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INNERNESS OF HIGHER DERIVATIONS

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Abstract. Let $\mathcal{A}$ be an algebra. A sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ is called a higher derivation if $d_n(ab) = \sum_{k=0}^{n} d_k(a)d_{n-k}(b)$ for each $a, b \in \mathcal{A}$ and each nonnegative integer n. In this paper a notion of an inner higher derivation is given. We characterize all uniformly bounded inner higher derivations on Banach algebras and show that each uniformly bounded higher derivation on a Banach algebra $\mathcal{A}$ is inner provided that each derivation on $\mathcal{A}$ is inner.

1. Introduction

Let $\mathcal{A}$ be an algebra. A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a derivation if it satisfies the Leibniz rule $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. A typical example of a derivation is $\delta_{a_0} : \mathcal{A} \to \mathcal{A}$ given by $\delta_{a_0}(a) = a_0a - aa_0$, where $a_0 \in \mathcal{A}$. A derivation of this form is called inner. One of the important questions in the theory of derivations is that "When are all bounded derivations on a Banach algebra inner?" Forty years ago, R. V. Kadison [3] and S. Sakai [10] independently proved that every derivation on a von Neumann algebra $\mathfrak{M}$ is inner; see also [8]. Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a homomorphism. As a generalization of the notion of a derivation, a linear mapping $D : \mathcal{A} \to \mathcal{A}$ is called a (σ, σ)-derivation if it satisfies the generalized Leibniz rule $D(ab) = D(a)\sigma(b) + \sigma(a)D(b)$ for all $a, b \in \mathcal{A}$ (see [7]).

If we define a sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ by $d_0 = I$ and $d_n = \frac{\delta^n}{n!}$, where I is the identity mapping on $\mathcal{A}$ and δ is a derivation on $\mathcal{A}$, then the Leibniz

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rule ensures us that d_n's satisfy the condition

$$d_n(ab) = \sum_{j=0}^{n} d_j(a)d_{n-j}(b) \quad (*)$$

for each $a, b \in \mathcal{A}$ and each nonnegative integer n. This motivates us to consider the sequences $\{d_n\}$ of linear mappings on an algebra $\mathcal{A}$ satisfying $(*)$. Such a sequence is called a higher derivation. Higher derivations were introduced by Hasse and Schmidt [1], and algebraists sometimes call them Hasse-Schmidt derivations. If $\delta : \mathcal{A} \to \mathcal{A}$ is a derivation then $d_n = \frac{\delta^n}{n!}$ is a higher derivation, though this is not the only example of a higher derivation. Let $\mathcal{A}$ be a unital algebra. A higher derivation $\{d_n\}$ is called inner in the sense of Roy and Sridharan, RS-inner, if $d_0 = I$ and there is a sequence $\{a_n\}$ in $\mathcal{A}$ with $a_0 = 1$, such that $\sum_{k=0}^{n} d_k(a)a_{n-k} = a_n a$ for each $a \in \mathcal{A}$ (see [9]). In this paper we give an alternative definition of innerness.

Among higher derivations we are interested in uniformly bounded higher derivations on a Banach algebra. A higher derivation $\{d_n\}$ is called uniformly bounded if there is an $M > 0$ such that $\|d_n\| \leq M$ for each n. A natural question is "When are all uniformly bounded higher derivations on a given Banach algebra inner?"

Indeed many mathematicians have shown that higher derivations are bounded (but possibly not uniformly bounded) in special cases. Loy [4] proved that if $\mathcal{A}$ is an (F)-algebra which is a subalgebra of a Banach algebra $\mathcal{B}$ of power series, then every higher derivation $\{d_n\} : \mathcal{A} \to \mathcal{B}$ is automatically continuous. Jewell [2], showed that a higher derivation from a Banach algebra onto a semisimple Banach algebra is continuous provided that $\ker(d_0) \subseteq \ker(d_n)$, for all $n \geqslant 1$. Villena [11], proved that every higher derivation from a unital Banach algebra $\mathcal{A}$ into $\mathcal{A}/\mathcal{P}$, where $\mathcal{P}$ is a primitive ideal of $\mathcal{A}$ with infinite codimension, is continuous. As a consequence of the Jewell theorem [2], each higher derivation on a C^*-algebra is automatically continuous. Also in [5] and [6], the first-named author gives a characterization of higher derivations and prime higher derivations on an algebra $\mathcal{A}$ in terms of derivations on $\mathcal{A}$, provided that d_0 is the identity mapping on $\mathcal{A}$. A sequence $\{d_n\}$ of linear mappings on an algebra $\mathcal{A}$ is called a prime higher derivation if $d_n(ab) = \sum_{k|n} d_k(a)d_{\frac{n}{k}}(b)$ for each $a, b \in \mathcal{A}$ and each $n \in \mathbb{N}$.

In the first section, we use the generating function of a uniformly bounded higher derivation to find some elementary facts concerning uniformly bounded higher derivations. We give a notion of innerness and state a characterization of uniformly bounded inner higher derivations in terms of their generating functions. In the second section, we show that each uniformly bounded higher derivation on an algebra $\mathcal{A}$ is inner provided that each derivation on $\mathcal{A}$ is inner.

2. Characterization of Uniformly Bounded Inner Higher Derivations

Throughout the paper, $\mathcal{A}$ is a unital Banach algebra with unit 1 and I is the identity mapping on $\mathcal{A}$. If $\{T_n\}$ is a uniformly bounded sequence of linear mappings on $\mathcal{A}$, then the function $\psi(t) = \sum_{n=0}^{\infty} T_n t^n$ is well defined for

$|t| < 1$. Moreover, the m-th derivative of ψ exists and obtained by $\psi^{(m)}(t) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} T_n t^{n-m}$. We use these facts during the paper.

Definition 2.1. Let $\mathcal{A}$ be a Banach algebra and $d = \{d_n\}$ be a uniformly bounded higher derivation on $\mathcal{A}$. The generating function of $\{d_n\}$, denoted by α, is defined for $|t| < 1$ by $\alpha_t = \sum_{n=0}^{\infty} d_n t^n$.

Recall that the Cauchy product of two sequences $\{a_n\}$ and $\{b_n\}$ is the sequence $\{c_n\}$ defined by $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. We denote c_n by $(a*b)_n$. Note that we formally have

$$\left(\sum_{n=0}^{\infty} a_n t^n\right)\left(\sum_{n=0}^{\infty} b_n t^n\right) = \sum_{n=0}^{\infty} (a*b)_n t^n.$$

Lemma 2.2. *Let $\mathcal{A}$ be a Banach algebra and $\{d_n\}$ be a uniformly bounded higher derivation on $\mathcal{A}$ with the generating function α. Then α_t is a homomorphism on $\mathcal{A}$ for $|t| < 1$.*

Proof. For each $a, b \in \mathcal{A}$, we have

$$\alpha_t(ab) = \sum_{n=0}^{\infty} d_n(ab) t^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} d_k(a) d_{n-k}(b)\right) t^n = \alpha_t(a)\alpha_t(b).$$

□

Recall that if a_0 is a fixed member of $\mathcal{A}$, then the inner derivation δ_{a_0} constructed via a_0 is defined by $\delta_{a_0}(a) = a_0 a - a a_0$ for all $a \in \mathcal{A}$.

Example 2.3. Let $\{d_n\}$ be the sequence defined recursively on $\mathcal{A}$ by $n d_n = \sum_{k=1}^{n} \delta_{a_k} d_{n-k}$ with $d_0 = I$, where $\{a_k\}$ is a sequence in $\mathcal{A}$. Then $\{d_n\}$ is a higher derivation.

To show this we use induction on n. For $n = 0$ we have $d_0(ab) = ab = d_0(a)d_0(b)$. Let us assume that $d_k(ab) = \sum_{i=0}^{k} d_i(a) d_{k-i}(b)$ for $k < n$. Thus we have

$$\begin{aligned} n d_n(ab) &= \sum_{k=1}^{n} \delta_{a_k} d_{n-k}(ab) \\ &= \sum_{k=1}^{n} \delta_{a_k} \sum_{i=0}^{n-k} d_i(a) d_{n-k-i}(b) \\ &= \sum_{k=1}^{n} \sum_{i=0}^{n-k} [a_k d_i(a) d_{n-k-i}(b) - d_i(a) d_{n-k-i}(b) a_k] \\ &= \sum_{k=1}^{n} \sum_{i=0}^{n-k} [a_k d_i(a) d_{n-k-i}(b) - d_i(a) a_k d_{n-k-i}(b)] \\ &\quad + \sum_{k=1}^{n} \sum_{i=0}^{n-k} [d_i(a) a_k d_{n-k-i}(b) - d_i(a) d_{n-k-i}(b) a_k]. \end{aligned}$$

We therefore have

$$\begin{aligned} nd_n(ab) &= \sum_{i=0}^{n}\left(\sum_{k=1}^{n-i}\delta_{a_k}d_{n-k-i}(a)\right)d_i(b) \\ &\quad +\sum_{i=0}^{n}d_i(a)\left(\sum_{k=1}^{n-i}\delta_{a_k}d_{n-k-i}(b)\right) \\ &= \sum_{i=0}^{n}(n-i)d_{n-i}(a)d_i(b) \\ &\quad +\sum_{i=0}^{n}d_i(a)(n-i)d_{n-i}(b) \\ &= \sum_{i=0}^{n}id_i(a)d_{n-i}(b)+\sum_{i=0}^{n}(n-i)d_i(a)d_{n-i}(b) \\ &= n\sum_{i=0}^{n}d_i(a)d_{n-i}(b). \end{aligned}$$

Remark 2.4. The first five terms of $\{d_n\}$ are

$$\begin{aligned} d_0 &= I, d_1=\delta_{a_1}, d_2=\frac{1}{2}\delta_{a_1}^2+\frac{1}{2}\delta_{a_2}, \\ d_3 &= \frac{1}{6}\delta_{a_1}^3+\frac{1}{6}\delta_{a_1}\delta_{a_2}+\frac{1}{3}\delta_{a_2}\delta_{a_1}+\frac{1}{3}\delta_{a_3}, \\ d_4 &= \frac{1}{24}\delta_{a_1}^4+\frac{1}{24}\delta_{a_1}^2\delta_{a_2}+\frac{1}{12}\delta_{a_1}\delta_{a_2}\delta_{a_1}+\frac{1}{12}\delta_{a_1}\delta_{a_3} \\ &\quad +\frac{1}{8}\delta_{a_2}\delta_{a_1}^2+\frac{1}{8}\delta_{a_2}^2+\frac{1}{4}\delta_{a_3}\delta_{a_1}+\frac{1}{4}\delta_{a_4}. \end{aligned}$$

Taking idea from Example 2.3, we give an alternative definition of inner higher derivations.

Definition 2.5. A uniformly bounded higher derivation $\{d_n\}$ on an algebra $\mathcal{A}$ is called inner if there is a bounded sequence $\{a_n\}$ in $\mathcal{A}$ such that $nd_n = \sum_{k=1}^{n}\delta_{a_k}d_{n-k}$. In this case we say that $\{d_n\}$ is constructed via $\{a_n\}$.

Proposition 2.6. *Let $\{d_n\}$ be a uniformly bounded higher derivation on a unital Banach algebra $\mathcal{A}$ with the generating function α. Then $\{d_n\}$ is inner if and only if there is a sequence $\{a_n\}$ in $\mathcal{A}$ with $a_0 = 0$ such that $\alpha'_t = \frac{1}{t}\left(\sum_{n=1}^{\infty}\delta_{a_n}t^n\right)\alpha_t$ for $|t| < 1$.*

Proof. We have

$$\frac{1}{t}\left(\sum_{n=1}^{\infty}\delta_{a_n}t^n\right)\alpha_t=\frac{1}{t}\left(\sum_{n=0}^{\infty}\delta_{a_n}t^n\right)\left(\sum_{n=0}^{\infty}d_nt^n\right)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\delta_{a_k}d_{n-k}\right)t^{n-1},$$

and the latter is equal to $\alpha'_t = \sum_{n=1}^{\infty} nd_nt^{n-1}$ if and only if d is inner. $\square$

Theorem 2.7. *Let $\{d_n\}$ be an inner higher derivation on a Banach algebra $\mathcal{A}$ constructed via a mutually commuting bounded sequence $\{a_n\}$ in $\mathcal{A}$ with $a_0 = 0$. Then*

$$d_n = \sum_{m=1}^{n} \sum_{\sum_{i=1}^{m} k_i = n} \frac{\delta_{a_{k_1}} \ldots \delta_{a_{k_m}}}{m! k_1 \ldots k_m}.$$

Proof. Since a_n's are mutually commuting, so are δ_{a_n}'s. We can therefore deduce that $\exp(\sum_{n=1}^{\infty} \frac{\delta_{a_n}}{n} t^n)$ satisfies the differential equation $\alpha_t' = \frac{1}{t} \left(\sum_{n=1}^{\infty} \delta_{a_n} t^n\right) \alpha_t$, for $|t| < 1$, mentioned in Proposition 2.6. Thus we have

$$\alpha_t = \exp\left(\sum_{n=1}^{\infty} \frac{\delta_{a_n}}{n} t^n\right) = \sum_{m=0}^{\infty} \frac{(\sum_{n=1}^{\infty} \frac{\delta_{a_n}}{n} t^n)^m}{m!}.$$

But

$$\left(\sum_{n=1}^{\infty} \frac{\delta_{a_n}}{n} t^n\right)^m = \sum_{n=m}^{\infty} \left(\sum_{\sum_{i=1}^{m} k_i = n} \frac{\delta_{a_{k_1}} \ldots \delta_{a_{k_m}}}{k_1 \ldots k_m}\right) t^n.$$

Hence

$$\begin{aligned}
\sum_{n=0}^{\infty} d_n t^n &= \alpha_t \\
&= \sum_{m=0}^{\infty} \frac{(\sum_{n=1}^{\infty} \frac{\delta_{a_n}}{n} t^n)^m}{m!} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=m}^{\infty} \left(\sum_{\sum_{i=1}^{m} k_i = n} \frac{\delta_{a_{k_1}} \ldots \delta_{a_{k_m}}}{k_1 \ldots k_m}\right) t^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=1}^{n} \sum_{\sum_{i=1}^{m} k_i = n} \frac{\delta_{a_{k_1}} \ldots \delta_{a_{k_m}}}{m! k_1 \ldots k_m}\right) t^n.
\end{aligned}$$

□

Example 2.8. Let $\{d_n\}$ be an inner higher derivation on a Banach algebra $\mathcal{A}$ constructed via a mutually commuting bounded sequence $\{a_n\}$ in $\mathcal{A}$. The first five terms of $\{d_n\}$ are

$$\begin{aligned}
d_0 &= I, d_1 = \delta_{a_1}, d_2 = \frac{1}{2}\delta_{a_1}^2 + \frac{1}{2}\delta_{a_2}, \\
d_3 &= \frac{1}{6}\delta_{a_1}^3 + \frac{1}{4}\delta_{a_1}\delta_{a_2} + \frac{1}{4}\delta_{a_2}\delta_{a_1} + \frac{1}{3}\delta_{a_3} = \frac{1}{6}\delta_{a_1}^3 + \frac{1}{2}\delta_{a_1}\delta_{a_2} + \frac{1}{3}\delta_{a_3}, \\
d_4 &= \frac{1}{24}\delta_{a_1}^4 + \frac{1}{12}\delta_{a_1}^2\delta_{a_2} + \frac{1}{12}\delta_{a_1}\delta_{a_2}\delta_{a_1} + \frac{1}{12}\delta_{a_2}\delta_{a_1}^2 \\
&\quad + \frac{1}{8}\delta_{a_2}^2 + \frac{1}{6}\delta_{a_1}\delta_{a_3} + \frac{1}{6}\delta_{a_3}\delta_{a_1} + \frac{1}{4}\delta_{a_4} \\
&= \frac{1}{24}\delta_{a_1}^4 + \frac{1}{4}\delta_{a_1}^2\delta_{a_2} + \frac{1}{8}\delta_{a_2}^2 + \frac{1}{3}\delta_{a_1}\delta_{a_3} + \frac{1}{4}\delta_{a_4}.
\end{aligned}$$

3. The Result

Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a homomorphism. A linear mapping $D : \mathcal{A} \to \mathcal{A}$ is called a (σ,σ)-derivation if it satisfies the generalized Leibniz rule $D(ab) = D(a)\sigma(b) + \sigma(a)D(b)$ for all $a, b \in \mathcal{A}$. It is called inner if there is an $a_0 \in \mathcal{A}$ such that $D(a) = a_0\sigma(a) - \sigma(a)a_0$ for all $a \in \mathcal{A}$.

Lemma 3.1. *Let $\mathcal{A}$ be a Banach algebra and $\{d_n\}$ be a uniformly bounded higher derivation on $\mathcal{A}$ with the generating function α. Then α'_t is an (α_t, α_t)-derivation.*

Proof. By Lemma 2.2, for each $a, b \in \mathcal{A}$ we have $\alpha_t(ab) = \alpha_t(a)\alpha_t(b)$. Taking the derivative we have the result. □

More generally, we have the following useful result. Note that if $\sigma : \mathcal{A} \to \mathcal{A}$ is an isomorphism then for each (σ,σ)-derivation D, the mapping $D\sigma^{-1}$ is an ordinary derivation. Thus if σ is an isomorphism and each derivation on $\mathcal{A}$ is inner, then each (σ,σ)-derivation on $\mathcal{A}$ is inner. We use this fact in the following theorem. In fact, since α_t is an isomorphism we can therefore deduce that if each derivation on $\mathcal{A}$ is inner, then each (α_t, α_t)-derivation is also inner.

Theorem 3.2. *Let $\mathcal{A}$ be a Banach algebra and $\{d_n\}$ be a uniformly bounded higher derivation on $\mathcal{A}$ with the generating function α. Then $\alpha_0^{(m)} = m!d_m$. Moreover, if each derivation on $\mathcal{A}$ is inner then*

$$\alpha_t^{(m+1)} = \sum_{i=0}^{m} \binom{m}{i} \delta_{a_{i,t}} \alpha_t^{(m-i)},$$

for some sequence $\{a_{m,t}\}$ in $\mathcal{A}$.

Proof. We use induction on m. Note that $\alpha_t^{(0)} = \alpha_t$ and $\alpha_t^{(1)} = \alpha'_t$. Thus Lemma 3.1 implies that $\alpha_t^{(1)}$ is an $(\alpha_t^{(0)}, \alpha_t^{(0)})$-derivation and the assumption guarantees the existence of a mapping φ from the real numbers into $\mathcal{A}$ such that $\alpha_t^{(1)}(a) = \varphi(t)\alpha_t^{(0)}(a) - \alpha_t^{(0)}(a)\varphi(t)$. Choosing a mapping φ satisfying the later equation and taking $a_{0,t} = \varphi(t)$ we have the result in the case $m = 0$.

Now suppose that the result holds for $m - 1$. Define β_t by

$$\beta_t = \alpha_t^{(m+1)} - \sum_{i=0}^{m-1} \binom{m}{i} \delta_{a_{i,t}} \alpha_t^{(m-i)}.$$

Let $a, b \in \mathcal{A}$. Taking the consecutive derivatives from $\alpha_t(ab) = \alpha_t(a)\alpha_t(b)$ we have

$$\alpha_t^{(m+1)}(ab) = \sum_{i=0}^{m+1} \binom{m+1}{i} \alpha_t^{(i)}(a)\alpha_t^{(m+1-i)}(b).$$

We therefore can write

$$\begin{aligned} \beta_t(ab) &= \sum_{i=0}^{m+1} \binom{m+1}{i} \alpha_t^{(i)}(a)\alpha_t^{(m+1-i)}(b) \\ &\quad - \sum_{i=0}^{m-1} \binom{m}{i} \delta_{a_{i,t}} \Big[\sum_{j=0}^{m-i} \binom{m-i}{j} \alpha_t^{(j)}(a)\alpha_t^{(m-i-j)}(b)\Big]. \end{aligned}$$

Since $\delta_{a_{i,t}}$'s are derivations, we have

$$\beta_t(ab) = \sum_{i=0}^{m+1} \binom{m+1}{i} \alpha_t^{(i)}(a)\alpha_t^{(m+1-i)}(b)$$
$$- \sum_{i=0}^{m-1} \binom{m}{i} \sum_{j=0}^{m-i} \binom{m-i}{j} \big[\delta_{a_{i,t}}(\alpha_t^{(j)}(a))\alpha_t^{(m-i-j)}(b) + \alpha_t^{(j)}(a)\delta_{a_{i,t}}(\alpha_t^{(m-i-j)}(b))\big].$$

We write

$$\beta_t(ab) = A\alpha_t(b) + \alpha_t(a)B + C\alpha_t'(b) + \alpha_t'(a)D + \sum_{r=2}^{m-1} E_r\alpha_t^{(r)}(b) + \sum_{s=2}^{m-1} \alpha_t^{(s)}(a)F_s$$

and evaluate the coefficients. We have

$$A = \binom{m+1}{m+1}\alpha_t^{(m+1)}(a) - \sum_{i=0}^{m-1} \binom{m}{i}\binom{m-i}{m-i}\delta_{a_{i,t}}\alpha_t^{(m-i)}(a) = \beta_t(a).$$

By the same argument $B = \beta_t(b)$. Also

$$C = \binom{m+1}{m}\alpha_t^{(m)}(a) - \sum_{i=0}^{m-1} \binom{m}{i}\binom{m-i}{m-i-1}\delta_{a_{i,t}}\alpha_t^{(m-i-1)}(a) - \alpha_t^{(m)}(a).$$

Note that the last term is obtained from $i = 0$ and $j = m$, since $\delta_{a_{0,t}}\alpha_t(b) = \alpha_t'(b)$. By the inductive hypothesis we thus have

$$\begin{aligned} C &= (m+1)\alpha_t^{(m)}(a) - \sum_{i=0}^{m-1} \binom{m}{i}\binom{m-i}{m-i-1}\delta_{a_{i,t}}\alpha_t^{(m-i-1)}(a) - \alpha_t^{(m)}(a) \\ &= m\big[\alpha_t^{(m)}(a) - \sum_{i=0}^{m-1} \binom{m-1}{i}\delta_{a_{i,t}}\alpha_t^{(m-1-i)}(a)\big] = 0. \end{aligned}$$

By the same argument $D = 0$. To evaluate E_r's we split the first summation and write

$$\beta_t(ab) = \sum_{i=0}^{m+1} \binom{m}{i-1}\alpha_t^{(i)}(a)\alpha_t^{(m+1-i)}(b) + \sum_{i=0}^{m+1} \binom{m}{i}\alpha_t^{(i)}(a)\alpha_t^{(m+1-i)}(b)$$
$$- \sum_{i=0}^{m-1} \binom{m}{i} \sum_{j=0}^{m-i} \binom{m-i}{j} \big[\delta_{a_{i,t}}(\alpha_t^{(j)}(a))\alpha_t^{(m-i-j)}(b) + \alpha_t^{(j)}(a)\delta_{a_{i,t}}(\alpha_t^{(m-i-j)}(b))\big].$$

Looking to the first and the last summation we have

$$\begin{aligned} E_r &= \binom{m}{m-r}\alpha_t^{(m+1-r)}(a) - \sum_{i=0}^{m-r} \binom{m}{i}\binom{m-i}{m-i-r}\delta_{a_{i,t}}\alpha_t^{(m-i-r)}(a) \\ &= \binom{m}{m-r}\big[\alpha_t^{(m-r+1)}(a) - \sum_{i=0}^{m-r} \frac{\binom{m}{i}\binom{m-i}{m-i-r}}{\binom{m}{m-r}}\delta_{a_{i,t}}\alpha_t^{(m-r-i)}(a)\big] \\ &= \binom{m}{m-r}\big[\alpha_t^{(m-r+1)}(a) - \sum_{i=0}^{m-r} \binom{m-r}{i}\delta_{a_{i,t}}\alpha_t^{(m-r-i)}(a)\big] = 0. \end{aligned}$$

By the same argument $F_s = 0$. This implies $\beta_t(ab) = \beta_t(a)\alpha_t(b) + \alpha_t(a)\beta_t(b)$, i.e. β_t is an (α_t, α_t)-derivation. Hence there is an $a_{m,t} \in \mathcal{A}$ such that $\beta_t = \delta_{a_{m,t}}\alpha_t$. We therefore have the result. □

Corollary 3.3. *Let $\mathcal{A}$ be a Banach algebra with the property that each derivation on $\mathcal{A}$ is inner. Then each uniformly bounded higher derivation on $\mathcal{A}$ is inner.*

Proof. Put $t = 0$ in Theorem 3.2. Then for $a_{k+1} = \frac{a_{k,0}}{k!}$ we have the result. □

Remark 3.4. The Kadison–Sakai theorem ensures us that if $\mathfrak{M}$ is a von Neumann algebra then each derivation on $\mathfrak{M}$ is inner. We can therefore deduce that each uniformly bounded higher derivation on a von Neumann algebra is inner.

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