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THE HARDY INEQUALITY WITH ONE NEGATIVE PARAMETER

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Dedicated to Professor Josip E. Pečarić

Submitted by C. Park

Abstract. In this paper, necessary and sufficient conditions for the validity of the Hardy inequality for the case $q < 0$, $p > 0$ and for the case $q > 0$, $p < 0$ are derived.

1. Introduction and preliminaries

The classical Hardy inequality

$$\left(\int_a^b \left(\int_a^x f(t)v(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}} \tag{1.1}$$

for all $f \geq 0$, where u, v are weight functions, is almost completely described for p, q such that

$$p \geq 1,\ q > 0$$

(see [3], [4], [5]), while for p, q such that

$$0 < p < 1,\ q > 1$$

it is known that inequality (1.1) doesn't hold (see [4], p.46).

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The so called *reverse Hardy inequality*

$$\left(\int_a^b \left(\int_a^x f(t)v(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \geq C\left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}} \tag{1.2}$$

was studied in [1] for

$$0 < p, q < 1 \quad \text{and} \quad p, q < 0;$$

the second case was described in [6] and the case for

$$-\infty < q \leq p < 0$$

was described in [2].

Here, we want to consider parameters p, q which satisfy either

$$p < 0,\ q > 0$$

or

$$p > 0,\ q < 0.$$

It will be shown that in the first case, the reverse inequality (1.2) hold (see Theorem 2.1) while in the second case, the reverse inequality (1.2) holds for $0 < p < 1$, $q < 0$ (see Theorem 2.2) and the Hardy inequality (1.1) holds for $p \geq 1, q < 0$ (see Theorem 2.4). The results can be extended to the "adjoint" inequalities

$$\left(\int_a^b \left(\int_x^b f(t)v(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}} \tag{1.3}$$

and

$$\left(\int_a^b \left(\int_x^b f(t)v(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \geq C\left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}} \tag{1.4}$$

(see Remark 2.6).

The negative powers p, q force us to work with functions having values from the interval $[0, +\infty]$. Therefore, we define the following arithmetics:

$$\begin{cases} 0 + (+\infty) = a + (+\infty) = a \cdot (+\infty) = \frac{a}{0} = +\infty, & a \in (0, +\infty]; \\ 0 \cdot (+\infty) = \frac{a}{+\infty} = 0, & a \in [0, +\infty); \\ 0^{-\alpha} = (+\infty)^{\alpha} = +\infty, \quad 0^{\alpha} = (+\infty)^{-\alpha} = 0, & \alpha \in (0, +\infty). \end{cases}$$

2. Main results

Let us denote

$$A(t) := \left(\int_a^t v^{p'}(x)dx\right)^{\frac{1}{p'}} \left(\int_t^b u(x)dx\right)^{\frac{1}{q}}, \quad p' = \frac{p}{p-1}.$$

Then we can formulate the following theorems:

Theorem 2.1. *Let $p < 0$ and $q > 0$. Then inequality (1.2) holds if and only if there exists $\tau \in (a,b)$ such that*

$$A(\tau) > 0. \tag{2.1}$$

Moreover,

i) *if*
$$0 < A^* := \sup_{(a,b)} A(t) < \infty,$$
and C is the best possible constant of inequality (1.2) then $A^* \leq C$;

ii) *if*
$$A^* = \infty,$$
then the best constant of inequality (1.2) does not exist, more precisely, the left hand side of (1.2) is infinite for all positive functions for which $\left(\int_a^b f^p(t)dt\right)^{\frac{1}{p}} > 0$.

Proof. Let $\tau \in (a,b)$ be arbitrary. Then

$$\begin{aligned} J &:= \int_a^b \left(\int_a^x f(t)v(t)dt\right)^q u(x)dx \geq \int_\tau^b \left(\int_a^x f(t)v(t)dt\right)^q u(x)dx \\ &\geq \int_\tau^b \left(\int_a^\tau f(t)v(t)dt\right)^q u(x)dx = \int_\tau^b u(x)dx \left(\int_a^\tau f(t)v(t)dt\right)^q. \end{aligned}$$

Applying the reverse Hölder inequality with powers p and $p' = \frac{p}{p-1}$ to the second integral of the last expression, we get that

$$\begin{aligned} J &\geq \int_\tau^b u(x)dx \left(\int_a^\tau v(t)^{p'}dt\right)^{\frac{q}{p'}} \left(\int_a^\tau f(t)^p dt\right)^{\frac{q}{p}} \\ &\geq \int_\tau^b u(x)dx \left(\int_a^\tau v(t)^{p'}dt\right)^{\frac{q}{p'}} \left(\int_a^b f(t)^p dt\right)^{\frac{q}{p}}. \end{aligned}$$

Thus, we obtain that

$$\int_a^b \left(\int_a^x f(t)v(t)dt\right)^q u(x)dx \geq A(\tau)^q \left(\int_a^b f(t)^p dt\right)^{\frac{q}{p}}. \tag{2.2}$$

It is easy to see that the condition (2.1) is equivalent with the validity of inequality (1.2), i.e. (2.2). If we suppose that condition (2.1) is satisfied, i.e. if there exists $\tau \in (a,b)$ such that $A(\tau) > 0$, then from (2.2) we have inequality (1.2) with $C \geq A(\tau)$. Conversely, let us suppose that inequality (1.2) holds, which means that for positive functions f such that

$$\left(\int_a^b f(t)^p dt\right)^{\frac{1}{p}} > 0,$$

the expression on the left hand side of inequality (1.2) is greater than zero, i.e.

$$\left(\int_a^b \left(\int_a^x f(t)v(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} > 0. \tag{2.3}$$

If we define

$$a^* = \sup\{t \in [a,b), \quad \int_a^t v^{p'}(s)ds = 0\}; \quad b^* = \inf\{t \in (a,b], \quad \int_t^b u(s)ds = 0\},$$

then

$$A(t) = 0 \quad \text{for} \quad \text{all } t \in (a,b) \quad \text{if and only if} \quad b^* \leq a^*.$$

This together with (2.3) implies that $A(t)$ is positive for some $t \in (a,b)$.

From (2.2) we have that

$$A(\tau) \leq C = \inf_{f>0} \frac{\left(\int_a^b \left(\int_a^x f(t)v(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}}}{\left(\int_a^b f(t)^p dt\right)^{\frac{1}{p}}}.$$

The right hand side of the last estimate is independent on τ, so we get that

$$A^* = \sup_{(a,b)} A(\tau) \leq C.$$

This ends the proof of i).

If $A^* = \infty$ then inequality (1.2) holds, since its left hand side is infinite for functions f such that

$$\left(\int_a^b f^p(t)dt\right)^{\frac{1}{p}} > 0,$$

which follows from (2.2). □

Theorem 2.2. *Let $0 < p < 1$ and $q < 0$. Then inequality (1.2) holds for all functions $f > 0$ if and only if the following condition is satisfied:*

$$A_* := \inf_{(a,b)} A(t) > 0.$$

Moreover, if C is the best possible constant in (1.2), then

$$\left(1 + \frac{p'}{q}\right)^{\frac{1}{p'}} \left(1 + \frac{q}{p'}\right)^{\frac{1}{q}} A_* \leq C \leq A_*.$$

Proof. (**Sufficiency**) Let $\alpha \in (0, -\frac{1}{p'})$ be a parameter and denote

$$V(t) := \int_a^t v^{p'}(\tau)d\tau.$$

For

$$\begin{aligned} J &:= \int_a^b \left(\int_a^x f(t)v(t)dt\right)^q u(x)dx \\ &= \int_a^b \left(\int_a^x f(t)v(t)dt\right)^p \left(\int_a^x f(t)v(t)dt\right)^{q-p} u(x)dx \\ &= \int_a^b \left(\int_a^x f(t)V^{-\alpha}(t)V^{\alpha}(t)v(t)dt\right)^p \left(\int_a^x f(t)v(t)dt\right)^{q-p} u(x)dx \end{aligned}$$

applying the reverse Hölder inequality with powers p and $p' = \frac{p}{p-1}$ to the integral in the first brackets, we get,

$$\begin{aligned} J \geq & \int_a^b \left(\int_a^x f^p(t) V^{-\alpha p}(t) dt \right) \left(\int_a^x V^{\alpha p'}(t) v^{p'}(t) dt \right)^{\frac{p}{p'}} \\ & \times \left(\int_a^x f(t) v(t) dt \right)^{q-p} u(x) dx \\ = & \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \int_a^b \left[\left(\int_a^x f^p(t) V^{-\alpha p}(t) dt \right) V^{(1+\alpha p')\frac{p}{p'}}(x) \right] \\ & \times \left(\int_a^x f(t) v(t) dt \right)^{q-p} u(x) dx. \end{aligned}$$

Now we again apply the reverse Hölder inequality with powers $\frac{q}{p}$ and $\frac{q}{q-p}$ which yields

$$\begin{aligned} J \geq & \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_a^b \left(\int_a^x f^p(t) V^{-\alpha p}(t) dt \right)^{\frac{q}{p}} V^{(1+\alpha p')\frac{q}{p'}}(x) u(x) dx \right)^{\frac{p}{q}} \\ & \times \left(\int_a^b \left(\int_a^x f(t) v(t) dt \right)^q u(x) dx \right)^{1-\frac{p}{q}} \\ = & \frac{J^{1-\frac{p}{q}}}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_a^b \left(\int_a^x f^p(t) V^{-\alpha p}(t) dt \right)^{\frac{q}{p}} V^{(1+\alpha p')\frac{q}{p'}}(x) u(x) dx \right)^{\frac{p}{q}}. \end{aligned}$$

The reverse Minkowski integral inequality with power $r = \frac{q}{p}$ yields

$$\begin{aligned} J \geq & \frac{J^{1-\frac{p}{q}}}{(1+\alpha p')^{\frac{p}{p'}}} \int_a^b f^p(t) V^{-\alpha p}(t) \left(\int_t^b V^{(1+\alpha p')\frac{q}{p'}}(x) u(x) dx \right)^{\frac{p}{q}} dt \\ \geq & \frac{J^{1-\frac{p}{q}} \ \mathbb{A}_*^p(\alpha)}{(1+\alpha p')^{\frac{p}{p'}}} \int_a^b f^p(t) dt, \end{aligned}$$

where

$$\mathbb{A}_*(\alpha) := \inf_{(a,b)} \mathbb{A}(t,\alpha) = \inf_{(a,b)} V^{-\alpha}(t) \left(\int_t^b V^{(1+\alpha p')\frac{q}{p'}}(x) u(x) dx \right)^{\frac{1}{q}}.$$

Therefore, we obtain that

$$J^{\frac{1}{q}} \geq \frac{\mathbb{A}_*(\alpha)}{(1+\alpha p')^{\frac{1}{p'}}} \left(\int_a^b f^p(t) dt \right)^{\frac{1}{p}}. \tag{2.4}$$

Now we show that

$$\mathbb{A}_*(\alpha) \geq C_1 \, A_*,$$

where C_1 depends only on α. Integration by parts leads to the estimate

$$\begin{aligned}
J_1(t,\alpha) :=& \int_t^b V^{(1+\alpha p')\frac{q}{p'}}(x)u(x)dx \\
=& \int_t^b V^{(1+\alpha p')\frac{q}{p'}}(x)d\left(-\int_x^b u(s)ds\right) \\
=& V^{(1+\alpha p')\frac{q}{p'}}(t)\int_t^b u(s)ds - \lim_{x\to b-} V^{(1+\alpha p')\frac{q}{p'}}(x)\int_x^b u(s)ds \\
&+ \frac{(1+\alpha p')q}{p'}\int_t^b\left(\int_x^b u(s)ds\right)V^{(1+\alpha p')\frac{q}{p'}-1}(x)dV(x) \\
\le& V^{\alpha q}(t)A^q(t) + \frac{(1+\alpha p')q}{p'}\int_t^b A^q(x)V^{\alpha q-1}(x)dV(x) \\
\le& A_*^q\left[V^{\alpha q}(t) + \frac{(1+\alpha p')q}{p'}\int_t^b V^{\alpha q-1}(x)dV(x)\right] \\
\le& -\frac{1}{\alpha p'}A_*^q V^{\alpha q}(t).
\end{aligned}$$

Since $J_1(t,\alpha) = \mathbb{A}^q(t,\alpha)V^{\alpha q}(t)$ due to the definition of $\mathbb{A}(t,\alpha)$, we finally obtain that

$$\mathbb{A}(t,\alpha) \ge (-\alpha p')^{-\frac{1}{q}}A_*,$$

i.e.

$$\mathbb{A}_*(\alpha) \ge (-\alpha p')^{-\frac{1}{q}}A_*,$$

and from (2.4) it follows that

$$J^{\frac{1}{q}} \ge \frac{(-\alpha p')^{-\frac{1}{q}}}{(1+\alpha p')^{\frac{1}{p'}}}A_*\left(\int_a^b f^p(t)dt\right)^{\frac{1}{p}}.$$

For the best constant C we have

$$\sup_{\alpha\in(0,-\frac{1}{p'})}\frac{(-\alpha p')^{-\frac{1}{q}}}{(1+\alpha p')^{\frac{1}{p'}}}A_* = \left(1+\frac{p'}{q}\right)^{\frac{1}{q}}\left(1+\frac{q}{p'}\right)^{\frac{1}{q}}A_* \le C.$$

The sufficiency part is proved.

(**Necessity**) From inequality (1.2) we get that

$$\begin{aligned}
C \le& \left(\int_a^b\left(\int_a^x f(t)v(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}}\left(\int_a^b f(t)^p dt\right)^{-\frac{1}{p}} \\
\le& \left(\int_\tau^b\left(\int_a^x f(t)^p dt\right)^{-\frac{q}{p}}\left(\int_a^x f(t)v(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}}.
\end{aligned}$$

If we choose

$$f(t) = v^{p'-1}(t),$$

then we get

$$C \leq \left(\int_\tau^b \left(\int_a^x v(t)^{p'} dt \right)^{\frac{q}{p'}} u(x)dx \right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^\tau v(t)^{p'} dt \right)^{\frac{1}{p'}} \left(\int_\tau^b u(x)dx \right)^{\frac{1}{q}} = A(\tau),$$

and consequently,

$$A(\tau) \geq C,$$

which proves the necessity of the condition. □

Proposition 2.3. *Let the assumptions of Theorem 2.2 be satisfied. Then the best constant C of inequality (1.2) satisfies*

$$\sup_{\alpha \in (0, -\frac{1}{p'})} \frac{\mathbb{A}_*(\alpha)}{(1+\alpha p')^{\frac{1}{p'}}} \leq C.$$

Proof. The proof follows from (2.4). □

Let us denote

$$B(t) := \begin{cases} \left(\underset{(a,t)}{supess}\, v(x) \right) \left(\int_a^t u(x)dx \right)^{\frac{1}{q}} & if \quad p = 1, \\ \\ \left(\int_a^t v^{p'}(x)dx \right)^{\frac{1}{p'}} \left(\int_a^t u(x)dx \right)^{\frac{1}{q}} & if \quad p > 1. \end{cases}$$

Then we can formulate the following theorem:

Theorem 2.4. *Let $p \geq 1$ and $q < 0$. Then inequality (1.1) holds if and only if there exists $\tau \in (a,b)$ such that*

$$B(\tau) < \infty.$$

Moreover,

i) *if*

$$0 < B := \inf_{(a,b)} B(t) < \infty,$$

and C is the best constant of inequality (1.1) then $C \leq B$;

ii) *if*

$$B = 0$$

then the best constant of the inequality does not exist, more precisely, the left hand side of (1.1) is zero for all nonnegative functions f.

Proof. Let $\tau \in (a,b)$ be arbitrary. Then

$$J := \int_a^b \left(\int_a^x f(t)v(t)dt \right)^q u(x)dx \geq \int_0^\tau \left(\int_a^x f(t)v(t)dt \right)^q u(x)dx$$

$$\geq \int_0^\tau \left(\int_a^\tau f(t)v(t)dt \right)^q u(x)dx = \int_0^\tau u(x)dx \left(\int_a^\tau f(t)v(t)dt \right)^q.$$

We estimate the second integral in the last expression as follows:

If $p = 1$ then

$$\int_a^\tau f(t)v(t)dt \le \underset{(a,\tau)}{supess}\, v(t) \int_a^\tau f(t)dt.$$

If $p > 1$ then we apply the Hölder inequality

$$\int_a^\tau f(t)v(t)dt \le \left(\int_a^\tau v(t)^{p'}dt\right)^{\frac{1}{p'}} \left(\int_a^\tau f(t)^p dt\right)^{\frac{1}{p}}.$$

Consequently, we have that

$$\int_a^b \left(\int_a^x f(t)v(t)dt\right)^q u(x)dx \ge B(\tau)^q \left(\int_a^b f(t)^p dt\right)^{\frac{q}{p}},$$

i.e.

$$\left(\int_a^b \left(\int_a^x f(t)v(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \le B(\tau) \left(\int_a^b f(t)^p dt\right)^{\frac{1}{p}}.$$

The rest of the proof follows analogously as in the proof of Theorem 2.1. □

Remark 2.5. In Theorem 2.2 we supposed that $f > 0$, which is important, since we can construct a nonnegative function f for which inequality (1.2) does not hold.

Remark 2.6. If we denote

$$A(t) := \left(\int_t^b v^{p'}(x)dx\right)^{\frac{1}{p'}} \left(\int_a^t u(x)dx\right)^{\frac{1}{q}}$$

and

$$B(t) := \begin{cases} \left(\underset{(t,b)}{supess}\, v(x)\right) \left(\int_t^b u(x)dx\right)^{\frac{1}{q}} & if \quad p = 1, \\ \\ \left(\int_t^b v^{p'}(x)dx\right)^{\frac{1}{p'}} \left(\int_t^b u(x)dx\right)^{\frac{1}{q}} & if \quad p > 1 \end{cases}$$

then we are able to formulate results analogous to Theorems 2.1, 2.2 and 2.4 for inequalities (1.3) and (1.4). The formulation and the proofs are left to the reader.

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