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On convolution equivalence with applications

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A distribution F on $(-\infty, \infty)$ is said to belong to the class $\mathcal{S}(\gamma)$ for some $\gamma \geqslant 0$ if $\lim_{x\to\infty} \overline{F}(x-u)/\overline{F}(x) = \mathrm{e}^{\gamma u}$ holds for all u and $\lim_{x\to\infty} \overline{F^{*2}}(x)/\overline{F}(x) = 2m_F$ exists and is finite. Let X and Y be two independent random variables, where X has a distribution in the class $\mathcal{S}(\gamma)$ and Y is non-negative with an endpoint $\hat{y} = \sup\{y : \mathrm{P}(Y \leqslant y) < 1\} \in (0, \infty)$. We prove that the product XY has a distribution in the class $\mathcal{S}(\gamma/\hat{y})$. We further apply this result to investigate the tail probabilities of Poisson shot noise processes and certain stochastic equations with random coefficients.

Keywords: asymptotics; class $\mathcal{S}(\gamma)$; endpoint; Poisson shot noise; rapid variation; stochastic equation; uniformity

1. Introduction and the main result

Throughout this paper, all limit relationships are for $x \to \infty$ unless stated otherwise; for two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \lesssim (x)$ if $\limsup a(x)/b(x) \leqslant 1$, $a(x) \gtrsim (x)$ if $\liminf a(x)/b(x) \geqslant 1$, and $a(x) \sim b(x)$ if both limits apply.

A distribution F on $[0, \infty)$ is said to belong to the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ if its right tail satisfies $\overline{F}(x) = 1 - F(x) > 0$ for all $x \geqslant 0$ and the relation

$$\lim_{x\to\infty} \frac{\overline{F}(x-u)}{\overline{F}(x)} = \mathrm{e}^{\gamma u} \tag{1.1}$$

holds for all u; F is said to belong to the class $\mathcal{S}(\gamma)$ if $F \in \mathcal{L}(\gamma)$ and the limit

$$\lim_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2m_F \tag{1.2}$$

exists and is finite, where F^{*2} denotes the two-fold convolution of F. Note that the convergence in (1.1) is automatically uniform on u in every finite interval. A typical example of a distribution in the class $\mathcal{S}(\gamma)$ with $\gamma > 0$ is the generalized inverse Gaussian distribution; see Embrechts (1983). More generally, a distribution F on $(-\infty, \infty)$ is said to belong to the class $\mathcal{L}(\gamma)$ or $\mathcal{S}(\gamma)$ if its right-hand distribution $F^+(x) = F(x)1_{(x\geqslant 0)}$ belongs to this class, where 1_A denotes the indicator function of a set A. As we go along we shall often suppress the phrase 'for some $\gamma \geqslant 0$', but it remains in place.

These classes were introduced by Chistyakov (1964) and Chover *et al.* (1973a, 1973b); they have been extensively investigated by many researchers and have been applied to many fields of probability theory. Recently, Pakes (2004) extended the classes to cover distributions on $(-\infty, \infty)$ in another way equivalent to the above. A more recent account is Shimura and Watanabe (2005). It is well known that the constant m_F in (1.2) is equal to $m_F = \int_{0-}^{\infty} \exp\{\gamma x\} F(\mathrm{d}x) < \infty$; see Chover *et al.* (1973a), Cline (1987), Rogozin (2000), and references therein. In the general case, (1.2) still holds with $m_F = \int_{-\infty}^{\infty} \exp\{\gamma x\} F(\mathrm{d}x) < \infty$; see Corollary 2.1(ii) of Pakes (2004) or Lemma 2.4 below.

Clearly, for $F \in \mathcal{L}(\gamma)$ for some $\gamma > 0$, its tail $\overline{F}$ is rapidly varying in the sense that

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 0, \qquad \text{for all } y > 1. \tag{1.3}$$

For simplicity, we designate by $F \in \mathcal{R}_{-\infty}$ the fact in (1.3). The rapid variation has been investigated in the literature; we refer the reader to Tang and Tsitsiashvili (2004) and references therein.

If a random variable X has a distribution in the class $\mathcal{S}(\gamma)$, then for every constant $y > 0$ it is easy to check by definition that the random variable yX has a distribution in the class $\mathcal{S}(\gamma/y)$. Motivated by this simple observation, in this paper we prove the following:

Theorem 1.1. *For two independent random variables X and Y, if X has a distribution $F \in \mathcal{S}(\gamma)$ and Y is non-negative with an (upper) endpoint*

$$0 < \hat{y} = \sup\{y : \mathrm{P}(Y \leqslant y) < 1\} < \infty, \tag{1.4}$$

then the product XY has a distribution in the class $\mathcal{S}(\gamma/\hat{y})$.

For $\gamma = 0$, the result of Theorem 1.1 has already been given in Corollary 2.5 of Cline and Samorodnitsky (1994), who extensively discussed the problem for that case. For $\gamma > 0$ and $\mathrm{P}(Y = \hat{y}) = \hat{p} > 0$, the result is also immediate. Actually, since in this case $F \in \mathcal{R}_{-\infty}$, it is easy to check by (1.3) that

$$\mathrm{P}(XY > x) \sim \hat{p}\mathrm{P}(X > x/\hat{y}); \tag{1.5}$$

see also Lemma A.3 of Tang and Tsitsiashvili (2004). Hence by Lemma 2.4(i) of Pakes (2004), the distribution of the product XY belongs to the class $\mathcal{S}(\gamma/\hat{y})$, as does that of $\hat{y}X$. However, Theorem 1.1 in its current form, though rather intuitive, is not trivial.

The proof of Theorem 1.1 is left to Section 3.

2. Applications

The study of asymptotic tail probabilities of quantities containing products is usually reduced to the study of the subtle tail behaviour of the product of two independent random variables. From this point of view, to investigate the tail behaviour of the products of independent random variables is one of the basic tasks in many applied as well as theoretical fields of probability theory. In their introduction, Cline and Samorodnitsky

(1994) proposed potential applications of the study to infinite-variance regressions, infinite-variance time series, and infinitely divisible stochastic processes. In this section, we give two other applications of Theorem 1.1.

2.1. Tail behaviour of Poisson shot noise

In this subsection we consider the tail probability of a Poisson shot noise process $\{S(t),\ t \geqslant 0\}$ defined by

$$S(t) = \sum_{k=1}^{N(t)} X_k h(t - \tau_k), \qquad t \geqslant 0, \tag{2.1}$$

where a summation over an empty set of index values is considered to be 0. In (2.1), $N(\cdot)$ is a Poisson process with arrival times τ_k, $k = 1, 2, \ldots,$ and intensity $\lambda > 0$; $\{X, X_k, k = 1, 2, \ldots\}$ is a sequence of independent and identically distributed (i.i.d.) shot marks with common distribution F on $(-\infty, \infty)$ and independent of $N(\cdot)$; and the shot function $h(\cdot)$ is assumed to be measurable, non-negative, bounded on every finite interval, non-zero on a set of positive Lebesgue measure, and null on $(-\infty, 0)$.

Many papers in the literature have been devoted to shot noise processes and their applications. Among these we refer to Klüppelberg and Mikosch (1995), McCormick (1997), Samorodnitsky (1998), Brémaud (2000), Klüppelberg *et al.* (2003), Lund *et al.* (2004) for recent advances.

The result below gives an explicit asymptotic formula for the tail probability of the shot noise process (2.1). Denote the uniform distribution on (0, 1) by $U(0, 1)$.

Theorem 2.1. *Consider the shot noise process* (2.1). *If $F \in \mathcal{S}(\gamma)$, then for every $t > 0$ such that the set $\{0 < u < t : h(u) > 0\}$ has a positive Lebesgue measure,*

$$\mathrm{P}(S(t) > x) \sim \lambda t \exp\left\{\lambda t\left(\mathrm{E}\left[\mathrm{e}^{(\gamma/\hat{h}(t))Xh(tU)}\right] - 1\right)\right\}\mathrm{P}(Xh(tU) > x), \tag{2.2}$$

where $\hat{h}(t)$ denotes the essential supremum (with respect to the Lebesgue measure) of the shot function $h(\cdot)$ over $[0, t]$ and U denotes a random variable distributed as $U(0, 1)$ and independent of X.

For the proof of Theorem 2.1 we shall need some preliminaries. The following lemma is well known; see, for example, Theorem 2.3.1 of Ross (1983).

Lemma 2.1. *Let $N(\cdot)$ be a Poisson process with arrival times τ_k, $k = 1, 2, \ldots.$ Given $N(t) = n$ for arbitrary $t > 0$ and $n = 1, 2, \ldots,$ the random vector $(\tau_1, \ldots, \tau_n)$ is equal in distribution to the random vector $(tU_{(1,n)}, \ldots, tU_{(n,n)})$, with $U_{(1,n)} \ldots, U_{(n,n)}$ being the order statistics of n independent and $U(0, 1)$ distributed random variables $U_1, \ldots, U_n$.*

In the next lemma, when F is on $[0, \infty)$ the results are well known and can be found, for example, in Chover *et al.* (1973: 665), while when F is on $(-\infty, \infty)$ they can be found in Lemmas 5.2 and 5.3 of Pakes (2004).

Lemma 2.2. *If $F \in \mathcal{S}(\gamma)$, then the relation*

$$\overline{F^{*n}}(x) \sim n m_F^{n-1} \overline{F}(x) \tag{2.3}$$

holds for each fixed $n = 1, 2, \ldots,$ and, moreover, for every $\varepsilon > 0$ there exists some constant $C_\varepsilon > 0$ such that the inequality

$$\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \leqslant C_\varepsilon [1 \vee (m_F + \varepsilon)^n] \tag{2.4}$$

holds for all $n = 1, 2, \ldots$ and all x.

Proof of Theorem 2.1. We write

$$\mathrm{P}(S(t) > x) = \sum_{n=1}^{\infty} \mathrm{P}\left(\sum_{k=1}^{n} X_k h(t - \tau_k) > x \middle| N(t) = n \right) \mathrm{P}(N(t) = n).$$

By Lemma 2.1, we obtain that

$$\mathrm{P}(S(t) > x) = \sum_{n=1}^{\infty} \mathrm{P}\left(\sum_{k=1}^{n} X_k h(t - tU_{(k,n)}) > x \right) \mathrm{P}(N(t) = n),$$

where $U_{(k,n)}$, $k = 1, \ldots, n$, denote the order statistics of n independent and $U(0, 1)$ distributed random variables U_k, $k = 1, \ldots, n$, which are independent of $\{X_k, k = 1, 2, \ldots\}$. Therefore,

$$\begin{aligned}
\mathrm{P}(S(t) > x) &= \sum_{n=1}^{\infty} \mathrm{P}\left(\sum_{k=1}^{n} X_k h(t - tU_k) > x \right) \mathrm{P}(N(t) = n) \\
&= \sum_{n=1}^{\infty} \mathrm{P}\left(\sum_{k=1}^{n} X_k h(tU_k) > x \right) \mathrm{P}(N(t) = n).
\end{aligned}$$

Clearly, the products $X_k h(tU_k)$, $k = 1, 2, \ldots,$ are i.i.d., and, by Theorem 1.1, their common distribution belongs to the class $\mathcal{S}(\gamma/\hat{h}(t))$. Notice $\mathrm{E}\left[K^{N(t)}\right] < \infty$ for all $K > 0$ and recall Lemma 2.2. An application of the dominated convergence theorem gives that

$$\mathrm{P}(S(t) > x) \sim \mathrm{P}(Xh(tU) > x) \sum_{n=1}^{\infty} n \left(\mathrm{E}\left[\mathrm{e}^{(\gamma/\hat{h}(t))Xh(tU)} \right] \right)^{n-1} \mathrm{P}(N(t) = n),$$

which, upon a trivial computation, implies relation (2.2). This completes the proof. $\square$

2.2. Tail behaviour of stochastic equations with random coefficients

Let $\{X, X_n, n = 1, 2, \ldots\}$ and $\{Y, Y_n, n = 1, 2, \ldots\}$ be two sequences of i.i.d. random variables with common distributions F on $(-\infty, \infty)$ and G on $[0, \infty)$, respectively, and let these two sequences be mutually independent. In this subsection we consider the tail probabilities of the stochastic equation

$$S_0 = 0,$$

$$S_n = Y_n(S_{n-1} + X_n), \qquad n = 1, 2, \ldots, \tag{2.5}$$

and its maxima

$$U_n = \max_{0 \leqslant k \leqslant n} S_k, \qquad n = 1, 2, \ldots .$$

Theorem 2.2. *Consider the stochastic equation* (2.5), *where* $F \in \mathcal{S}(\gamma)$ *and* G *has an endpoint* $0 < \hat{y} < \infty$. *The following assertions hold for each* $n = 1, 2, \ldots$*:*

(i) *The distributions of* S_n *and* U_n *belong to* $\mathcal{S}(\gamma \hat{y}^{-n})$.

(ii) *If* $0 < \hat{y} \leqslant 1$ *then*

$$P(S_n > x) \sim \sum_{i=1}^{n} (E[\exp\{\gamma X\}])^{i-1} E[\exp\{\gamma S_{n-i}\}] P\left(X \prod_{j=1}^{i} Y_j > x \right) \tag{2.6}$$

and

$$P(U_n > x) \sim \sum_{i=1}^{n} (E[\exp\{\gamma X\}])^{i-1} E[\exp\{\gamma U_{n-i}\}] P\left(X \prod_{j=1}^{i} Y_j > x \right). \tag{2.7}$$

(iii) *If* $1 < \hat{y} < \infty$ *then*

$$P(U_n > x) \sim P(S_n > x) \sim \sum_{i=1}^{n} \left(\prod_{j=n-i+1}^{n-1} E\left[\exp\{\gamma \hat{y}^{-j} X\}\right] \right) P\left(X \prod_{j=1}^{i} Y_j > x \right), \tag{2.8}$$

where a product over an empty set of index values is considered to be 1.

The stochastic equation (2.5) is one of the basic models in mathematical finance; see, for example, Section 8.4 of Embrechts *et al.* (1997). The reader is further referred to Goldie (1991) and Embrechts and Goldie (1994) for the study of the tail behaviour of perpetuity sequences. We give here an actuarial explanation. Introduce

$$S'_0 = 0,$$

$$S'_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j, \qquad n = 1, 2, \ldots .$$

Clearly, $(S_1, \ldots, S_n) \overset{d}{=} (S'_1, \ldots, S'_n)$, where $\overset{d}{=}$ denotes 'equal in distribution'. According to Tang and Tsitsiashvili (2004), X_i represents an insurer's net loss within period *i*, Y_j

represents the discount factor from time j to time $j-1$, and, therefore, S'_n represents the insurer's aggregated discounted losses by time n if each loss is calculated at the end of the period. Hence, $\mathrm{P}(\max_{0\leqslant k\leqslant n} S'_k > x) = \mathrm{P}(U_n > x)$ can further be thought of as the probability of ruin within the finite horizon n of the insurer whose current wealth is $x \geqslant 0$. In this stochastic economic environment, the distribution G has a finite endpoint $\hat{y}$ if the insurer always invests a certain proportion of his wealth in a risk-free asset (a bank), and invests the remainder of his wealth in a risky asset (a stock market). For a related discussion, see Example 4.1 of Tang and Tsitsiashvili (2004). The last item of Theorem 2.2 improves Theorem 4.3 of Tang and Tsitsiashvili (2004) by removing the condition $\mathrm{P}(Y = \hat{y}) > 0$.

Discussion of the stochastic equation

$$\begin{aligned} &T_0 = 0, \\ &T_n = Y_n T_{n-1} + X_n, \qquad n = 1, 2, \ldots, \end{aligned} \tag{2.9}$$

completely parallels the above, so we omit the details here.

To prove Theorem 2.2 we need two lemmas. The first is from Rogozin and Sgibnev (1999).

Lemma 2.3. *Let F, F_1, and F_2 be three distributions such that $F \in \mathcal{S}(\gamma)$ and that the limit $k_i = \lim_{x\to\infty} \overline{F}_i(x)/\overline{F}(x)$ exists and is finite for $i = 1, 2$. Then*

$$\lim_{x\to\infty} \frac{\overline{F_1 * F_2}(x)}{\overline{F}(x)} = k_1 \int_{-\infty}^{\infty} \exp\{\gamma x\} F_2(\mathrm{d}x) + k_2 \int_{-\infty}^{\infty} \exp\{\gamma x\} F_1(\mathrm{d}x).$$

The next lemma is from Tang and Tsitsiashvili (2003, Lemma 3.2).

Lemma 2.4. *Let F_1 and F_2 be two distributions on $(-\infty, \infty)$. If $F_1 \in \mathcal{S}(\gamma)$, $F_2 \in \mathcal{L}(\gamma)$, and $\overline{F_2}(x) = O\big(\overline{F_1}(x)\big)$, then $F \in \mathcal{S}(\gamma)$ and*

$$\overline{F_1 * F_2}(x) \sim \int_{-\infty}^{\infty} \exp\{\gamma x\} F_2(\mathrm{d}x) \overline{F_1}(x) + \int_{-\infty}^{\infty} \exp\{\gamma x\} F_1(\mathrm{d}x) \overline{F_2}(x). \tag{2.10}$$

Proof of Theorem 2.2. We only prove that if $1 < \hat{y} < \infty$ then, for each $n = 1, 2, \ldots,$ the distributions of S_n and U_n belong to $\mathcal{S}(\gamma \hat{y}^{-n})$ and relations (2.8) hold, and we point out that the other assertions can be proven similarly. Actually, (2.7) has been proven in Theorem 4.1 of Tang and Tsitsiashvili (2004).

First we show that, for each $n = 1, 2, \ldots,$ the distribution of S_n belongs to $\mathcal{S}(\gamma \hat{y}^{-n})$ and the relation

$$\mathrm{P}(S_n > x) \sim \sum_{i=1}^{n} \left(\prod_{j=n-i+1}^{n-1} \mathrm{E}\big[\exp\{\gamma \hat{y}^{-j} X\}\big] \right) \mathrm{P}\left(X \prod_{j=1}^{i} Y_j > x \right) \tag{2.11}$$

holds. Trivially, relation (2.11) holds for $n = 1$ since $\mathrm{P}(S_1 > x) = \mathrm{P}(Y_1 X_1 > x)$. By Theorem 1.1, we also know that the distribution of S_1 belongs to $\mathcal{S}(\gamma \hat{y}^{-1})$.

Now we inductively assume that (2.11) holds for $n = m-1$ for some integer $m \geqslant 2$ and that the distribution of S_{m-1} belongs to $\mathcal{S}(\gamma\hat{y}^{1-m})$.

If $\gamma > 0$ then $\overline{F}(x) = o(\mathrm{P}(S_{m-1} > x))$. Using Lemma 2.3, we obtain that

$$\begin{aligned}
\mathrm{P}(X_m + S_{m-1} > x) &\sim \mathrm{E}\big[\exp\{\gamma\hat{y}^{1-m}X\}\big]\mathrm{P}(S_{m-1} > x) \\
&\sim \mathrm{E}\big[\exp\{\gamma\hat{y}^{1-m}X\}\big]\sum_{i=1}^{m-1}\left(\prod_{j=m-i}^{m-2}\mathrm{E}\big[\exp\{\gamma\hat{y}^{-j}X\}\big]\right)\mathrm{P}\left(X\prod_{j=1}^{i}Y_j > x\right) \\
&= \sum_{i=1}^{m-1}\left(\prod_{j=m-i}^{m-1}\mathrm{E}\big[\exp\{\gamma\hat{y}^{-j}X\}\big]\right)\mathrm{P}\left(X\prod_{j=1}^{i}Y_j > x\right).
\end{aligned}$$

Moreover, using the first step above and Lemma 2.4(i) of Pakes (2004), we see that the distribution of $X_m + S_{m-1}$ belongs to $\mathcal{S}(\gamma\hat{y}^{1-m})$. Hence by Theorem 1.1, the distribution of S_m belongs to $\mathcal{S}(\gamma\hat{y}^{-m})$, and it follows that

$$\begin{aligned}
\mathrm{P}(S_m > x) &= \int_0^{\hat{y}}\mathrm{P}\left(X_m + S_{m-1} > \frac{x}{y}\right)G(\mathrm{d}y) \\
&\sim \sum_{i=1}^{m-1}\left(\prod_{j=m-i}^{m-1}\mathrm{E}\big[\exp\{\gamma\hat{y}^{-j}X\}\big]\right)\mathrm{P}\left(X\prod_{j=1}^{i+1}Y_j > x\right) \\
&= \sum_{i=2}^{m}\left(\prod_{j=m-i+1}^{m-1}\mathrm{E}\big[\exp\{\gamma\hat{y}^{-j}X\}\big]\right)\mathrm{P}\left(X\prod_{j=1}^{i}Y_j > x\right) \\
&\sim \sum_{i=1}^{m}\left(\prod_{j=m-i+1}^{m-1}\mathrm{E}\big[\exp\{\gamma\hat{y}^{-j}X\}\big]\right)\mathrm{P}\left(X\prod_{j=1}^{i}Y_j > x\right),
\end{aligned}$$

showing that (2.11) holds for $n = m$.

If $\gamma = 0$, then using Lemma 2.4 and the fact $\overline{F}(x) = O(\mathrm{P}(S_{m-1} > x))$, we see that the distribution of $X_m + S_{m-1}$ belongs to $\mathcal{S}(0)$ and

$$\mathrm{P}(X_m + S_{m-1} > x) \sim \mathrm{P}(X_m > x) + \mathrm{P}(S_{m-1} > x) \sim \sum_{i=0}^{m-1}\mathrm{P}\left(X\prod_{j=1}^{i}Y_j > x\right).$$

Hence by Theorem 1.1, the distribution of S_m belongs to $\mathcal{S}(0)$, and it follows that

$$\mathrm{P}(S_m > x) = \int_0^{\hat{y}}\mathrm{P}\left(X_m + S_{m-1} > \frac{x}{y}\right)G(\mathrm{d}y) \sim \sum_{i=1}^{m}\mathrm{P}\left(X\prod_{j=1}^{i}Y_j > x\right),$$

showing once again that (2.11) holds for $n = m$.

The mathematical induction method concludes that, for each $n = 1, 2, \ldots,$ the distribution of S_n belongs to $\mathcal{S}(\gamma\hat{y}^{-n})$ and (2.11) holds.

For the rest of the proof, using the identity $\max_{0\leqslant k\leqslant n} S'_k \overset{d}{=} U_n$ for $n = 1, 2, \ldots,$ as explained above, and Theorem 2.1 of Tang and Tsitsiashvili (2003), we find that

$$U_n \overset{d}{=} V_n, \qquad n = 1, 2, \ldots,$$

where V_n, $n = 1, 2, \ldots,$ constitute a Markov chain defined by

$$\begin{aligned} &V_0 = 0, \\ &V_n = Y_n \max\{0, X_n + V_{n-1}\}, \qquad n = 1, 2, \ldots . \end{aligned} \tag{2.12}$$

Starting from (2.12) and proceeding along the same lines as above, we obtain that, for each $n = 1, 2, \ldots,$ the distribution of U_n belongs to $\mathcal{S}(\gamma \hat{y}^{-n})$ and the relation

$$\mathrm{P}(U_n > x) \sim \sum_{i=1}^{n} \left(\prod_{j=n-i+1}^{n-1} \mathrm{E}\left[\exp\{\gamma \hat{y}^{-j} X\}\right] \right) \mathrm{P}\left(X \prod_{j=1}^{i} Y_j > x \right)$$

holds. This completes the proof. □

3. Proof of Theorem 1.1

To prove Theorem 1.1, we need several lemmas.

Lemma 3.1. *Let X and Y be two independent random variables distributed as F and G, respectively, where $F \in \mathcal{R}_{-\infty}$, G has an endpoint $0 < \hat{y} \leqslant \infty$, and $F(0-)G(0-) = 0$. We have the following assertions:*

(i) *For every $y_0 \in (0, \hat{y})$,*

$$\mathrm{P}(XY > x) \sim \int_{y_0}^{\hat{y}} \overline{F}(x/y) G(\mathrm{d}y).$$

(ii) *For every $y_0 \in (0, \hat{y})$,*

$$\mathrm{P}(XY > x) \sim \int_{x/\hat{y}}^{x/y_0} \overline{G}(x/u) F(\mathrm{d}u),$$

where $x/\hat{y} = 0$ when $\hat{y} = \infty$.

Proof. Assertion (i) is known from Lemma A.3 of Tang and Tsitsiashvili (2004). For assertion (ii), in view of the relation

$$\mathrm{P}(XY > x) = \int_{x/\hat{y}}^{x/y_0} \overline{G}(x/u) F(\mathrm{d}u) + \int_{x/y_0}^{\infty} \overline{G}(x/u) F(\mathrm{d}u),$$

it suffices to prove that

$$\lim_{x\to\infty}\frac{\int_{x/y_0}^{\infty}\overline{G}(x/u)F(\mathrm{d}u)}{\int_{x/\hat{y}}^{x/y_0}\overline{G}(x/u)F(\mathrm{d}u)}=0.$$

Actually, for some $y_* \in (y_0, \hat{y})$ we have

$$\frac{\int_{x/y_0}^{\infty}\overline{G}(x/u)F(\mathrm{d}u)}{\int_{x/\hat{y}}^{x/y_0}\overline{G}(x/u)F(\mathrm{d}u)}\leqslant\frac{\overline{F}(x/y_0)}{\int_{x/y_*}^{x/y_0}\overline{G}(x/u)F(\mathrm{d}u)}\leqslant\frac{\overline{F}(x/y_0)}{\overline{G}(y_*)\big(\overline{F}(x/y_*)-\overline{F}(x/y_0)\big)}\to 0,$$

where the last step is due to the facts $F\in\mathcal{R}_{-\infty}$ and $\overline{G}(y_*)>0$ (see (1.4)). This completes the proof. □

Note that in the following lemma and its proof a key word is 'uniformity', which is the spirit of the result. Let us take an example to clarify its meaning. For two positive bivariate functions $a(\cdot;\,\cdot)$ and $b(\cdot;\,\cdot)$, we say that the asymptotic relation $a(x;\,s)\lesssim(x;\,s)$ holds uniformly over all s in a non-empty set Δ if

$$\limsup_{x\to\infty}\sup_{s\in\Delta}\frac{a(x;\,s)}{b(x;\,s)}\leqslant 1.$$

Note also that the presence of an arbitrarily fixed constant A in the lemma is one of the tricks used in the proof of Theorem 1.1.

Lemma 3.2. *Let X_1 and X_2 be two i.i.d. random variables with common distribution $F\in\mathcal{S}(\gamma)$. Then for arbitrarily fixed $0<\delta\leqslant 1$ and $A>0$, the relation*

$$\mathrm{P}(X_1+sX_2>x)\lesssim\exp\{\gamma A/\delta\}\overline{F}(x/s)+\left(2\int_A^{\infty}\mathrm{e}^{\gamma u}F(\mathrm{d}u)+\mathrm{E}\big[\mathrm{e}^{s\gamma X_1}\big]\right)\overline{F}(x) \tag{3.1}$$

holds uniformly over all $s\in[\delta,\,1]$.

Proof. Let us first establish a preliminary. For an arbitrarily fixed number $A>0$ and all $x>2A$,

$$\begin{aligned}\int_A^{x-A}\overline{F}(x-u)F(\mathrm{d}u)&=\left(\int_{-\infty}^{\infty}-\int_{-\infty}^{A}-\int_{x-A}^{\infty}\right)\overline{F}(x-u)F(\mathrm{d}u)\\&=\overline{F^{*2}}(x)-2\int_{-\infty}^{A}\overline{F}(x-u)F(\mathrm{d}u)-\overline{F}(A)\overline{F}(x-A).\end{aligned}$$

Hence, by the definition of $F\in\mathcal{S}(\gamma)$ and the dominated convergence theorem, we obtain

$$\lim_{x\to\infty} \frac{\int_A^{x-A} \overline{F}(x-u)F(\mathrm{d}u)}{\overline{F}(x)} = \left(2\mathrm{E}\left[\mathrm{e}^{\gamma X_1}\right] - 2\int_{-\infty}^{A} \mathrm{e}^{\gamma u}F(\mathrm{d}u) - \overline{F}(A)\mathrm{e}^{\gamma A}\right). \tag{3.2}$$

We can now start proving relation (3.1). For every $s \in [\delta, 1]$ and $A > 0$, we have for all $x > 2A$ that

$$\mathrm{P}(X_1 + sX_2 > x) = \left(\int_{-\infty}^{A} + \int_A^{x-A} + \int_{x-A}^{\infty}\right)\overline{F}\left(\frac{x-u}{s}\right)F(\mathrm{d}u) = I_1 + I_2 + I_3. \tag{3.3}$$

For I_1, by the definition of $F \in \mathcal{L}(\gamma)$ we have, uniformly over all $s \in [\delta, 1]$,

$$\frac{I_1}{\overline{F}(x/s)} \leqslant \frac{\overline{F}(x/s - A/\delta)}{\overline{F}(x/s)} \to \exp\{\gamma A/\delta\}. \tag{3.4}$$

For I_2, by (3.2) we have, uniformly over all $s \in [\delta, 1]$,

$$\frac{I_2}{\overline{F}(x)} \leqslant \frac{1}{\overline{F}(x)}\int_A^{x-A} \overline{F}(x-u)F(\mathrm{d}u) \to 2\int_A^{\infty} \mathrm{e}^{\gamma u}F(\mathrm{d}u) - \overline{F}(A)\mathrm{e}^{\gamma A}. \tag{3.5}$$

Finally, we deal with I_3. We have

$$I_3 = \overline{F}(A/s)\overline{F}(x-A) + \int_{-\infty}^{A/s} \overline{F}(x-sv)F(\mathrm{d}v).$$

For every $\varepsilon > 0$, we choose some $M > 0$ such that $\int_{-\infty}^{-M} \mathrm{e}^{\delta\gamma v}F(\mathrm{d}v) \leqslant \mathrm{e}^{-\delta\gamma M}F(-M) \leqslant \varepsilon$. Using the local uniformity of the convergence of (1.1), we have, uniformly over all $s \in [\delta, 1]$,

$$\int_{-\infty}^{A/s} \frac{\overline{F}(x-sv)}{\overline{F}(x)}F(\mathrm{d}v) \geqslant \int_{-M}^{A/s} \frac{\overline{F}(x-sv)}{\overline{F}(x)}F(\mathrm{d}v) \to \int_{-M}^{A/s} \mathrm{e}^{s\gamma v}F(\mathrm{d}v) \geqslant \int_{-\infty}^{A/s} \mathrm{e}^{s\gamma v}F(\mathrm{d}v) - \varepsilon$$

and

$$\begin{aligned}\int_{-\infty}^{A/s} \frac{\overline{F}(x-sv)}{\overline{F}(x)}F(\mathrm{d}v) &\leqslant \int_{-M}^{A/s} \frac{\overline{F}(x-sv)}{\overline{F}(x)}F(\mathrm{d}v) + \frac{\overline{F}(x+sM)}{\overline{F}(x)}F(-M) \\ &\to \int_{-M}^{A/s} \mathrm{e}^{s\gamma v}F(\mathrm{d}v) + \mathrm{e}^{-\delta sM}F(-M) \leqslant \int_{-\infty}^{A/s} \mathrm{e}^{s\gamma v}F(\mathrm{d}v) + \varepsilon.\end{aligned}$$

We conclude that, uniformly over all $s \in [\delta, 1]$,

$$\frac{I_3}{\overline{F}(x)} \to \overline{F}(A/s)\mathrm{e}^{\gamma A} + \int_{-\infty}^{A/s} \mathrm{e}^{s\gamma v}F(\mathrm{d}v). \tag{3.6}$$

Substituting (3.4), (3.5), and (3.6) into (3.3), we obtain that (3.1) holds uniformly over all $s \in [\delta, 1]$. This completes the proof. □

Lemma 3.3. *Let Y_1 and Y_2 be two i.i.d. non-negative random variables with common distribution G and endpoint $0 < \hat{y} \leqslant \infty$, and let X be a random variable distributed as $F \in \mathcal{R}_{-\infty}$ and independent of Y_1 and Y_2. Then*

$$\lim_{x\to\infty} \frac{\mathrm{P}(X(Y_1 \wedge Y_2) > x)}{\mathrm{P}(X(Y_1 \vee Y_2) > x)} = \frac{\mathrm{P}(Y_1 = \hat{y})}{2 - \mathrm{P}(Y_1 = \hat{y})}, \tag{3.7}$$

where $\mathrm{P}(Y_1 = \hat{y}) = 0$ *when* $\hat{y} = \infty$.

Proof. By Lemma 3.1(ii), for every $y_0 \in (0, \hat{y})$, we have that

$$\mathrm{P}(X(Y_1 \wedge Y_2) > x) \sim \int_{x/\hat{y}}^{x/y_0} \overline{G}^2(x/u) F(\mathrm{d}u)$$

and that

$$\mathrm{P}(X(Y_1 \vee Y_2) > x) \sim \int_{x/\hat{y}}^{x/y_0} \big(2\overline{G}(x/u) - \overline{G}^2(x/u)\big) F(\mathrm{d}u).$$

Hence,

$$\frac{\mathrm{P}(X(Y_1 \wedge Y_2) > x)}{\mathrm{P}(X(Y_1 \vee Y_2) > x)} \sim \frac{\displaystyle\int_{x/\hat{y}}^{x/y_0} (\overline{G}(x/u)/(2 - \overline{G}(x/u)))\big(2\overline{G}(x/u) - \overline{G}^2(x/u)\big) F(\mathrm{d}u)}{\displaystyle\int_{x/\hat{y}}^{x/y_0} \big(2\overline{G}(x/u) - \overline{G}^2(x/u)\big) F(\mathrm{d}u)}.$$

It follows that

$$\inf_{y_0 \leqslant y < \hat{y}} \frac{\overline{G}(y)}{2 - \overline{G}(y)} \leqslant \frac{\mathrm{P}(X(Y_1 \wedge Y_2) > x)}{\mathrm{P}(X(Y_1 \vee Y_2) > x)} \leqslant \sup_{y_0 \leqslant y < \hat{y}} \frac{\overline{G}(y)}{2 - \overline{G}(y)}.$$

Letting $y_0 \nearrow \hat{y}$ yields (3.7). □

Proof of Theorem 1.1. Denote by H the distribution of the product XY. To prove $H \in \mathcal{S}(\gamma/\hat{y})$, we notice the following facts:

- A normalization gives that $XY = (\hat{y}X)(Y/\hat{y})$, where $\hat{y}X$ has a distribution in the class $\mathcal{S}(\gamma/\hat{y})$ and $Y/\hat{y}$ has an endpoint 1.
- When $\gamma = 0$ the result is known from Corollary 2.5 of Cline and Samorodnitsky (1994).
- If $\mathrm{P}(Y = \hat{y}) = \hat{p} > 0$, then relation (1.5) holds and hence $H \in \mathcal{S}(\gamma/\hat{y})$.
- If we have proven the result for the case where X is non-negative, then applying Corollary 2.1(i) of Pakes (2004), the result holds for the general case where X is real-valued.

Therefore, in what follows we can assume (i) $\hat{y} = 1$, (ii) $\gamma > 0$ (hence $F \in \mathcal{R}_{-\infty}$), (iii) $\mathrm{P}(Y = 1) = 0$, and (iv) $\mathrm{P}(X \geqslant 0) = 1$. Let us show that $H \in \mathcal{S}(\gamma)$.

By Lemma A.4 of Tang and Tsitsiashvili (2004) we know that $H \in \mathcal{L}(\gamma)$. An application of Fatou's lemma leads to

$$\overline{H^{*2}}(x) \geqslant 2\int_{0-}^{x/2} \overline{H}(x-u) H(\mathrm{d}u) 2\mathrm{E}\big[\mathrm{e}^{\gamma XY}\big]\, \overline{H}(x).$$

Hence, it suffices to prove that

$$\overline{H^{*2}}(x) \leqslant 2\mathrm{E}\left[\mathrm{e}^{\gamma XY}\right]\overline{H}(x). \tag{3.8}$$

Let (X_i, Y_i), $i = 1, 2$, be i.i.d. copies of (X, Y). We formulate the proof of relation (3.8) in three steps.

1. First we assume $\mathrm{P}(\delta < Y < 1) = 1$ for some $0 < \delta < 1$. We follow the proof of Theorem 2.1 of Cline and Samorodnitsky (1994) to divide the probability $\overline{H^{*2}}(x) = \mathrm{P}(X_1Y_1 + X_2Y_2 > x)$ into two parts,

$$J_1 + J_2 = \mathrm{P}(X_1Y_1 + X_2Y_2 > x,\ Y_2 \leqslant Y_1) + \mathrm{P}(X_1Y_1 + X_2Y_2 > x,\ Y_1 < Y_2).$$

Conditioning on (Y_1, Y_2) and applying Lemma 3.2, for arbitrarily fixed $A > 0$,

$$\begin{aligned} J_1 &= \iint_{(\delta<y_2\leqslant y_1<1)} \mathrm{P}\left(X_1 + \frac{y_2}{y_1}X_2 > \frac{x}{y_1}\right) G(\mathrm{d}y_1)G(\mathrm{d}y_2) \\ &\leqslant \exp\{\gamma A/\delta\} \iint_{(\delta<y_2\leqslant y_1<1)} \overline{F}(x/y_2)G(\mathrm{d}y_1)G(\mathrm{d}y_2) \\ &\quad + \iint_{(\delta<y_2\leqslant y_1<1)} \left(2\int_A^\infty \mathrm{e}^{\gamma u}F(\mathrm{d}u) + \mathrm{E}\left[\exp\left\{\frac{y_2}{y_1}\gamma X\right\}\right]\right)\overline{F}(x/y_1)G(\mathrm{d}y_1)G(\mathrm{d}y_2). \end{aligned}$$

A similar relation can be derived for J_2. Therefore,

$$\begin{aligned} &\mathrm{P}(X_1Y_1 + X_2Y_2 > x) \\ &\quad \leqslant \exp\{\gamma A/\delta\}\mathrm{P}(X(Y_1 \wedge Y_2) > x) \\ &\quad\quad + \iint_{(\delta<y_2\leqslant y_1<1)} \left(2\int_A^\infty \mathrm{e}^{\gamma u}F(\mathrm{d}u) + \mathrm{E}\left[\exp\left\{\frac{y_2}{y_1}\gamma X\right\}\right]\right)\overline{F}(x/y_1)G(\mathrm{d}y_1)G(\mathrm{d}y_2) \\ &\quad\quad + \iint_{(\delta<y_1<y_2<1)} \left(2\int_A^\infty \mathrm{e}^{\gamma u}F(\mathrm{d}u) + \mathrm{E}\left[\exp\left\{\frac{y_1}{y_2}\gamma X\right\}\right]\right)\overline{F}(x/y_2)G(\mathrm{d}y_1)G(\mathrm{d}y_2) \\ &\quad = K_1 + K_2 + K_3. \end{aligned}$$

Clearly,

$$K_2 + K_3 \geqslant \mathrm{E}[\exp\{\delta\gamma X\}]\mathrm{P}(X(Y_1 \vee Y_2) > x).$$

Since $F \in \mathcal{R}_{-\infty}$, $\hat{y} = 1$, and $\mathrm{P}(Y = 1) = 0$, by Lemma 3.3 we have

$$\mathrm{P}(X(Y_1 \wedge Y_2) > x) = o(\mathrm{P}(X(Y_1 \vee Y_2) > x)).$$

Hence,

$$\mathrm{P}(X_1Y_1 + X_2Y_2 > x) \lesssim K_2 + K_3. \tag{3.9}$$

Letting $A \to \infty$ in (3.9) yields that

$$
\begin{aligned}
&\mathrm{P}(X_1Y_1 + X_2Y_2 > x) \\
&\leqslant \iint_{(\delta < y_2 \leqslant y_1 < 1)} \mathrm{E}\left[\exp\left\{\frac{y_2}{y_1}\gamma X\right\}\right] \overline{F}(x/y_1)G(\mathrm{d}y_1)G(\mathrm{d}y_2) \\
&\quad + \iint_{(\delta < y_1 < y_2 < 1)} \mathrm{E}\left[\exp\left\{\frac{y_1}{y_2}\gamma X\right\}\right] \overline{F}(x/y_2)G(\mathrm{d}y_1)G(\mathrm{d}y_2) \\
&= \int_{\delta}^{1-} \mathrm{E}\left[\exp\{\gamma X(Y/y_1)\}1_{(Y/y_1 \leqslant 1)}\right] \overline{F}(x/y_1)G(\mathrm{d}y_1) \\
&\quad + \int_{\delta}^{1-} \mathrm{E}\left[\exp\{\gamma X(Y/y_2)\}1_{(Y/y_2 < 1)}\right] \overline{F}(x/y_2)G(\mathrm{d}y_2).
\end{aligned}
$$

Since $F \in \mathcal{R}_{-\infty}$, for every $y_0 \in (\delta, 1)$, we apply Lemma 3.1(i) (as well as the idea behind it) twice to obtain

$$
\begin{aligned}
\mathrm{P}(X_1Y_1 + X_2Y_2 > x) &\leqslant 2\int_{y_0}^{1-} \mathrm{E}\left[\exp\{\gamma X(Y/y)\}1_{(Y/y \leqslant 1)}\right] \overline{F}(x/y)G(\mathrm{d}y) \\
&\leqslant 2 \sup_{y_0 < y < 1} \mathrm{E}\left[\exp\{\gamma X(Y/y)\}1_{(Y/y \leqslant 1)}\right] \int_{y_0}^{1-} \overline{F}(x/y)G(\mathrm{d}y) \\
&\sim 2 \sup_{y_0 < y < 1} \mathrm{E}\left[\exp\{\gamma X(Y/y)\}1_{(Y/y \leqslant 1)}\right] \mathrm{P}(XY > x).
\end{aligned}
$$

Note that, by the dominated convergence theorem,

$$
\lim_{y_0 \nearrow 1} \sup_{y_0 < y < 1} \mathrm{E}\left[\exp\{\gamma X(Y/y)\}1_{(Y/y \leqslant 1)}\right] = \lim_{y \nearrow 1} \mathrm{E}\left[\exp\{\gamma X(Y/y)\}1_{(Y/y \leqslant 1)}\right] = \mathrm{E}\left[\mathrm{e}^{\gamma XY}\right].
$$

We obtain relation (3.8).

2. Next we assume $\mathrm{P}(0 < Y < 1) = 1$. We arbitrarily choose a constant $0 < \delta < 1$ such that $\mathrm{P}(0 < Y \leqslant \delta) < 0.5\mathrm{P}(\delta < Y < 1)$ and then derive

$$
\begin{aligned}
\mathrm{P}(X_1Y_1 + X_2Y_2 > x) &= \mathrm{P}(X_1Y_1 + X_2Y_2 > x,\ \delta < Y_i < 1 \text{ for } i = 1 \text{ and } 2) \\
&\quad + \mathrm{P}(X_1Y_1 + X_2Y_2 > x,\ 0 < Y_i \leqslant \delta \text{ for } i = 1 \text{ or } 2). \qquad (3.10)
\end{aligned}
$$

The second term above is not larger than

$$
\begin{aligned}
2\mathrm{P}(X_1\delta + X_2Y_2 > x)\mathrm{P}(0 < Y_1 \leqslant \delta) &= 2\mathrm{P}(X_1\delta + X_2Y_2 > x,\ \delta < Y_1 < 1)\frac{\mathrm{P}(0 < Y_1 \leqslant \delta)}{\mathrm{P}(\delta < Y_1 < 1)} \\
&\leqslant 2\mathrm{P}(X_1Y_1 + X_2Y_2 > x)\frac{\mathrm{P}(0 < Y \leqslant \delta)}{\mathrm{P}(\delta < Y < 1)}.
\end{aligned}
$$

Substituting this into (3.10) and rearranging the resulting inequality leads to

$$
\begin{aligned}
&\mathrm{P}(X_1Y_1 + X_2Y_2 > x) \\
&\quad \leqslant \left(1 - \frac{2\mathrm{P}(0 < Y \leqslant \delta)}{\mathrm{P}(\delta < Y < 1)}\right)^{-1} \mathrm{P}(X_1Y_1 + X_2Y_2 > x,\ \delta < Y_i < 1 \text{ for } i = 1 \text{ and } 2) \\
&\quad \lesssim \left(1 - \frac{2\mathrm{P}(0 < Y \leqslant \delta)}{\mathrm{P}(\delta < Y < 1)}\right)^{-1} 2\mathrm{E}\left[\mathrm{e}^{\gamma XY} 1_{(\delta < Y < 1)}\right] \mathrm{P}(XY > x,\ \delta < Y < 1) \\
&\quad \leqslant \left(1 - \frac{2\mathrm{P}(0 < Y \leqslant \delta)}{\mathrm{P}(\delta < Y < 1)}\right)^{-1} 2\mathrm{E}\left[\mathrm{e}^{\gamma XY} 1_{(\delta < Y < 1)}\right] \overline{H}(x),
\end{aligned}
$$

where in the last but one line we used the result proven in step 1. Since $\delta > 0$ can be arbitrarily small, we again obtain (3.8).

3. Finally we assume $\mathrm{P}(0 \leqslant Y < 1) = 1$. The extension from step 2 to this step is straightforward. Actually,

$$
\begin{aligned}
&\mathrm{P}(X_1Y_1 + X_2Y_2 > x) \\
&\quad = \mathrm{P}(X_1Y_1 + X_2Y_2 > x,\ 0 < Y_i < 1 \text{ for } i = 1 \text{ and } 2) + 2\mathrm{P}(XY > x)\mathrm{P}(Y = 0) \\
&\quad \lesssim 2\mathrm{E}\left[\mathrm{e}^{\gamma XY} 1_{(0 < Y < 1)}\right] \overline{H}(x) + 2\overline{H}(x)\mathrm{P}(Y = 0) \\
&\quad = 2\mathrm{E}\left[\mathrm{e}^{\gamma XY}\right] \overline{H}(x),
\end{aligned}
$$

where in the last but one line we used the result proven in step 2. This completes the proof. □

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