}_\tau(V)$. In the same vein of ideas, we can consider the solution a_{TS}, through $S \in \mathcal{L}_T(V)$, instead of $S \in \mathcal{L}_T(V)^\bullet$,

$$a_{TS}(t) := Sa_T(t),\ t \in \mathbb{R},\ T \in \mathcal{L}_\tau(V),\ S \in \mathcal{L}_T(V)$$

(see also (1.2)),

$$\beta_{TS}(t) := S + \beta_T + S\beta_T,\ T \in \mathcal{L}_\tau(V),\ S \in \mathcal{L}_T(V).$$

Now, we try to solve (1.2) in (V, τ), using β_T in order to define the curves $\beta_{xT}(t) := \beta_T(t)(x) \in V$, $x \in V$. By an easy computation we obtain $\dot{\beta}_{xT} = \dot{a}_{xT} = Ta_{xT}$ a relation which does not constitute a solution of (1.2). Therefore the 1-parameter subgroup β_T does not give us solutions of (1.2) in the locally convex space (V, τ). In the following we see that in the case of a left A-convex algebra $(\mathbb{A}, \Gamma) \equiv (\mathbb{A}, \tau)$ (in particular of an lmc-algebra $(\mathbb{A}, \Gamma)$) both a_u, β_u (observe that (1.10) and (1.9) give solutions of (1.1) and (1.2), respectively.

4. Solutions of basic equations in unital and non unital algebras (continued)

We start by first considering a *unital algebra* $\mathbb{A}$. Then its left regular representation (: LRR) l gives an embedding of $\mathbb{A}$ into the algebra $L(\mathbb{A}) := \{T : \mathbb{A} \to \mathbb{A},\ \text{linear}\}$ of operators on $\mathbb{A}$, by the relation

$$\mathbb{A} \cong l(\mathbb{A}) \underset{\to}{\subset} L(\mathbb{A}).$$

In case $(\mathbb{A}, \Gamma)$ is moreover a left *A-convex topological algebra* (in particular, an lmc-algebra $(\mathbb{A}, \Gamma)$) we put:

$$\tilde{p}(T) := \inf\{\varepsilon > 0 : p(Tx) \leq \varepsilon p(x), x \in \mathbb{A}\},\ T \in \mathcal{L}_\tau(\mathbb{A}),\ \tau = \tau_\Gamma.$$

Thus, we obtain the *"isometry"* (see also [12]).

$$\overline{p}(x) = \tilde{p}(l_x),\ x \in \mathbb{A},\ p \in \Gamma,$$

where we have put

$$\overline{p}(x) := \inf\{\varepsilon > 0 : p(xy) \leq \varepsilon \cdot p(y), y \in \mathbb{A}\} \geq 0, p \in \Gamma,\ x \in \mathbb{A}$$

(see also [13] for details). Each $\overline{p}$ is thus an *algebra seminorm* on $\mathbb{A}$ (: $\overline{p}(xy) \leq \overline{p}(x)\overline{p}(y),\ x, y \in \mathbb{A}$) and by putting $\overline{\Gamma} \equiv \{\overline{p} : p \in \Gamma\}$ we obtain a corresponding lmc algebra $(\mathbb{A}, \overline{\Gamma})$ of the initial left A-convex algebra $(\mathbb{A}, \Gamma)$, which is *σ-complete, if* $(\mathbb{A}, \Gamma)$ *is*, because $\Gamma(\mathbb{A})$ is σ-complete if $(\mathbb{A}, \Gamma)$ is). Thus

$$l(\mathbb{A}) \underset{\to}{\subset} \mathcal{L}_\Gamma(\mathbb{A}) \underset{\to}{\subset} \mathcal{L}_\tau(\mathbb{A})$$

(see also (1.6)). Now, we can use Section 1 above by considering a_T, β_T (see (1.3) and (1.4)) for $T = l_u$, $u \in \mathbb{A}$, $l_u(x) \equiv ux$, $u, x \in \mathbb{A}$. But we can also consider the curves $a_u \in \mathrm{Hom}^\infty(\mathbb{R}, \mathbb{A}^\bullet)$, $\beta_u \in \mathrm{Hom}^\infty(\mathbb{R}, \mathbb{A}^\circ)$ (see (1.9) and (1.10)), for which we obtain (by an easy computation):

$$\dot{a}_u = ua_u = a_u u,\ \dot{\beta}_u = u + u\beta_u = u + \beta_u u,\ u \in \mathbb{A}.$$

These curves in $\mathbb{A}$ satisfy the initial conditions (1.11). Now, for each $x \in \mathbb{A}$, we define the curves,

$$a_{xu} : \mathbb{R} \to \mathbb{A} : a_{xu}(t) := a_u(t)x = x + \sum_{n=1}^{\infty} \frac{t^n}{n!} u^n x \equiv r_x \circ a_u(t),\ \textit{with } x, u \textit{ in } \mathbb{A}.$$

If, in particular, $x \in \mathbb{A}^\bullet$, then $a_{xu}(\mathbb{R}) \subseteq \mathbb{A}^\bullet$. Then we can easily find the relations

$$r_x \circ a_u = a_{x,u} = a_{xl_u},\ \textit{with } x, u \textit{ in } \mathbb{A},$$

$$\frac{d}{dt}(r_x \circ a_u) = \dot{a}_{x,u} = \dot{a}_{xl_u} = ua_{x,u} = l_u a_{xl_u},\ \textit{with } x, u \textit{ in } \mathbb{A},$$

where $r_x(u) := ux$ is the right R.R. of $\mathbb{A}$. This means that we have found solutions of the differential equation (1.1) satisfying the initial conditions $\alpha_{x,u}(0) = x$, $x \in \mathbb{A}$. (For $x \in \mathbb{A}^\bullet$ the solutions are also in $\mathbb{A}^\bullet$). Although β_T have failed to give solutions of (1.2) in (V, τ), β_u can give us solutions of (1.2) in $(\mathbb{A}, \tau = \tau_\Gamma)$. Thus we define the family $\beta_{x,u}$ of curves by the relations,

$$\beta_{x,u} : \mathbb{R} \to \mathbb{A} : \beta_{x,u}(t) := x \circ \beta_u(t) \equiv x + \beta_u(t) + x\beta_u(t),\ \textit{with } x, u \textit{ in } \mathbb{A} \quad (4.1)$$

where in particular, for $x \in \mathbb{A}^\circ$ we also obtain;

$$\beta_{x,u}(\mathbb{R}) \subseteq \mathbb{A}^\circ,\ u \in \mathbb{A},\ x \in \mathbb{A}^\circ.$$

Then using (4.1) we compute

$$\begin{aligned}\dot{\beta}_{x,u}(t) &= \dot{x} + \dot{\beta}_u(t) + x\dot{\beta}_u(t) = \dot{\beta}_u(t) + x\dot{\beta}_u(t)\\ &= u + u\beta_u(t) + xu + xu\beta_u(t)\\ &= u + (x + \beta_u(t) + x\beta_u(t))u = u + \beta_{x,u}(t)u\end{aligned}$$

Therefore, $\beta_{x,u}$ is a solution of (1.2) in $\mathbb{A}$

$$\dot{\beta}_{x,u} = u + \beta_{x,u}u,\ x, u \in \mathbb{A},\ \beta_{x,u}(0) = x,\ x \in \mathbb{A}.$$

If in particular, we take $x \in \mathbb{A}^\circ$, we obtain

$$\beta_{x,u}(\mathbb{R}) \subseteq \mathbb{A}^\circ,\ x \in \mathbb{A}^\circ,\ u \in \mathbb{A}. \tag{4.2}$$

¿From the above, we can now notice the advantage of an algebra-structure that, in addition, we have defined on a vector space V. In other words, the geometry of an algebra is richer than the geometry of the underlying vector space. Now, we consider the case of a *non-unital algebra* $\mathbb{A}$. Obviously we can also consider the 1-parameter subgroup βu in the group $\mathbb{A}^\circ$ of quasi-invertible elements and also the curves $\beta_{x,u}$ as above, getting thus the solutions of (1.2) through each $x \in \mathbb{A}$ (and also in particular of $x \in \mathbb{A}^\circ$) for all $u \in \mathbb{A}$. But what happens with the differential equation (1.1)? In this case *we can solve* (1.1) *"locally"*. In fact, given the $0 \neq e = e^2 \in \mathbb{A}$, we can solve (1.1) for each $u \in e\mathbb{A}e$ (which is a closed subalgebra of $(\mathbb{A}, \tau)$ for each algebra topology τ on $\mathbb{A}$). In this case we define;

$$a_{e,u} : \mathbb{R} \to (e\mathbb{A}e)^\bullet : a_{e,u}(t) := e + \sum_{n=1}^{\infty} \frac{t^n}{n!}u^n,\ u \in e\mathbb{A}e,\ t \in \mathbb{R},$$

$$a_{x,u} : \mathbb{R} \to e\mathbb{A}e : a_{x,u}(t) := a_{e,u}(t)x,\ x \in e\mathbb{A}e,\ u \in e\mathbb{A}e.$$

getting in this way a solution of (1.1), through each $x \in e\mathbb{A}e$ (in the sense that $a_{x,u}(0) = x$). In particular for $x \in (e\mathbb{A}e)^\bullet$ we get $a_{x,u}(\mathbb{R}) \subseteq (e\mathbb{A}e)^\bullet$. Therefore, we can consider (1.1) for each $u \in \mathbb{A}$ for which there exists $0 \neq e = e^2 \in \mathbb{A} : u \in e\mathbb{A}e$. In the latter case we can consider *the algebra* $e\mathbb{A}e$ *as the tangent space of the group* $(e\mathbb{A}e)^\bullet$ (of the invertible elements of the unital algebra (eAe, e)) *at the unit* e. We thus conclude that the differential equations (1.2) can be considered for any $u \in (\mathbb{A}, \tau)$, for a left A-convex algebra $(\mathbb{A}, \tau)$ and we also obtain the solutions (4.2) using the 1-parameter subgroup β_u (see (1.9)) of the group $\mathbb{A}^\circ$ of quasi-invertible elements of $\mathbb{A}$. Therefore, the circle-exponent function (see (1.7)) makes it possible to *present* $\mathbb{A}$ *as the tangent space of the group* $\mathbb{A}^\circ$ at $0 \in \mathbb{A}$ *iff there exists a left A-convex family* $\Gamma \subseteq Sem(\mathbb{A})$ *making* $\mathbb{A}$ *a left A-convex algebra.*

Note. We write (1.2) in the form $T = \dot{a} - T \circ a \equiv \dot{a} - T|_a$; so we express T, through $\dot{a}$ and its restriction on a.

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