Bernoulli **5**(5), 1999, 779–795

Taylor expansions of curve-crossing probabilities

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Using an approach based on the Cameron–Martin–Girsanov theorem, we obtain a Taylor expansion for the probability that Brownian motion hits a smooth nonlinear boundary which grows at a suitable rate. The structure and probabilistic meaning of the terms in the expansion are studied in some detail.

Keywords: Brownian motion; Cameron–Martin–Girsanov theorem; curve-crossing probabilities; harmonic functions; Lévy-Khinchine operator

1. Introduction and summary

1.1. Some motivation

Peter Hall and ATAW considered the following statistical problem (viewed as a prototype for a class of problems involving asymptotic expansions for probabilities in infinite-dimensional settings). Exact, but discontinuous, confidence bands for a continuous distribution function can be constructed using the (known) percentage points of the Kolmogorov–Smirnov statistic. Suppose that one wishes to present smoothed confidence bands (perhaps for aesthetic reasons). The simplest way to do this is to obtain a smoothed estimate of the distribution function, but still use a percentage point of the Kolmogorov–Smirnov statistic to construct the bands. The resulting bands are smooth but the nominal coverage probability is no longer exactly correct. The question considered by Peter Hall and ATAW was the following: what is the size of the coverage error incurred by smoothing?

An important paper by Götze (1985) suggests a way in which one might approach problems of this type. His key development is to describe Edgeworth-type expansions in abstract settings in terms of Taylor expansions of the relevant 'probability functional'. However, although very illuminating on a conceptual level, this approach is often rather difficult to justify rigorously (and these difficulties are certainly apparent in the confidence-bands problem described above). Nevertheless, consideration of Götze's approach does lead to some interesting questions concerning the existence and nature of Taylor expansions for curve-crossing probabilities associated with Brownian motion. In the present paper, we focus exclusively on this last-mentioned topic.

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1.2. General problems for Brownian motion

Let $\{W_t: 0 \leqslant t < \infty\}$ be a Brownian motion with $W_0 = 0$. Let a and c be continuous functions on $[0, \infty)$ with $a(0) > 0$. Let

$$\Psi(\mathrm{a}) := \mathbb{P}[W \text{ hits } a] := \mathbb{P}[W_t = a(t) \text{ for some } t]. \tag{1.1}$$

Problem 1. *When can we write, for $|\varepsilon| < \varepsilon_0$ (ε_0 some positive constant),*

$$\mathbb{P}[W \text{ hits } a + \varepsilon c] = \mathbb{P}[W \text{ hits } a] \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \gamma_n, \tag{1.2}$$

the series being absolutely convergent? If $\varepsilon c(\cdot) \geqslant 0$, then, of course, the sum will be $\mathbb{P}[W \text{ hits } a + \varepsilon c \mid W \text{ hits } a]$.

Problem 2. *What is the clearest intuitive probabilistic (that is, sample-path) meaning of γ_n?* Think about the case when $c(\cdot) \geqslant 0$ and when ε is small and positive. Write

$$\tau := \inf\{t: W_t = a(t)\}, \qquad \tilde{\tau} := \inf\{t: W_t = a(t) + \varepsilon c(t)\},$$

with $\inf(\emptyset) := \infty$ as usual. Intuitively, the γ_1 term should focus on first-order effects in the situation where τ and $\tilde{\tau}$ are very close. The γ_2 term should comprise two terms: one a second-order effect for the situation just mentioned, the other the product of two first-order effects in the situation where W makes a 'significant' downward excursion from the curve a between times τ and $\tilde{\tau}$. The γ_n terms will become rapidly more complicated as n increases, and it is already difficult to obtain complete understanding of the γ_3 term, even when we know what it is.

In terms of analysis, we should of course be able to regard $\Psi(\mathrm{a})\gamma_n$ as in some sense an nth order Fréchet derivative of Ψ at a along c.

1.3 The harmonic-function method

Let m be a measure on $(0, \infty)$ with $m(0, \infty) < 1$ and let b be the function on $[0, \infty)$ such that

$$\int_0^\infty \exp\{\theta b(t) - \tfrac{1}{2}\theta^2 t\} m(\mathrm{d}\theta) = 1. \tag{1.3}$$

As Lerche (1986, p. 34) and Karatzas and Shreve (1988, Section 4.3C) remark, b is strictly increasing and concave. Then,

$$\tilde{H}(t, x) := \int_0^\infty \exp\{\theta x - \tfrac{1}{2}\theta^2 t\} m(\mathrm{d}\theta)$$

is *space-time harmonic* on $\{(t, x): x \leqslant b(t)\}$ in that

$$\frac{\partial \tilde{H}}{\partial t} + \frac{1}{2}\frac{\partial^2 \tilde{H}}{\partial x^2} = 0$$

in this region. Moreover, for fixed x, $\tilde{H}(t, x)$ tends to zero as t tends to infinity, and $\tilde{H}(t, b(t)) = 1$. If $\sigma \equiv \inf\{t\colon W_t = b(t)\}$, then $\tilde{H}(t \wedge \sigma, W_{t\wedge\sigma})$ is a bounded martingale, whence

$$\mathbb{P}(W \text{ hits } b) = \tilde{H}(0, 0) = m(0, \infty).$$

(Note that if $\sigma = \infty$, then, for any x, W_t will equal x for a sequence of t values tending to ∞, so that the limit of the martingale $\tilde{H}(t \wedge \sigma, W_{t\wedge\sigma})$ will be 0.)

The harmonic-function method, also called the weighted-likelihood-function method, was used by Robbins and Siegmund (1970; 1973) to calculate crossing probabilities of certain classes of curved boundaries for Brownian motion. The harmonic-function method is related by time inversion to the method of images which has been used effectively in similar contexts by Daniels (1982; 1996). These methods can be extended to deduce the law of the first-passage time, and this has applications to the construction of sequential tests of unit power (Robbins and Siegmund 1973; Daniels 1982) and to the pricing of barrier options (Roberts and Shortland 1997). A third approach, suggested by Durbin (1985; 1988; 1992), yields a series expansion for the first-passage density to a general boundary. Unfortunately, none of these methods seem well suited to our consideration of perturbing a to $a + \varepsilon c$, though we shall use them as a cross-check on some of our 'non-perturbative' results.

1.4. The Cameron–Martin–Girsanov (CMG) method

This method is in one sense a probabilistic counterpart to the harmonic-function method just described, but it has greater flexibility. A full account of this method may be found in Rogers and Williams (1987).

As previously, let a be a continuous function on $[0, \infty)$, with $a(0) > 0$. Let c now be an absolutely continuous function on $[0, \infty)$ such that

$$c(0) = 0 \quad \text{and} \quad \int_0^t c'(s)^2 \mathrm{d}s < \infty \text{ for every } t \text{ in } [0, \infty). \tag{1.4}$$

Let W be our Brownian motion started at 0. The CMG formula shows that with $\tau := \inf\{t\colon W_t = a(t)\}$ as before,

$$\begin{aligned}
\mathbb{P}[W \text{ hits } a + \varepsilon c] &= \mathbb{P}[W - \varepsilon c \text{ hits } a] \\
&= \mathbb{E}\left(\exp\left\{-\varepsilon \int_0^\tau c'(t)\,\mathrm{d}W_t - \tfrac{1}{2}\varepsilon^2 \int_0^\tau c'(t)^2\,\mathrm{d}t\right\};\, \tau < \infty\right) \\
&= \mathbb{E}\left(\exp\left\{-\varepsilon \int_0^\tau c'(t)\,\mathrm{d}W_t - \tfrac{1}{2}\varepsilon^2 \int_0^\tau c'(t)^2\,\mathrm{d}t\right\} \,\middle|\, \tau < \infty\right)\mathbb{P}[W \text{ hits } a], \quad (1.5)
\end{aligned}$$

since $\mathbb{P}[W \text{ hits } a] = \mathbb{P}(\tau < \infty)$. That this is rigorously true given only (1.4) will be confirmed later. Thus, at least formally,

$$\gamma_n = \mathbb{E}\left\{ \tilde{h}_n \left(\int_0^\tau c'(t)^2 \mathrm{d}t, \ -\int_0^\tau c'(t) \mathrm{d}W_t \right) \middle| \tau < \infty \right\}, \tag{1.6}$$

where we define the Hermite polynomial $\tilde{h}_n$ on $[0, \infty) \times \mathbb{R}$ (note the (t, x) order) via

$$\sum_{n=0}^{\infty} \frac{\theta^n}{n!} \tilde{h}_n(t, x) = \mathrm{e}^{\theta x - \frac{1}{2}\theta^2 t}.$$

1.5. The linearity assumption on *a*

To make progress, we shall henceforth in this paper restrict attention to the 'linear boundary' case:

$$a(t) = \alpha + \beta t, \qquad \alpha > 0, \beta > 0. \tag{1.7}$$

It is well known that

$$\mathbb{P}[W \text{ hits } a] = \mathrm{e}^{-2\alpha\beta}. \tag{1.8}$$

In our study of the 'linear boundary' case, we unashamedly use shift-invariance properties and the like, which will not hold in general. Our definitions of Δ and $\partial\Delta$ are not the 'proper' ones: they are those which allow the special case to be treated most simply.

The process $\{W_t \colon t < \tau\}$ conditioned by the event $\{\tau < \infty\}$ has the same law as

$$\{B_t + 2\beta t \colon t < T\}, \qquad \text{where } T := T_{\alpha,\beta} := \inf\{t \colon B_t = \alpha - \beta t\}, \tag{1.9}$$

in which B is a Brownian motion started at 0. This corresponds to the analytic fact that if, for $x < \alpha + \beta t$, we let $p(t, x)$ be the probability that for some $s > 0$, $x + W_{t+s} - W_t$ equals $\alpha + \beta(t + s)$, so that

$$p(t, x) = \exp\{-2(\alpha + \beta t - x)\beta\},$$

then we have the operator identity

$$p^{-1}\left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) p = \frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2} + 2\beta\frac{\partial}{\partial x}.$$

It follows from (1.5) and (1.9) that if we define the martingale N via

$$N_t := \int_{s=0}^{t} c'(s)\, \mathrm{d}B_s, \tag{1.10}$$

then

$$\mathbb{P}[W \text{ hits } a + \varepsilon c] = \mathbb{P}[W \text{ hits } a]\mathbb{E}\left(G_T^{(\varepsilon c)} \mathrm{e}^{-2\beta\varepsilon c(T)} \right), \tag{1.11}$$

where $G^{(\varepsilon c)}$ is the martingale with

$$G_t^{(\varepsilon c)} := \exp\{-\varepsilon N_t - \tfrac{1}{2}\varepsilon^2 \langle N \rangle_t\}. \tag{1.12}$$

Here, $\langle N \rangle$ is the quadratic-variation process of N, so that

$$\langle N \rangle_t = \int_0^t c'(s)^2 \, \mathrm{d}s.$$

1.6. The operator $\mathscr{A}$

Define the sets Δ and $\partial\Delta$ and the projection π: $\partial\Delta \to [0, \infty)$ as follows:

$$\Delta := \{(t, x) \in [0, \infty) \times \mathbb{R}: x < \alpha - \beta t\},$$

$$\partial\Delta := \{(t, x) \in [0, \infty) \times \mathbb{R}: x = \alpha - \beta t\},$$

$$\pi(t, \alpha - \beta t) := t.$$

Note that T is the exit time of B from Δ. It is well known that T has probability density function

$$f(t) := f_{\alpha,\beta}(t) = \frac{\alpha}{\sqrt{2\pi t^3}} \exp\left(-\frac{\alpha^2}{2t}\right) \exp\left(\beta\alpha - \tfrac{1}{2}\beta^2 t\right) \qquad t > 0, \tag{1.13}$$

and that, for $\lambda \geqslant -\frac{1}{2}\beta^2$,

$$\mathbb{E}\mathrm{e}^{-\lambda T} = \mathrm{e}^{-\theta(\lambda)\alpha}, \qquad \text{where } \theta(\lambda) := \sqrt{\beta^2 + 2\lambda} - \beta.$$

Let h be a nice function on $[0, \infty)$, and define the function $\tilde{h}$ on Δ via

$$\tilde{h}(t, x) := \mathbb{E}h(t + T_{\alpha - \beta t - x, \beta}), \qquad (t, x) \in \Delta. \tag{1.14}$$

Then with $\mathscr{F}_t = \sigma\{B_s: s \leqslant t\}$ as usual,

$$H_t := \mathbb{E}(h(T) \,|\, \mathscr{F}_t) = \tilde{h}(t \wedge T, B_{t \wedge T}),$$

and since H is a martingale, Itô's formula shows that

$$H_t = \mu + \int_0^{t \wedge T} \frac{\partial \tilde{h}}{\partial x}(s, B_s) \, \mathrm{d}B_s \tag{1.15}$$

(where $\mu := \mathbb{E}h(T)$) and that $\tilde{h}$ is space-time-harmonic on Δ. Thus $\tilde{h}$ is the space-time-harmonic extension of $h \circ \pi$ from $\partial\Delta$ to Δ. We note that

$$h(T) = H_T = \mu + \int_0^T \frac{\partial \tilde{h}}{\partial x}(s, B_s) \, \mathrm{d}B_s. \tag{1.16}$$

For $k = 0, 1, 2, \ldots$, we define for $x = \alpha - \beta t$, so that $(t, x) \in \partial\Delta$,

$$(\mathscr{A}_k h)(t) := \lim_{\Delta \ni (u, y) \to (t, x)} \frac{\partial^k \tilde{h}}{\partial y^k}(u, y). \tag{1.17}$$

Lemma 1.1. *We have:*

(a) $$\mathscr{A}_0 = \mathrm{id}, \quad \mathscr{A}_k = \mathscr{A}^k, \qquad \textit{where } \mathscr{A} := \mathscr{A}_1.$$

Consideration of the space-time-harmonic function $\tilde{h}(t, x) = \exp(\theta x - \frac{1}{2}\theta^2 t)$ *shows that, for* $\theta \geqslant -\beta$ *(but not for* $\theta < -\beta$*),*

(b) $$\mathscr{A}h = \theta h \quad \textit{when} \quad h(t) = \exp\{\theta(\alpha - \beta t) - \tfrac{1}{2}\theta^2 t\},$$

whence, if $h_n(t)$ *denotes the Hermite function* $\tilde{h}_n(t, \alpha - \beta t)$ *(in agreement with the tilde convention), then*

(c) $$\mathscr{A}h_n = nh_{n-1}.$$

If $e_\lambda(t) := \mathrm{e}^{-\lambda t}$*, then, for* $\lambda \geqslant -\frac{1}{2}\beta^2$*,*

(d) $$\mathscr{A}e_\lambda = \theta(\lambda)\mathrm{e}_\lambda, \ \textit{where} \ \theta(\lambda) := \sqrt{\beta^2 + 2\lambda} - \beta.$$

As is explained later, the operator $\mathscr{A}$ *is the Lévy–Khinchine operator associated with the hitting-time process for drifting Brownian motion with drift* β*:*

(e) $$(\mathscr{A}h)(t) = \int_{u=0}^{\infty} \{h(t) - h(t+u)\} \frac{\mathrm{e}^{-\frac{1}{2}\beta^2 u}}{\sqrt{2\pi u^3}} \, \mathrm{d}u.$$

The space-time-harmonic property of $\tilde{h}$ *is reflected in the fact that*

(f) $$\mathscr{A}^2 = -2D - 2\beta\mathscr{A}, \ \textit{where} \ D = \mathrm{d}/\mathrm{d}t;$$

*and this agrees with the 'symbol' formula (*d*).*

1.7. Main result

Our main result, is the following.

Theorem 1.1. *We continue to work with an arbitrary perturbation* $a + \varepsilon c$ *(*c *satisfying (1.4)) of the linear boundary* $a(t) = \alpha + \beta t$*, where* $\alpha, \beta > 0$*. Recall that* W *and* B *are Brownian motions started at* 0*, and that we define*

$$N_t := \int_0^t c'(s) \, \mathrm{d}B_s, \qquad \langle N \rangle_t := \int_0^t c'(s)^2 \, \mathrm{d}s, \qquad G_t^{(\varepsilon c)} := \exp\{-\varepsilon N_t - \tfrac{1}{2}\varepsilon^2 \langle N \rangle_t\},$$

and

$$T := \inf\{t \colon B_t + \beta t = \alpha\}.$$

Recall that

(a) $$\mathbb{P}[W \text{ hits } a + \varepsilon c] = \mathbb{P}[W \text{ hits } a]\mathbb{E}\Big(G_T^{(\varepsilon c)} \mathrm{e}^{-2\beta\varepsilon c(T)}\Big).$$

Suppose that there exists $\varphi > 0$ *such that*

(b) $$\mathbb{E} \exp(\tfrac{1}{2}\varphi^2 \langle N \rangle_T) < \infty.$$

Then, for $|\varepsilon| < (\sqrt{10} - 3)\varphi$*, we have the absolutely convergent Taylor expansion*

(c)
$$\mathbb{E}\Big(G_T^{(\varepsilon c)} \mathrm{e}^{-2\beta\varepsilon c(T)}\Big) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \gamma_n,$$

where

(d)
$$\gamma_n = \mathrm{e}^{-2\beta\alpha} \mathbb{E}\tilde{h}_n(\langle N \rangle_T, -2\beta c(T) - N_T).$$

For nice c (say, smooth and with nth-order derivative $c^{(n)}$ bounded on $[0, \infty)$ *for each* $n \geqslant 1$, *so that in particular* (b) *holds for some* φ), *we have*

(e)
$$\gamma_n = \mathbb{E}g_n(T),$$

the functions g_n being obtained recursively via the fact that $g_0 \equiv 1$ and, for $n \geqslant 1$,

(f)
$$\sum_{k=0}^{n} \binom{n}{k} c^k \mathscr{A}^k g_{n-k} = (-2\beta c)^n.$$

Thus, for example, $g_1(t) = -2\beta c(t)$, $g_2(t) = 4\beta(c\mathscr{A}c)(t) + 4\beta^2 c(t)^2$ *and*

$$g_3 = -8\beta^3 c^3 - 12\beta c\mathscr{A}(c\mathscr{A}c) - 12\beta^2 c\mathscr{A}(c^2) + 6\beta c^2 \mathscr{A}^2 c.$$

The above results may be derived formally as the special case when $h = \sum \varepsilon^n g_n / n!$ *of the following formula for a nice function h on* $[0, \infty)$:

(g)
$$\mathbb{E}G_T^{(\varepsilon c)} \sum_{k=0}^{\infty} \frac{\varepsilon^k c(T)^k}{k!} (\mathscr{A}^k h)(T) = \mathbb{E}h(T).$$

We can reformulate (f) *as the formal-power-series result:*

(h)
$$\left\{ \mathrm{e}^{\varepsilon c(t)\mathscr{A}} \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} g_n(\cdot) \right\}(t) = \mathrm{e}^{-2\beta\varepsilon c(t)}.$$

1.8. Boundaries below *a*

Define

$$S_t^\beta := \inf\{u \colon B_u + \beta u > t\}, \qquad t \geqslant 0,$$

so that $\{S_t^\beta\}$ is the hitting-time process (in its right-continuous version) for Brownian motion with drift β. The process $\{S_t^\beta\}$ is a subordinator (increasing process with independent increments) with

$$\mathbb{E}\mathrm{e}^{-\lambda S_t^\beta} = \exp\{-\theta(\lambda)t\}, \qquad \text{for } \lambda > -\tfrac{1}{2}\beta^2. \tag{1.18}$$

The operator $(-\mathscr{A})$ is the infinitesimal generator of the transition semigroup $\{P_t^\beta\}$ of $\{S_t^\beta\}$, so that, for a nice function h on $[0, \infty)$ and for $u \in [0, \infty)$, we have

$$\sum \left\{ \frac{(-u\mathscr{A})^k h}{k!} \right\}(t) = \{(\mathrm{e}^{-u\mathscr{A}})h\}(t) = (P_u^\beta h)(t) = \mathbb{E}h(t + S_u^\beta). \tag{1.19}$$

On comparing this with parts (a) and (g) of Theorem 1.1 with $\varepsilon = -1$, we would guess the 'equality' case of the following theorem – at least when c satisfies (1.4).

Theorem 1.2. *Again take* $a(t) = \alpha + \beta t$ *where* $\alpha, \beta > 0$. *Assume that* c *is a right-continuous non-negative function on* $[0, \infty)$, *with* $c(0) < a(0)$. *Suppose that we can find a function* h *on* $[0, \infty)$ *such that*

$$\text{(a)} \qquad \textit{for } t \geqslant 0, \qquad e^{2\beta c(t)} - \mathbb{E}h(t + S^{\beta}_{c(t)}) = (\geqslant, \leqslant) 0.$$

Then

$$\text{(b)} \qquad \mathbb{P}[W \text{ hits } a - c] - \mathbb{P}[W \text{ hits } a]\mathbb{E}h(T) = (\geqslant, \leqslant) 0.$$

Because the boundary $a - c$ lies below the boundary a, this theorem might seem rather weird at first. The result was first discovered as indicated above. Once found, it looked as if it should have a direct proof; and indeed it does.

Example 1.1. Suppose that $c(t) = \eta t$, where $0 \leqslant \eta \leqslant \frac{1}{2}\beta$. If we put $h = e_\lambda$ in Theorem 1.2(a), then we see from (1.18) that we must have $-\lambda - \theta(\lambda)\eta = 2\beta\eta$, whence $\lambda = -2\beta\eta + 2\eta^2 \in [-\frac{1}{2}\beta^2, 0]$, and

$$\mathbb{E}h(T) = e^{-\theta(\lambda)\alpha} = e^{2\alpha\eta},$$

in agreement with (1.8).

The restriction $\eta \leqslant \frac{1}{2}\beta$ is necessary for the square root involved in $\theta(\lambda)$, namely $(\beta^2 - 4\beta\eta + 4\eta^2)^{1/2}$, to equal $+(\beta - 2\eta)$. From some points of view, it is a strange restriction. We note that, since $\varphi = \beta/\eta$, in order to apply Theorem 1.1 we need $\eta < (\sqrt{10} - 3)\beta$.

We can 'proceed in the time-honoured way' by choosing a suitable answer h and then finding the c to which it corresponds under Theorem 1.2(a). Indeed, choosing h to be a positive mixture of exponentials (in other words, a Laplace transform) amounts to using the harmonic-function method. The simplest new example is given by $h(t) = 1 + t$. This leads to the following result.

Lemma 1.2. *As usual,* $a(t) = \alpha + \beta t$ *where* $\alpha, \beta > 0$. *If* $2\beta^2 > 1$, *then*

$$\text{(a)} \qquad \mathbb{P}[W \text{ hits } a - c] = \left(1 + \frac{\alpha}{\beta}\right) e^{-2\alpha\beta}$$

in the case when $c(t)$ *is the unique solution in* $[0, \infty)$ *of*

$$\text{(b)} \qquad 1 + t + \beta^{-1}c(t) = e^{2\beta c(t)}.$$

The condition that $2\beta^2 > 1$ is needed to guarantee that (b) has a unique solution $c(t)$ in $[0, \infty)$. It also forces the right-hand side of (a) to be less than 1 for all α(!) Note that $c(t)$ is roughly $(\log t)/(2\beta)$ for large t.

1.9. Discussion of Theorem 1.2 and Theorem 1.1

We note that Theorem 1.2 gives a partial sample-path explanation of Theorem 1.1 for the case when $c(\cdot) \geqslant 0$ and $\varepsilon < 0$ (and $|\varepsilon| < (\sqrt{10} - 3)\varphi$); for then Theorem 1.1(h) and equation (1.19) show that if $(-\varepsilon c)$ becomes the 'c' in Theorem 1.2, then the 'h' in the equality case of Theorem 1.2(a) may be taken to be $\sum(-\varepsilon)^n g_n/n!$. However, this explanation does not provide sample-path identification of the individual g_n functions.

1.10. Boundaries above *a*

We now concentrate on the case when $\varepsilon = 1$ and $c(\cdot) \geqslant 0$.

Example 1.2. First take $\varepsilon = 1$ and $c(t) = \eta t$, where $\eta > 0$. Comparing parts (a) and (g) of Theorem 1.1 suggests that we try to arrange that

$$\sum_{k=0}^{\infty} \frac{c(t)^k}{k!}(\mathscr{A}^k h)(t) = \mathrm{e}^{-2\beta c(t)} \qquad t \geqslant 0; \tag{1.20}$$

and we see that we can do this if we take $h = e_\lambda$, so that $\mathscr{A}^k h = \theta(\lambda)^k h$, and then pick λ so that $\eta\theta(\lambda) - \lambda = -2\beta\eta$. Thus, we choose $\lambda = 2\beta\eta + 2\eta^2$, and find that $\theta(\lambda) = 2\eta$, so that we have $\mathbb{E}h(T) = \mathrm{e}^{-2\alpha\eta}$. It all tallies with (1.18): we have arrived at the correct answer.

Once more it is the case that choosing h to be a mixture of exponentials amounts to using the harmonic-function method.

Example 1.3. Let us try once more to cheat in the time-honoured way. We take $\varepsilon = +1$ and $h(t) = 1 + t$, whence $\mathscr{A}h \equiv -\beta^{-1}$ and $\mathscr{A}^2 h = 0$. We assume that $2\beta^2 < 1$, and define $c(t)$ to be the unique root in $[0, \infty)$ of

$$1 + t - \beta^{-1} c(t) = \mathrm{e}^{-2\beta c(t)}.$$

Thus, (1.20) holds with our new notation, and, on comparing parts (a) and (g) of Theorem 1.1, it looks as though we should have

$$\mathbb{P}[W \text{ hits } a + c] = \mathbb{P}[W \text{ hits } a]\mathbb{E}h(T) = \mathbb{P}[W \text{ hits } a](1 + \alpha/\beta).$$

But this is clearly nonsensical since $\mathbb{P}[W \text{ hits } a + c]$ is certainly less than $\mathbb{P}[W \text{ hits } a]$.

Why have things gone wrong in Example 1.3? We note that $c(t)$ is close to βt for large t. In the case when $c(t)$ is exactly βt and $\varepsilon = 1$, we have $N_t = \beta B_t$, $\langle N \rangle_t = \beta^2 t$ and

$$G_T := G_T^{(\varepsilon c)} = \exp\{-\beta B_T - \tfrac{1}{2}\beta^2 T\} = \exp\{-\beta\alpha + \tfrac{1}{2}\beta^2 T\},$$

since $B_T = \alpha - \beta T$. We note from (1.13) that $\mathbb{E}(G_T T) = \infty$ in this case. It is therefore to be expected that the terms in Theorem 1.1(g) will explode for Example 1.3. It is perhaps a little strange in regard to (1.20) that $h = 1 + t$ corresponds to a c with $c(t) \sim \beta t$, whereas, as we saw in Example 1.2, $h = \exp(-4\beta^2 t)$ corresponds to $c(t) = \beta t$.

If we try to use Theorem 1.1 on Example 1.3, we find that, since $c(t) \sim \beta t$, the supremum value for φ is 1, so that the theorem applies if $|\varepsilon| < \sqrt{10} - 3$; hence, the value $\varepsilon = 1$ is not allowed in the theorem.

1.11. What is the significance of the functions g_n?

We give one explanation – not the excursion one on this occasion(!) – at the end of this paper.

2. Proofs of Lemma 1.1 and Theorem 1.1

Formal verification of Lemma 1.1. 'Formal verification' here signifies that we check the 'algebra' rather than concerning ourselves with, for example, the precise degree of regularity required of the h functions.

The fact that $\mathscr{A}_2 = \mathscr{A}^2$ follows because $\partial\tilde{h}/\partial x$ is space-time-harmonic on Δ with boundary function $(\mathscr{A}h) \circ \pi$. Proving (b), (c) and (d) in the statement of Lemma 1.1 is straightforward. The agreement of (d) and (e) is a well-known Laplace-transform result. We have

$$\theta(\lambda)^2 + 2\beta\theta(\lambda) - 2\lambda = 0,$$

whence, from (d),

$$(\mathscr{A}^2 + 2\beta\mathscr{A} + 2D)\mathrm{e}_\lambda = 0,$$

and (f) clearly holds on suitably nice functions. A better way of seeing (f) is to observe that, for a nice function h on $[0, \infty)$,

$$\begin{aligned}(Dh)(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\tilde{h}(t, \alpha - \beta t) = \left(\frac{\partial\tilde{h}}{\partial t} - \beta\frac{\partial\tilde{h}}{\partial x}\right)(t, \alpha - \beta t)\\ &= \left(-\frac{1}{2}\frac{\partial^2\tilde{h}}{\partial x^2} - \beta\frac{\partial\tilde{h}}{\partial x}\right)(t, \alpha - \beta t) = (-\tfrac{1}{2}\mathscr{A}^2 h - \beta\mathscr{A}h)(t).\end{aligned}$$

The reader might be amused to derive (f) directly from (e). □

Proof of the first parts of Theorem 1.1. We begin by checking equation (1.5). For any finite t, the law $\tilde{\mathbb{P}}$ (say) of $W - \varepsilon c$ has likelihood ratio relative to the Wiener law of W given by

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = R_t := \exp\left\{-\varepsilon\int_0^t c'(s)\,\mathrm{d}W_s - \tfrac{1}{2}\varepsilon^2\int_0^t c'(s)^2\,\mathrm{d}s\right\} \qquad \text{on } \mathscr{W}_t,$$

where $\mathscr{W}_t$ is the augmented natural filtration of W. Recall that $\tau := \inf\{t\colon W(t) = a(t)\}$. Since the martingale $R_s\colon 0 \leqslant s \leqslant t\}$ is uniformly integrable, we have

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = R_{t\wedge\tau} \qquad \text{on } \mathscr{W}_{t\wedge\tau}.$$

Now the event $\{\tau < t\}$ is an element of $\mathscr{W}_{t\wedge\tau}$, so that

$$\tilde{\mathbb{P}}(\tau < t) = \mathbb{E}(\mathrm{R}_{t\wedge\tau};\, \tau < t) = \mathbb{E}(\mathrm{R}_\tau I_{\{\tau<\infty\}};\, \tau < t). \tag{2.1}$$

However, by Fatou's lemma, $R_\tau I_{\{\tau<\infty\}}$ is in L^1 (with expectation at most 1). Hence, on letting $t\uparrow\infty$ in (2.1), we obtain

$$\tilde{\mathbb{P}}(\tau < \infty) = \mathbb{R}(R_\tau;\, \tau < \infty),$$

which is equation (1.5).

We now make the linearity assumption: $a(t) = \alpha + \beta t$, where $\alpha, \beta > 0$. The equivalence of law stated around (1.9) is standard. There is therefore no problem in obtaining part (a) of Theorem 1.1, so we have

$$\mathbb{P}[W \text{ hits } a + \varepsilon c] = \mathbb{P}[W \text{ hits } a]\mathbb{E}\exp\{-\varepsilon N_T - \tfrac{1}{2}\varepsilon^2\langle N\rangle_T - 2\beta\varepsilon c(T)\}.$$

Recall that we next assume that there exists a $\varphi > 0$ such that

$$\mathbb{E}\exp(\tfrac{1}{2}\varphi^2\langle N\rangle_T) < \infty. \tag{2.2}$$

Lemma 2.1. *For* $0 \leqslant |\varepsilon| < (\sqrt{10} - 3)\varphi$,

$$\mathbb{E}\exp\{|\varepsilon|\,|N_T| + \tfrac{1}{2}|\varepsilon|^2\langle N\rangle_T + 2\beta|\varepsilon|\,|c(T)|\} < \infty.$$

We assume this lemma for the time being. Then, for $0 \leqslant |\varepsilon| < (\sqrt{10} - 3)\varphi$, the terms in the expansion as a power series in ε of

$$\exp\{-\varepsilon N_T - \tfrac{1}{2}\varepsilon^2\langle N\rangle_T - 2\beta\varepsilon c(T)\}$$

are dominated by the (non-negative) terms in the expansion as a power series in $|\varepsilon|$ of

$$\exp\{|\varepsilon|\,|N_T| + \tfrac{1}{2}|\varepsilon|^2\langle N\rangle_T + 2\beta|\varepsilon|\,|c(T)|\}.$$

Lemma 2.1 therefore guarantees that, for $0 \leqslant |\varepsilon| < (\sqrt{10} - 3)\varphi$, we *do* have the absolutely convergent Taylor series expansion at (c) with the γ_n as at (d) in Theorem 1.1. □

Having settled that main analytic point, we now assume that c is smooth with derivatives (of all orders at least 1) bounded, and use *formal* power series to prove (e) in Theorem where the g_n are related via the recurrence relation (f). It is enough to prove the formal formula at (g),

$$\mathbb{E}G_T^{(\varepsilon c)}\sum_{k=0}^{\infty}\frac{\varepsilon^k c(T)^k}{k!}(\mathscr{A}^k h)(T) = \mathbb{E}h(T), \tag{2.3}$$

and to substitute the formal expression $h = \sum\varepsilon^n g_n/n!$. Formal power series make things neat, but the reader who is unhappy with their use can (as we did at first) derive the recurrence relation directly by an analogue of the method we now give.

Formal verification of (2.3). We use the $\tilde{h}$ notation around (1.15). We apply the 'differential' version of the stochastic calculus as described in Revuz and Yor (1991) or Rogers and Williams (1987). By the usual 'exponential martingale' formula,

$$\mathrm{d}G_t^{(\varepsilon c)} = -\varepsilon c'(t) G_t^{(\varepsilon c)} \, \mathrm{d}B_t.$$

Since $\partial^k \tilde{h}/\partial x^k$ is space-time-harmonic on Δ,

$$\mathrm{d}\left\{\frac{\partial^k \tilde{h}}{\partial x^k}(t, B_t)\right\} = \frac{\partial^{k+1}\tilde{h}}{\partial x^{k+1}}(t, B_t)\,\mathrm{d}B_t, \qquad t < T.$$

The '$t < T$' here and elsewhere is made rigorous by replacing t by $t \wedge T$ on the left-hand side and multiplying the right-hand side by $I_{\{t<T\}}$.

With '$\stackrel{\bullet}{=}$' denoting 'equality modulo the differential of a local martingale', we have, for two local martingales $M^{(1)}$ and $M^{(2)}$, and a smooth deterministic function f on $[0, \infty)$,

$$\mathrm{d}\{f(t) M_t^{(1)} M_t^{(2)}\} \stackrel{\bullet}{=} f'(t) M_t^{(1)} M_t^{(2)}\,\mathrm{d}t + f(t)\,\mathrm{d}M_t^{(1)}\,\mathrm{d}M_t^{(2)}.$$

As always, $\mathrm{d}B_t\,\mathrm{d}B_t = \mathrm{d}t$.

Hence, for $t < T$,

$$\mathrm{d}\left\{\frac{\varepsilon^k c(t)^k}{k!} G_t^{(\varepsilon c)} \frac{\partial^k \tilde{h}}{\partial x^k}(t, B_t)\right\}$$

$$\stackrel{\bullet}{=} G_t^{(\varepsilon c)} c'(t) \left\{\frac{\varepsilon^k c(t)^{k-1}}{(k-1)!}\frac{\partial^k \tilde{h}}{\partial x^k}(t, B_t) - \frac{\varepsilon^{k+1} c(t)^k}{k!}\frac{\partial^{k+1}\tilde{h}}{\partial x^{k+1}}(t, B_t)\right\}\mathrm{d}t,$$

the first term of the right-hand side being 0 if $k = 0$. Formal summation over k from 0 to ∞ now yields that

$$\sum_{k=0}^{\infty} \frac{\varepsilon^k c(t)^k}{k!} G_t^{(\varepsilon c)} \frac{\partial^k \tilde{h}}{\partial x^k}(t, B_t),$$

when stopped at T, defines a local martingale. The (formal) value of this local martingale when $t = T$ is

$$G_T^{(\varepsilon c)} \sum_{k=0}^{\infty} \frac{\varepsilon^k c(T)^k}{k!} (\mathscr{A}^k h)(T),$$

and the value when $t = 0$ is $\tilde{h}(0, 0) = \mathbb{E}h(T)$. Hence, if we ignore the difference between local martingale and martingale, and assume that the relevant optional-stopping theorem is valid at time T, the result (2.3) follows. □

Proof of Lemma 2.1. For $\varepsilon > 0$, define

$$A(\varepsilon) := \mathbb{E}\exp\{\varepsilon|N_T| + \tfrac{1}{2}\varepsilon^2\langle N\rangle_T + 2\beta\varepsilon|c(T)|\}.$$

By Hölder's inequality, we have for any $p, q, r > 0$ with $p^{-1} + q^{-1} + r^{-1} = 1$,

$$A(\varepsilon) \leqslant \{\mathbb{E}\exp(\varepsilon p|N_T|)\}^{1/p}\{\mathbb{E}\exp(\tfrac{1}{2}\varepsilon^2 q\langle N\rangle_T)\}^{1/q}\{\mathbb{E}\exp(2\beta\varepsilon r|c(T)|)\}^{1/r}. \tag{2.4}$$

We shall make good choices of p, q and r later.

For any $\psi > 0$,

$$\mathbb{E}\exp(\psi|\mathrm{N}_T|) \leqslant \mathbb{E}\exp(\psi N_T) + \mathbb{E}\exp(-\psi N_T). \tag{2.5}$$

But

$$\begin{aligned}\mathbb{E}\exp(\pm\psi N_T) &= \mathbb{E}\{\exp(\psi^2\langle N\rangle_T)\exp(\pm\psi N_T - \psi^2\langle N\rangle_T)\}\\ &\leqslant \{\mathbb{E}\exp(2\psi^2\langle N\rangle_T)\}^{1/2}\{\mathbb{E}\exp(\pm 2\psi\mathbb{N}_T - 2\psi^2\langle N\rangle_T)\}^{1/2}\\ &\leqslant \{\mathbb{E}\exp(2\psi^2\langle N\rangle_T)\}^{1/2}\end{aligned} \tag{2.6}$$

using the Cauchy–Schwarz inequality and the fact that $\exp(\pm 2\psi N_t - 2\psi^2\langle N\rangle_t)$ is a non-negative supermartingale. So, by (2.5) and (2.6) with $\psi = \varepsilon p$, we have

$$\mathbb{E}\exp(\varepsilon p|N_T|) \leqslant 2\{\mathbb{E}\exp(2\varepsilon^2 p^2\langle N\rangle_T)\}^{1/2}. \tag{2.7}$$

To establish a bound on $\mathbb{E}\exp(2\beta\varepsilon r|c(T)|)$, we argue as follows. First,

$$\begin{aligned}|c(t)| &= \left|\int_0^t c'(s)\,\mathrm{d}s\right| \leqslant \int |c'(s)|\,\mathrm{d}s\\ &\leqslant t^{1/2}\langle N\rangle_t^{1/2} \leqslant \max\{t/\gamma,\, \gamma\langle N\rangle_t\}, \qquad \text{for any } \gamma > 0.\end{aligned}$$

Define $z(\eta) := \mathbb{E}\exp(\eta T)$. The largest η for which $z(\eta)$ is finite is $\frac{1}{2}\beta^2$. So choose $\eta = \frac{1}{2}\beta^2$ and $\gamma = 2\beta\varepsilon r/\eta$ to obtain

$$\begin{aligned}\mathbb{E}\exp(2\beta\varepsilon r|c(T)|) &\leqslant \mathbb{E}\exp(\eta T) + \mathbb{E}\exp(2\beta\varepsilon r\gamma\langle N\rangle_T)\\ &\leqslant z(\tfrac{1}{2}\beta^2) + \mathbb{E}\exp(8\varepsilon^2 r^2\langle N\rangle_T).\end{aligned} \tag{2.8}$$

From (2.4), (2.7) and (2.8), the sensible choice of (p, q, r), giving rise to the largest value $\varepsilon_0(\varphi)$ of ε, is such that

$$2\varepsilon_0(\varphi)^2 p^2 = \tfrac{1}{2}\varepsilon_0(\varphi)^2 q = 8\varepsilon_0(\varphi)^2 r^2 = \tfrac{1}{2}\varphi^2, \qquad p^{-1} + q^{-1} + r^{-1} = 1.$$

The solution is given by

$$p = 2r, \qquad q = 16r^2, \qquad r = (\sqrt{10} + 3)/4, \qquad \varepsilon_0(\varphi) = (\sqrt{10} - 3)\varphi.$$

Lemma 2.1 is proved. □

3. Theorem 1.2 and the g_n functions

Throughout the remainder of the paper, let B and W denote *independent* Brownian motions starting at 0.

Suppose that $a(t) = \alpha + \beta t$. Suppose that $c(\cdot) \geqslant 0$. Define

$$\tau := \inf\{u\colon W_u = a(u)\} \qquad \text{(as before)},$$

$$\sigma := \inf\{u\colon W_u = a(u) - c(u).\}$$

For $t \geqslant 0$, define

$$S_t^{\beta} := \inf\{u\colon B_u + \beta u > t\}, \qquad \text{(so that } S_t^{\beta} < \infty \text{ a.s.)},$$

$$S_t^{-\beta} := \inf\{u\colon B_u - \beta u > t\} \leqslant \infty,$$

$$T := S_\alpha^{\beta} \qquad \text{(as usual)}.$$

We use the following standard facts:

(a) conditional on $\{S_t^{-\beta} < \infty\}$, $S_t^{-\beta}$ has the law of S_t^{β};
(b) $\tau \stackrel{\mathscr{D}}{=} \sigma + S_{c(\sigma)}^{-\beta}$ ($\stackrel{\mathscr{D}}{=}$ signifying equality of law), by the strong Markov property of W at stopping time σ;
(c) $\tau \stackrel{\mathscr{D}}{=} S_t^{-\beta}$.

We shall use $\stackrel{\text{(a)}}{=}$ to signify 'is equal to because of (a)'.

Proof of Theorem 1.2. Rewrite the assumption that

$$\mathrm{e}^{2\beta c(t)} - \mathbb{E}h(t + S_{c(t)}^{\beta}) \geqslant 0 \qquad (t \geqslant 0)$$

as

$$\begin{aligned}
1 &\geqslant \mathrm{e}^{-2\beta c(t)}\mathbb{E}h(t + S_{c(t)}^{\beta}) \\
&= \mathbb{P}(S_{c(t)}^{-\beta} < \infty)\mathbb{E}h(t + S_{c(t)}^{\beta}) \\
&\stackrel{\text{(a)}}{=} \mathbb{E}[h(t + S_{c(t)}^{-\beta});\ S_{c(t)}^{-\beta} < \infty].
\end{aligned}$$

Since σ is independent of B (and hence of the S-process),

$$1 \geqslant \mathbb{E}[h(\sigma + S_{c(\sigma)}^{-\beta});\ S_{c(\sigma)}^{-\beta} < \infty | \sigma] \qquad \text{on } \{\sigma < \infty\},$$

whence

$$\begin{aligned}
&\mathbb{P}(\sigma < \infty) \\
&\quad \geqslant \mathbb{E}[h(\sigma + S_{c(\sigma)}^{-\beta});\ S_{c(\sigma)}^{-\beta} < \infty;\ \sigma < \infty] \\
&\quad = \mathbb{E}[h(\sigma + S_{c(\sigma)}^{-\beta});\ \sigma + S_{c(\sigma)}^{-\beta} < \infty] \\
&\quad \stackrel{\text{(b)}}{=} \mathbb{E}[h(\tau);\ \tau < \infty] \stackrel{\text{(c)}}{=} \mathbb{E}[h(S_\alpha^{-\beta});\ S_\alpha^{-\beta} < \infty] \\
&\quad \stackrel{\text{(a)}}{=} \mathbb{P}(S_\alpha^{-\beta} < \infty)\mathbb{E}h(S_\alpha^{\beta}) = \mathbb{P}(\tau < \infty)\mathbb{E}h(T).
\end{aligned}$$

Thus,

$$\mathbb{P}[W \text{ hits } a - c] \geqslant \mathbb{P}[W \text{ hits } a]\mathbb{E}h(T),$$

as required. The '$\leqslant$' and '$=$' cases are done similarly. □

3.1. Another formulation of the Taylor expansion

Our Taylor expansion is now perhaps best formulated as

$$\mathbb{P}[W \text{ hits } a + \varepsilon c] = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \mathbb{E}[g_n(\tau);\ \tau < \infty],$$

where $g_0 \equiv 1$ and, for $n \geqslant 1$,

$$\sum_{k=0}^{n} \binom{n}{k} (-c)^k \mathscr{G}^k g_{n-k} = 0.$$

Here, $\mathscr{G} := -\mathscr{A} - 2\beta$ denotes the generator of the killed subordinator $\{S_t^{-\beta}\colon\ t \geqslant 0\}$.

The g_n functions just mentioned are identical to those used previously, and one can derive this reformulation by simple algebra.

3.2. Discussion of the functions g_n

Suppose now that c *is strictly increasing and that* $\varepsilon > 0$. Define

$$\tau_0^\varepsilon := \tau := \inf\{t\colon\ W_t = \alpha + \beta t\} \leqslant \infty \qquad (\tau \text{ is as before}),$$

$$\tau_n^\varepsilon := \inf\{t\colon\ W_t = \alpha + \beta t + \varepsilon c(\tau_{n-1}^\varepsilon)\} \leqslant \infty \qquad (n \geqslant 1).$$

Then

$$\tau_n^\varepsilon \uparrow \tau_\infty^\varepsilon = \inf\{t\colon\ W_t = \alpha + \beta t + \varepsilon c(t)\} \leqslant \infty.$$

Of course, it does *not* necessarily follow that $\mathbb{P}(\tau_n^\varepsilon < \infty) \downarrow \mathbb{P}(\tau_\infty^\varepsilon < \infty)$.

Conditionally on $\tau_0^\varepsilon,\ \tau_1^\varepsilon,\ \ldots,\ \tau_{n-1}^\varepsilon$, we have (with $\tau_{-1} := 0$)

$$\tau_n^\varepsilon \overset{\mathscr{D}}{=} \tau_{n-1}^\varepsilon + S^{-\beta}_{\varepsilon[c(\tau_{n-1}^\varepsilon) - c(\tau_{n-2}^\varepsilon)]} \text{ on } \{\tau_{n-1}^\varepsilon < \infty\},$$

and, because of the relation between the $\{S^{-\beta}\}$ process and the $\{S^\beta\}$ process, the transition semigroup $\{P_t^\beta\}$ of which has generator $(-\mathscr{A})$, we can hope to prove that *on* $\{\tau < \infty\}$,

$$\mathbb{P}(\tau_n^\varepsilon < \infty | \tau) = \sum_{k=0}^{n} \frac{\varepsilon^k}{k!} g_k(\tau) + O(\varepsilon^{n+1}), \tag{3.1}$$

a result which would give some insight into the g_n functions.

Now, on $\{\tau < \infty\}$,

$$\mathbb{P}(\tau_1^\varepsilon < \infty | \tau) = \mathrm{e}^{-2\beta\varepsilon c(\tau)} = 1 - \varepsilon 2\beta c(\tau) + O(\varepsilon^2),$$

$$= (g_0 + \varepsilon g_1)(\tau) + O(\varepsilon^2),$$

so that (3.1) holds when $n = 1$.

Now consider (3.1) when $n = 2$. On $\{\tau < \infty\}$, we have

$$\begin{aligned}
\mathbb{P}(\tau_2^\varepsilon < \infty | \tau) &= \mathbb{E}\{\exp[-2\beta\varepsilon\{c(\tau_1^\varepsilon) - c(\tau_0^\varepsilon)\}];\ \tau_1^\varepsilon < \infty | \tau\} \\
&= \mathbb{E}\{\exp[-2\beta\varepsilon\{c(S^{-\beta}_{\varepsilon c(\tau)} + \tau) - c(\tau)\}];\ S^{-\beta}_{\varepsilon c(\tau)} < \infty | \tau\} \\
&= \mathbb{P}\{S^{-\beta}_{\varepsilon c(\tau)} < \infty\}\mathbb{E}\{\exp[-2\beta\varepsilon\{c(S^{\beta}_{\varepsilon c(\tau)} + \tau) - c(\tau)\}] | \tau\} \\
&= \mathrm{e}^{-2\beta\varepsilon c(\tau)}\mathrm{e}^{2\beta\varepsilon c(\tau)}\mathbb{E}\{\exp[-2\beta\varepsilon c(S^{\beta}_{\varepsilon c(\tau)} + \tau)] | \tau\}
\end{aligned}$$

Hence, we have the exact formula on $\{\tau < \infty\}$:

$$\mathbb{P}(\tau_2^\varepsilon < \infty | \tau) = (P^\beta_{\varepsilon c(\tau)}\mathrm{e}^{-2\beta\varepsilon c(\cdot)})(\tau).$$

Formal expansion in terms of the infinitesimal generator $(-\mathscr{A})$ of $\{P_t^\beta\}$ yields (using the fact that $\mathscr{A}1 = 0$) on $\{\tau < \infty\}$:

$$\begin{aligned}
\mathbb{P}(\tau_2^\varepsilon < \infty | \tau) &= (1 - \varepsilon c\mathscr{A} + \tfrac{1}{2}\varepsilon^2 c^2 \mathscr{A}^2)(1 - 2\beta\varepsilon c + 2\beta^2\varepsilon^2 c^2)(\tau) + O(\varepsilon^3) \\
&= 1 + \varepsilon(-2\beta c) + \varepsilon^2(2\beta c\mathscr{A}c + 2\beta^2 c^2)(\tau) + O(\varepsilon^3),
\end{aligned}$$

which is result (3.1) for $n = 2$.

Things become more complicated for $n \geqslant 3$; and we have to be careful and systematic in doing the calculations. One finds that on $\{\tau_1^\varepsilon < \infty\}$,

$$\mathbb{P}(\tau_3^\varepsilon < \infty | \tau_0^\varepsilon, \tau_1^\varepsilon) = \mathrm{e}^{2\beta\varepsilon c_0}(P^\beta_{\varepsilon(c_1 - c_0)}\mathrm{e}^{-2\beta\varepsilon c(\cdot)})(\tau_1^\varepsilon).$$

Here, c_1 stands for $c(\tau_0^\varepsilon)$ and c_0 for $c(\tau_0^\varepsilon = c(\tau)$. Since we shall always work conditionally on τ, we regard c_0 as a constant. Now, on $\{\tau < \infty\}$,

$$\tau_1^\varepsilon = S^{-\beta}_{\varepsilon c_0} + \tau.$$

After the usual $S^{-\beta}$ to S^β interchange, we find that on $\{\tau < \infty\}$,

$$\mathbb{P}(\tau_3^\varepsilon < \infty | \tau) = (P^\beta_{\varepsilon c_0} P^\beta_{\varepsilon[\mathrm{c}(\cdot) - c_0]}\mathrm{e}^{-2\beta\varepsilon c(\cdot)})(\tau).$$

We write this as the value at τ of

$$\{(1 - \varepsilon c_0\mathscr{A} + \tfrac{1}{2}\varepsilon^2 c_0^2\mathscr{A}^2)(1 - \varepsilon(c - c_0)\mathscr{A} + \tfrac{1}{2}\varepsilon^2(c - c_0)^2\mathscr{A}^2)\}\mathrm{e}^{-2\beta\varepsilon c} + O(\varepsilon^4).$$

Here, c_0 is initially regarded as a constant unaffected by $\mathscr{A}$; however, having worked things out in this way, we then put $c_0 = c$. If we employ this italicized strategy on the operator within the large $\{\}$ braces, we find that it becomes

$$1 - \varepsilon c\mathscr{A} + \varepsilon^2(\tfrac{1}{2}c^2\mathscr{A}^2 + c\mathscr{A}c\mathscr{A} - c^2\mathscr{A}^2) + \cdots$$
$$= 1 - \varepsilon c\mathscr{A} + \varepsilon^2(-\tfrac{1}{2}c^2\mathscr{A}^2 + c\mathscr{A}c\mathscr{A}) + \cdots .$$

If we apply this operator to

$$\mathrm{e}^{-2\beta\varepsilon c} = 1 - 2\beta\varepsilon c + 2\beta^2\varepsilon^2c^2 - \tfrac{4}{3}\beta^3\varepsilon^3c^3 + \cdots,$$

and remember that $\mathscr{A}1 = 0$, we obtain (3.1) for $n = 3$.

We emphasize that in Theorem 1.1, c can have fluctuating sign. Martingales do have their uses! However, the sample-path method just indicated applies in cases where (1.4) fails.

Acknowledgements

We thank Peter Hall and Chris Rogers for their help and inspiration.

References

Daniels, H.E. (1982) Sequential tests constructed from images. *Ann. Statist.*, **10**, 394–400.

Daniels, H.E. (1996) Approximating the first crossing-time density for a curved boundary. *Bernoulli*, **2**, 133–143.

Durbin, J. (1985) The first passage density of a continuous Gaussian process to a general boundary. *J. Appl. Probab.*, **22**, 99–122.

Durbin, J. (1988) A reconciliation of two different expressions for the first passage density of Brownian motion to a curved boundary. *J. Appl. Probab.*, **25**, 829–832.

Durbin, J. (1992) The first passage density of the Brownian motion process to a curved boundary. *J. Appl. Probab.*, **29**, 291–304.

Götze, F. (1985) Asymptotic expansions in functional limit theorems. *J. Multivariate Anal.*, **16**, 1–20.

Karatzas, I. and Shreve, S.E. (1988) *Brownian Motion and Stochastic Calculus*. Berlin: Springer-Verlag.

Lerche, H.R. (1986) *Boundary Crossing of Brownian Motion*, Lecture Notes in Statist, **40**. Berlin: Springer-Verlag.

Revuz, D. and Yor, M. (1991) *Continuous Martingales and Brownian Motion*. Berlin: Springer-Verlag.

Robbins, H. and Siegmund, D. (1970) Boundary crossing probabilities for the Wiener process and partial sums. *Ann. Math. Statist.*, **41**, 1410–1429.

Robbins, H. and Siegmund, D. (1973) Statistical tests of power one and the integral representation of solutions of certain partial differential equations. *Bull. Acad. Sinica*, **1**, 93–120.

Roberts, G.O. and Shortland, C.F. (1997) Pricing barrier options with time dependent coefficients. *J. Math. Finance*, **7**, 83–93.

Rogers, L.C.G and Williams, D. (1987) *Diffusions, Markov Processes and Martingales. Volume 2: Itô Calculus*. Chichester and New York: Wiley.

Received December 1996 and revised January 1998.