

Stochastic integral equations without probability

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A pathwise approach to stochastic integral equations is advocated. Linear extended Riemann–Stieltjes integral equations driven by certain stochastic processes are solved. Boundedness of the p -variation for some $0 < p < 2$ is the only condition on the driving stochastic process. Typical examples of such processes are infinite-variance stable Lévy motion, hyperbolic Lévy motion, normal inverse Gaussian processes, and fractional Brownian motion. The approach used in the paper is based on a chain rule for the composition of a smooth function and a function of bounded p -variation with $0 < p < 2$.

Keywords: chain rule; extended Riemann–Stieltjes integral; fractional Brownian motion; Lévy process; p -variation; stable process; stochastic integral equation

1. Introduction

Most parts of the current theory of stochastic differential or integral equations (we prefer here the latter notion because it is more appropriate) are based on the notion of stochastic integral with respect to a semimartingale. Given a local martingale M and a stochastic process V with sample paths of bounded variation, the stochastic integral with respect to the semimartingale $Y = M + V$ is the sum of the Itô integral with respect to M and the Lebesgue–Stieltjes integral with respect to V . The integrand in the stochastic integral must be a predictable stochastic process. In this paper we consider stochastic integral equations based on an extended Riemann–Stieltjes integral. It is defined for a large class of stochastic processes as integrands and integrators. Both, integrand and integrator, may have sample paths of unbounded variation. Moreover, there are no requirements on the type of filtration the processes are adopted to. In particular, extended Riemann–Stieltjes integral equations are perfectly suited for some classes of pure jump semimartingales, but also for certain non-semimartingales such as fractional Brownian motion with parameter $H \in (0.5, 1)$.

For the present approach to stochastic integral equations, the notion of p -variation plays a central role. The p -variation of the sample paths of a stochastic process is an indicator of its extended Riemann–Stieltjes integrability. The p -variation, $0 < p < \infty$, of a real-valued function f on $[a, b]$ is defined as

$$v_p(f) = v_p(f; [a, b]) = \sup_{\kappa} \sum_{i=1}^n |f(x_i) - f(x_{i-1})|^p, \quad (1.1)$$

where the supremum is taken over all subdivisions κ of $[a, b]$:

$$\kappa : a = x_0 < \dots < x_n = b, \quad n \geq 1. \tag{1.2}$$

If $v_p(f) < \infty$, f is said to have *bounded p -variation on $[a, b]$* . The case $p = 1$ corresponds to the usual definition of bounded variation of f . Recall the difference between 2-variation and the quadratic variation of a stochastic process. The latter is defined as the limit of the quantities $\sum_{i=1}^n |f(x_i) - f(x_{i-1})|^2$ along a given sequence of subdivisions, provided this limit exists almost surely or in probability.

Since L. C. Young’s (1936) paper on Stieltjes integration, it has been known that the Riemann–Stieltjes integral may exist even if both integrand and integrator have unbounded variation. Young (1936) proved that, if f has bounded p -variation and h has bounded q -variation with $p^{-1} + q^{-1} > 1$, then the integral $\int_a^b f \, dh$ exists: (1) in the Riemann–Stieltjes sense whenever f and h have no discontinuities at the same point; (2) in the Moore–Pollard–Stieltjes sense whenever f and h have no one-sided discontinuities at the same point; (3) always in the sense defined by Young. Integrability in the Riemann–Stieltjes sense means existence of the limit as the mesh of the subdivisions tends to zero, while existence of the limit under refinements of the subdivisions gives rise to the Moore–Pollard–Stieltjes integral. Dudley (1992) clarified the definition of Young’s integral first by complementing it at the end-points a, b , and then by giving its alternative variant. Dudley and Norvaiša (1999a) proved a number of properties of Young’s integral and further suggested two modifications, the left and right Young integrals. These two integrals are used in the present paper. Their definition can be found in Section 2.3. Both integrals are well suited for solving certain linear integral equations driven by possibly discontinuous functions.

We consider two forward linear equations with additive and multiplicative noise: for each $t \in [0, T]$,

$$Z(t) = Z(0) + c \int_0^t Z(s) \, ds + (\text{LY}) \int_0^t D(s) \, dX(s), \tag{1.3}$$

$$Z(t) = Z(0) + c \int_0^t Z(s) \, ds + (\text{LY}) \int_0^t \sigma Z(s) \, dX(s), \tag{1.4}$$

whenever the integrals with respect to $X = (X(t))_{t \in [0, T]}$ exist as left Young integrals, and the remaining two integrals exist in the Riemann sense. Here c, σ are constants and D is a suitable function or stochastic process. Right Young integrals are used for the corresponding backward linear integral equations. Typical examples of processes X are infinite-variance stable Lévy motion, fractional Brownian motion, hyperbolic Lévy motion or Lévy processes generated by normal inverse Gaussian processes. These and related processes are used to model turbulence in physics, stock price changes in mathematical finance, traffic in high-speed networks, failure-generating mechanisms in reliability theory as well as various phenomena running under the heading of ‘fractal’.

The left Young integral equations (1.3) and (1.4) are correctly defined and have unique solutions under conditions stated below in Sections 4 and 5. In particular, equation (1.4) with $c = 0$ and $\sigma = 1$ reduces to the Moore–Pollard–Stieltjes integral equation

$$Z(t) = 1 + (\text{MPS}) \int_0^t Z(s-) dX(s), \quad t \in [0, T], \tag{1.5}$$

whenever the sample paths of X are right-continuous. If in addition almost all sample paths of X have bounded p -variation with $0 < p < 2$, then equation (1.5) is correctly defined in the class of processes Z having sample paths of bounded q -variation with $p^{-1} + q^{-1} > 1$ due to the aforementioned results of Young. Its solution is then given by

$$Z(t) = e^{X(t)-X(0)} \prod_{0 < s \leq t} (1 + \Delta X(s)) e^{-\Delta X(s)},$$

where $\Delta X(t)$ describes the jump of X at t . See Section 4 for details. Recall that the corresponding stochastic integral equation driven by a semimartingale $Y = M + V$ has the form

$$Z(t) = 1 + (\text{I}) \int_0^t Z(s-) dM(s) + (\text{LS}) \int_0^t Z(s-) dV(s), \quad t \in [0, T], \tag{1.6}$$

where the first and second integrals are defined in the sense of Itô and Lebesgue–Stieltjes, respectively. Equation (1.6) is correctly defined in the class of processes Z for which the integrals exist. So, for example, Z must be adapted to an underlying filtration of M . By Theorem 1 of Doléans-Dade (1970), in the class of semimartingales the unique solution of this equation is given by

$$Z(t) = e^{Y(t)-Y(0)-[M^c](t)/2} \prod_{0 < s \leq t} (1 + \Delta Y(s)) e^{-\Delta Y(s)}, \quad t \in [0, T], \tag{1.7}$$

where $[M^c](t)$ denotes the quadratic variation of the continuous martingale part of M . If one assumes that X in (1.5) has sample paths of bounded variation and $M \equiv 0$ in (1.6), then $Y = V = X$ and (1.5) coincides with (1.6). Indeed, the integrals appearing in these equations exist and have the same value; see Proposition 2.7. below. In general, equations (1.5) and (1.6) are driven by processes from different classes, which have a non-empty intersection containing a large class of pure jump Lévy processes.

As already mentioned, the main condition for solving equation (1.5) path by path, as well as equations (1.3) and (1.4), is boundedness of p -variation with $p < 2$. This condition can be slightly weakened. However, it cannot be replaced by the boundedness of 2-variation. Recall that sample paths of standard Brownian motion have unbounded 2-variation and bounded p -variation for every $p > 2$; see Taylor (1972) for the exact result. Thus, in order to apply the present approach for solving (1.5), one needs to know whether the stochastic process X has sample paths with bounded p -variation for some $p < 2$. This is known for a variety of stochastic processes. Some of these results are given in Section 2.2.

The main result of this paper concerns the solution of Riemann–Stieltjes type integral equations driven by processes whose sample paths may have unbounded variation. A first related result is Theorem 4.1 of Freedman (1983). He solved (1.5) by an application of Banach’s fixed point theorem. He assumed that (1.5) is driven by a (deterministic) continuous function of bounded p -variation with $1 \leq p < 2$. Dudley and Norvaiša (1999a, Theorem 5.21), extended this result to discontinuous functions by proving Duhamel’s formula. The latter is used to find the Fréchet derivative of the indefinite product integral.

Lyons (1994) extended Freedman (1983) in a different direction, replacing $Z - dX$ in (1.5) by $\varphi(Z) dX$ for suitable nonlinear φ and X continuous with values in \mathbb{R}^d . The approach to solving linear integral equations which is advocated in the present paper is perhaps the most simple and natural one. To solve equations (1.3) and (1.4) we adapt an approach common in stochastic analysis. Namely, we prove a chain rule for the composition of a smooth function and a function of bounded p -variation with $p < 2$. Then, by applying this formula, we verify that a suitably chosen function solves the equation of interest.

The paper is organized as follows. In Section 2.1 we introduce p -variation and related quantities. In Section 2.2 we recall the definition and properties of some classes of stochastic processes which are relevant for our purposes. In Section 2.3 we define the extended Riemann–Stieltjes integrals and discuss their existence and relationship with other types of integrals. We also give some of their basic properties. The chain rule (Theorem 3.1) based on the left and right Young integrals is given in Section 3. This result is applied in Section 4 to solve both homogeneous and non-homogeneous linear integral equations. The deterministic theory of the preceding sections is used in Section 5 to solve the stochastic integral equations (1.3) and (1.4).

2. Preliminaries

2.1. Functions of bounded p -variation

This subsection contains notation and simple properties related to p -variation.

Let $a < b$ be two real numbers. A real-valued function f on $[a, b]$ is called *regulated*, for which we write $f \in \mathcal{R} = \mathcal{R}([a, b])$, if it has a left limit at each point of (a, b) and a right limit at each point of $[a, b)$. A regulated function is bounded and has at most countably many jumps of the first kind. Such a function can be redefined on the extended interval $\{a, a+\} \cup \{x-, x, x+\} \cup \{b-, b\}$ endowed with the natural linear ordering. We will often make use of this construction.

Define the following function on $[a, b]$:

$$\begin{aligned}
 f_b^+(x) &= f^+(x) = f(x+) = \lim_{y \downarrow x} f(y), & a \leq x < b, & & f_b^+(b) &= f(b), \\
 f_a^-(x) &= f^-(x) = f(x-) = \lim_{y \uparrow x} f(y), & a < x \leq b, & & f_a^-(a) &= f(a),
 \end{aligned}$$

Let $\tau \subset [a, b]$ be a non-degenerate interval, open or closed at either end. Define $\Delta_\tau^- f$ on τ by $\Delta_\tau^- f(x) = f(x) - f(x-)$ for each $x \in \tau$ which is not the left end-point of τ and $\Delta_\tau^- f(x) = 0$ at the left end-point x whenever τ is left-closed. Similarly, define $\Delta_\tau^+ f$ on τ by $\Delta_\tau^+ f(x) = f(x+) - f(x)$ for each $x \in \tau$ which is not the right end-point of τ and $\Delta_\tau^+ f(x) = 0$ at the right end-point x whenever τ is right-closed. For $\tau = [a, b]$ write $\Delta^- f = \Delta_{[a,b]}^- f$ and $\Delta^+ f = \Delta_{[a,b]}^+ f$. For a function F we will occasionally write

$$\sum_\tau F(\Delta f) = \sum_{x \in \tau} F([\Delta_\tau^- f](x)) + \sum_{x \in \tau} F([\Delta_\tau^+ f](x)).$$

For example, for $F(u) = |u|^p$, $u \in \mathbb{R}$, $0 < p < \infty$, let

$$\mathcal{O}_p(f; \tau) = \left(\sum_{\tau} |\Delta f|^p \right)^{1/p}. \tag{2.1}$$

If $\tau = [a, b]$ then write $\mathcal{O}_p(f) = \mathcal{O}_p(f; [a, b])$.

Recall from (1.1) the definition of p -variation $v_p(f)$, $0 < p < \infty$. All functions of bounded p -variation constitute the set

$$\mathcal{W}_p = \mathcal{W}_p([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} \text{ with } v_p(f) < \infty\}.$$

Note that $\mathcal{W}_q \subset \mathcal{W}_p$ for $0 < q < p < \infty$. Moreover, every function of bounded p -variation is regulated, i.e. $\mathcal{W}_p \subset \mathcal{R}$. Defining $V_p(f) = V_p(f; [a, b]) = v_p^{1/p}(f)$, one can show that $\mathcal{O}_p(f) \leq V_p(f)$. In general, this inequality cannot be replaced by an equality. However, if f has bounded p -variation for some $p < 1$, then f is a pure jump function, and for those f and p , $\mathcal{O}_p(f) = V_p(f)$.

Given a non-degenerate interval $\tau \subset [a, b]$, open or closed at either end, which also may be extended by points x_{\pm} , define the *oscillation* of f :

$$\text{Osc}(f; \tau) = \sup\{|f(x) - f(y)| : x, y \in \tau\}. \tag{2.2}$$

Below we will need a well-known property of regulated functions in a slightly modified form:

Lemma 2.1. *Assume $f \in \mathcal{R}([a, b])$. For every $\varepsilon > 0$, there are at most a finite number of points $x \in [a, b]$ for which $|\Delta^- f(x)| > \varepsilon$ or $|\Delta^+ f(x)| > \varepsilon$. Moreover, there exists a subdivision $\{a = x_0 < \dots < x_n = b\}$ such that*

$$\text{Osc}(f; [x_{i-1}+, x_i-]) < \varepsilon \quad \text{for } i = 1, \dots, n.$$

Proof. An application of the Bolzano–Weierstrass theorem yields the first statement. If $|\Delta^+ f(x)| \vee |\Delta^- f(x)| < \alpha$ for all $x \in (c, d) \subset [a, b]$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < 2\alpha$ for $x, y \in (c, d)$ with $|x - y| < \delta$; cf. Lebesgue (1973, p. 21). Thus, for given $\varepsilon > 0$ we can find a subdivision of $[a, b]$ such that $\max_i \text{Osc}(f; (x_{i-1}, x_i)) < \varepsilon$. The second statement now follows from the relation

$$\text{Osc}(f; [y+, z-]) = \lim_{u|_y, v|_z} \text{Osc}(f; (u, v)), \quad a \leq y < z \leq b. \quad \square$$

We refer to Section 2 of Dudley and Norvaiša (1999a) for further details on p -variation.

2.2. Stochastic processes and p -variation

In this subsection we collect some useful facts about the p -variation of several important classes of stochastic processes. These properties will be used in Section 5.

Here and in what follows, all stochastic processes $X = (X(t))_{t \geq 0}$ are supposed to be separable, continuous in probability and defined on a complete probability space. In this subsection, $[a, b] = [0, T]$ for an arbitrary but fixed $T \in (0, \infty)$. Then $v_p(X)$, $0 < p < \infty$,

is a random variable possibly assuming ∞ with positive probability. The zero–one law for the p -variation $v_p(X)$ and the question of its boundedness were established for major classes of stochastic processes X .

The results of the present paper are applicable to sample paths of stochastic processes having bounded p -variation with $0 < p < 2$. It is well known that standard Brownian motion does not satisfy this condition. However, there are several other classes of stochastic processes that enjoy this desirable p -variation property. Here we focus on two particular classes of stochastic processes which have attracted the attention of many researchers in applied mathematics.

A mean-zero Gaussian process $B_H = (B_H(t))_{t \geq 0}$ with $B_H(0) = 0$ is called (standard) *fractional Brownian motion with index* $H \in (0, 1)$ if it has covariance function

$$\text{cov}(B_H(t), B_H(s)) = 0.5(t^{2H} + s^{2H} - |t - s|^{2H}) \quad \text{for } t, s \geq 0.$$

If $H = 0.5$, the right-hand side is equal to $t \wedge s$, i.e. $B_{0.5}$ is Brownian motion.

The following claim follows by a combination of the results in Fernique (1964) and Theorem 3 of Kawada and Kôno (1973).

Proposition 2.2. *Let B_H be fractional Brownian motion with index $H \in (0, 1)$ and $p \in (H^{-1}, \infty)$. Then almost all sample paths of B_H are continuous and $v_p(B_H) < \infty$ with probability 1.*

Remarks. (1) Kawada and Konô (1973) give conditions for the boundedness of p -variation of continuous Gaussian processes X more general than fractional Brownian motion. Their conditions are in terms of a function b satisfying $E(X(s) - X(t))^2 \leq (\text{const.})b(|t - s|)$ for $t, s \geq 0$. The p -variation of arbitrary Gaussian processes was considered by Jain and Monrad (1983).

(2) Fractional Brownian motion with $H \in (0.5, 1)$ is a standard process for modelling long-range dependent phenomena; see, for example, Samorodnitsky and Taqqu (1994, Section 7.2). Because of that property it has recently attracted some attention in mathematical finance; see, for example, Cutland *et al.* (1995), Dai and Heyde (1996) or Lin (1995). However, B_H with $H \in (0.5, 1)$ is *not* a semimartingale (see Liptser and Shiryaev 1986, Section 4.9), and therefore standard stochastic calculus does not apply. To solve this problem, a non-standard analysis, as well as an extension of standard stochastic integrals, were used by the aforementioned authors. We show in Section 5 that Riemann–Stieltjes integral equations driven by sample paths of fractional Brownian motion are appropriate.

Another class of stochastic processes fits well into the framework of pathwise integration: the class of Lévy processes. A stochastic process $X = (X(t))_{t \geq 0}$ which is continuous in probability is called a *Lévy process* if it has independent, stationary increments, if almost all sample paths are right-continuous and have limits to the left and if $X(0) = 0$. Such a process has Lévy–Itô representation (see Itô 1969, Theorem 1.7.1)

$$X(t) = at + bB(t) + \lim_{\delta \downarrow 0} \left[\sum_{I(\delta; [0, t])} \Delta^- X - t \int_{|x| > \delta} \frac{x}{1 + x^2} \nu(dx) \right], \tag{2.3}$$

where the limit exists uniformly on bounded intervals with probability 1. Here B stands for standard Brownian motion, a, b are constants and ν is a Borel measure on $\mathbb{R} \setminus \{0\}$ satisfying

$$\int_{\mathbb{R} \setminus \{0\}} (1 \wedge |x|^2) \nu(dx) < \infty.$$

It is called the *Lévy measure* of X . Moreover, $I(\delta; [0, t])$ denotes the set of $s \in [0, t]$ satisfying $|\Delta^- X(s)| > \delta$. If the limit

$$d = \lim_{\delta \downarrow 0} \int_{|x| > \delta} \frac{x}{1 + x^2} \nu(dx) \tag{2.4}$$

exists, is finite and $a = d$ then we say that X does not have a drift.

The p -variation of Lévy processes was considered in various papers; see Bertoin (1996; Section I.6) or Dudley *et al.* (1999) for a list of references. For certain Lévy processes, Bretagnolle (1972) characterises the property $v_p(X) < \infty$ in terms of the finiteness of the integral $\int (1 \wedge |x|^p) \nu(dx)$.

A Lévy process X_α is called *α -stable Lévy motion with index α* , $0 < \alpha < 2$, if $b = 0$ in (2.3) and it has Lévy measure

$$\nu_\alpha(dx) = c_1 x^{-1-\alpha} dx I_{(0, \infty)}(x) + c_2 (-x)^{-1-\alpha} dx I_{(-\infty, 0)}(x),$$

where $c_1, c_2 \geq 0$ are constants with $c_1 + c_2 > 0$. If $\alpha < 1$ or the marginal distributions of X_α are symmetric, the limit (2.4) exists and is finite.

The p -variation of α -stable Lévy motion was studied by Fristedt and Taylor (1973). From their Theorem 2 one obtains the following result.

Proposition 2.3. *Let X_α be α -stable Lévy motion. Assume that X_α does not have a drift for $\alpha < 1$ and that the Lévy measure is symmetric for $\alpha = 1$. Then $v_p(X_\alpha)$ is finite or infinite with probability 1 according to whether $p > \alpha$ or $p \leq \alpha$.*

Remark. Note that α -stable processes with $0 < \alpha < 2$ are infinite-variance processes. Because their sample paths exhibit large jumps, they are considered as alternatives to Brownian motion. For various applications of α -stable processes in finance, physics, earth sciences and other fields, see, for example, Janicki and Weron (1993) or Samorodnitsky and Taqqu (1994).

Another well-studied subclass of Lévy processes consists of the normal inverse Gaussian processes and hyperbolic Lévy motion. They gained their name from the marginal distributions which are either normal inverse Gaussian or hyperbolic. Using the above-mentioned result of Bretagnolle (1972) and utilizing the form of the Lévy measure – see Eberlein and Keller (1995) in the hyperbolic case and Barndorff-Nielsen (1997) in the normal inverse Gaussian case – one can show that these processes have bounded p -variation for $p > 1$. Therefore they fit nicely into the framework of pathwise integration advocated in this paper. These processes were used to model turbulence in physics, stock

price changes in mathematical finance and failure-generating mechanisms in reliability theory; see Barndorff-Nielsen (1978; 1986) for the definition and properties as well as applications of these processes. Recently, these classes of Lévy processes were suggested as realistic models for stock returns; see Barndorff-Nielsen (1995; 1997), Eberlein and Keller (1995) and Küchler *et al* (1994).

In addition to the references on p -variation of stochastic processes given earlier, we should mention that Lépingle (1976) showed that every semimartingale X satisfies $v_p(X) < \infty$ for $p > 2$. A bibliography on p -variation with annotated references can be found in Dudley *et al.* (1999).

2.3. Extended Riemann–Stieltjes integrals

In this subsection we review the classical Riemann–Stieltjes integral and several of its extensions. A usual, for two real-valued functions f and h on $[a, b]$, a Riemann–Stieltjes sum is defined by

$$S(f, h, \kappa, \sigma) = \sum_{i=1}^n f(y_i)[h(x_i) - h(x_{i-1})].$$

Here κ is a subdivision of $[a, b]$ – see (1.2) – and $\sigma = \{y_1, \dots, y_n\}$ is an intermediate subdivision of κ , i.e. $x_{i-1} \leq y_i \leq x_i$ for $i = 1, \dots, n$. The function f is Riemann–Stieltjes integrable with respect to h on $[a, b]$ if there exists a number I satisfying the following property: given $\varepsilon > 0$, one can find a $\delta > 0$ such that

$$|S(f, h, \kappa, \sigma) - I| < \varepsilon \tag{2.5}$$

for all subdivisions κ with mesh $\max_i(x_i - x_{i-1}) < \delta$ and for all intermediate subdivisions σ of κ . The number I , if it exists, is unique and will be denoted by

$$(\text{RS}) \int_a^b f dh. \tag{2.6}$$

If f is Riemann–Stieltjes integrable with respect to h then f and h cannot have a jump at the same point. The Moore–Pollard–Stieltjes integral, an extension of the Riemann–Stieltjes integral, requires less restrictive necessary conditions at jump points. Its definition is the same as above with one exception: the convergence of the Riemann–Stieltjes sums as the mesh tends to zero is replaced by their convergence under refinements of subdivisions. More precisely, we say that κ is a *refinement* of a subdivision λ if $\kappa \supset \lambda$. Then the function f is Moore–Pollard–Stieltjes integrable, or *MPS integrable*, with respect to h on $[a, b]$ if there exists a number I satisfying the following property: given $\varepsilon > 0$ one can find a subdivision λ of $[a, b]$ such that (2.5) holds for all refinements κ of λ and for all intermediate subdivisions σ of κ . The number I , if it exists, is unique and will be denoted by

$$(\text{MPS}) \int_a^b f dh. \tag{2.7}$$

If f is MPS integrable with respect to h then f and h cannot have a jump at the same point

on the same side. In particular, this necessary condition is satisfied if f is right-continuous and h is left-continuous or vice versa.

It is well known that (2.6) exists, and so does (2.7), if h is of bounded variation and f is continuous. However, both integrals may exist when none of the two functions have bounded variation. This was proved by Young (1936):

Theorem 2.4. *Assume $h \in \mathcal{W}_p$ and $f \in \mathcal{W}_q$ for some $p, q > 0$ with $p^{-1} + q^{-1} > 1$. Then the following statements hold:*

- (i) (2.6) exists if f and h do not have a common discontinuity at the same point.
- (ii) (2.7) exists if f and h do not have a common discontinuity on the same side and at the same point.

Moreover, there exists a finite constant $K = K(p, q)$ such that, for any $y \in [a, b]$, the inequality

$$\left| \int_a^b f \, dh - f(y)[h(b) - h(a)] \right| \leq KV_p(h)V_q(f) \tag{2.8}$$

holds for both kinds of integral, provided it is defined.

In (1.5), the integrand in the MPS integral is left-continuous and the driving stochastic process is right-continuous. Therefore the notion of the MPS integral suffices for the applications presented in Section 5 below. If the sample paths of the driving stochastic process are only known to be regulated, the same results still hold if the MPS integral is replaced by another extension of the Riemann–Stieltjes integral. The following variants of the integral introduced by Young (1936) were proposed by Dudley and Norvaiša (1999a, Definition 3.11). First recall the notation f_a^-, f_b^+ from Section 2.1.

Definition 2.5. *Assume $f, h \in \mathcal{R}$. Define the left Young integral by*

$$(LY) \int_a^b f \, dh = (\text{MPS}) \int_a^b f_a^- \, dh_b^+ + [f(\Delta^+ h)](a) + \sum_{(a,b)} \Delta^- f \Delta^+ h \tag{2.9}$$

whenever the MPS integral exists and the sum converges absolutely. Define the right Young integral by

$$(RY) \int_a^b f \, dh = (\text{MPS}) \int_a^b f_b^+ \, dh_a^- + [f(\Delta^- h)](b) - \sum_{(a,b)} \Delta^+ f \Delta^- h \tag{2.10}$$

whenever the MPS integral exists and the sum converges absolutely. We say that f is LY integrable (or RY integrable) with respect to h on $[a, b]$ provided (2.9) (or (2.10)) is defined.

Notice that the left Young integral (2.9) is defined by the MPS integral of f_a^- whenever h is right-continuous. Similarly, the right Young integral (2.10) is defined by the MPS integral of f_b^+ whenever h is left-continuous. The left and right Young integrals have the usual properties of integrals. For example, they are bilinear and additive on adjacent intervals; see Dudley and

Norvaiša (1999a, Propositions 3.21 and 3.25) or Norvaiša (1998, Theorem 4). Moreover, we have:

Lemma 2.6. *For regulated functions f and h on $[a, b]$ the following hold:*

- (i) *If f is LY integrable with respect to h on $[a, b]$ then the indefinite integral $\Psi(\cdot) = (\text{LY}) \int_a^\cdot f dh$ is a regulated function on $[a, b]$ with jumps*

$$(\Delta^- \Psi)(x) = [f^-(\Delta^- h)](x) \quad \text{and} \quad (\Delta^+ \Psi)(y) = [f(\Delta^+ h)](y)$$

for $a \leq y < x \leq b$.

- (ii) *If f is RY integrable with respect to h on $[a, b]$ then the indefinite integral $\Phi(\cdot) = (\text{RY}) \int_a^\cdot f dh$ is a regulated function on $[a, b]$ with jumps*

$$(\Delta^- \Phi)(x) = -[f(\Delta^- h)](x) \quad \text{and} \quad (\Delta^+ \Phi)(y) = -[f^+(\Delta^+ h)](y)$$

for $a \leq y < x \leq b$.

Proof. This is a special case of Lemma 3.26 of Dudley and Norvaiša (1999a), where the corresponding lemma is proved for three function variants of the LY and RY integrals. The first statement follows by taking $g \equiv 1$ in Lemma 3.26 and using representation (3.42) from Dudley and Norvaiša (1999a). The second follows by taking $f \equiv 1$ in Lemma 3.26 and using representation (3.41) from Dudley and Norvaiša (1999a). A direct proof of Lemma 2.6 is given in Norvaiša (1998, Proposition 7). \square

In stochastic analysis, the Lebesgue–Stieltjes integral is used to integrate with respect to stochastic processes having sample paths of bounded variation. In this case, the values of the above extensions of the Riemann–Stieltjes integrals agree with the corresponding values of the Lebesgue–Stieltjes integral (or LS integral) as stated next:

Proposition 2.7. *If h is a right-continuous function of bounded variation and f is a regulated function on $[a, b]$ then the following three integrals exist and are equal:*

$$(\text{LY}) \int_a^b f dh = (\text{MPS}) \int_a^b f_a^- dh = (\text{LS}) \int_a^b f_a^- dh. \quad (2.11)$$

Proof. We may and do assume that h is a non-decreasing function. The MPS integral in (2.11) exists by Theorem 5.32 in Hildebrandt (1938). Thus the LY integral in (2.11) exists and the first equality holds by Definition 2.5. Since f_a^- is bounded and Borel measurable, the LS integral also exists. The proof of the second equality is given in the proof of Theorem 4.2 in Dudley and Norvaiša (1999b). \square

The following statement is a consequence of Theorem 2.4(ii) and Hölder's inequality.

Theorem 2.8. *Assume $h \in \mathscr{W}_p$ and $f \in \mathscr{W}_q$ for some $p, q > 0$ with $p^{-1} + q^{-1} > 1$. Then both, the left Young integral (2.9) and the right Young integral (2.10), exist.*

Notice that no restriction on the jumps of the functions h and f is required for the existence of Young's integrals.

We finish with an auxiliary statement used in Section 5 below.

Lemma 2.9. *Let $f \in \mathcal{W}_q$, $h \in \mathcal{W}_p$ for some $q, p > 0$ with $p^{-1} + q^{-1} > 1$ and assume h continuous. Then the integrals (RS) $\int_a^b f_a^- dh$ and (RS) $\int_a^b f dh$ exist and are equal.*

Proof. Given $\varepsilon > 0$, choose $\delta > 0$ such that $|h(x) - h(y)| < \varepsilon$ for $|x - y| < \delta$. Let $\{x_i : i = 0, \dots, n\}$ be a subdivision of $[a, b]$ with mesh less than δ and let $\{y_i : i = 1, \dots, n\}$ be an intermediate subdivision. Assume first that $q > 1$. Write $q^* = q/(q - 1)$. Then, by Hölder's inequality,

$$\left| \sum_{i=1}^n [f_a^-(y_i) - f(y_i)][h(x_i) - h(x_{i-1})] \right| \leq \varepsilon^{1-p/q^*} \mathcal{O}_q(f) v_p^{1/q^*}(h).$$

If $q \leq 1$ then a simpler bound holds because f is of bounded variation. The assertion now follows from Theorem 2.4.1. □

3. Chain rule

As usual, the composition $g \circ h$ on $[a, b]$ of two functions g and h is defined by $(g \circ h)(x) = g(h(x))$ whenever h lives on $[a, b]$ and g on the range of h . In this section a *chain rule* for $g \circ h$ is given under the assumptions that h is a function of bounded p -variation for some $p \in (0, 2)$ and g is a smooth function. Depending on whether the left or right Young integral is used, two variants of the chain rule are proved. They are analogous to Itô's formula for the composition of a smooth function and Brownian motion.

The following theorem is basic to this paper.

Theorem 3.1. *Let $h = (h_1, \dots, h_d) : [a, b] \rightarrow \mathbb{R}^d$, where for every $l = 1, \dots, d$, $h_l \in \mathcal{W}_p$ for some $p \in (0, 2)$. Let $g : \mathbb{R}^d \rightarrow \mathbb{R}$ be a differentiable function with locally Lipschitz partial derivatives g'_l , $l = 1, \dots, d$. Then the integrals (LY) $\int_a^b (g'_l \circ h) dh_l$ exist and satisfy the relation*

$$\begin{aligned} (g \circ h)(b) - (g \circ h)(a) &= \sum_{l=1}^d (\text{LY}) \int_a^b (g'_l \circ h) dh_l + \sum_{[a,b]} \left[\Delta^+(g \circ h) - \sum_{l=1}^d (g'_l \circ h) \Delta^+ h_l \right] \\ &\quad + \sum_{(a,b]} \left[\Delta^-(g \circ h) - \sum_{l=1}^d (g'_l \circ h)^- \Delta^- h_l \right], \end{aligned} \tag{3.1}$$

where the two sums in (3.1) converge absolutely. Similarly, the integrals (RY) $\int_a^b (g'_l \circ h) dh_l$ exist and satisfy the relation

$$\begin{aligned}
 (g \circ h)(b) - (g \circ h)(a) &= \sum_{l=1}^d (\text{RY}) \int_a^b (g'_l \circ h) dh_l + \sum_{(a,b)} \left[\Delta^+(g \circ h) - \sum_{l=1}^d (g'_l \circ h)^+ \Delta^+ h_l \right] \\
 &\quad + \sum_{(a,b)} \left[\Delta^-(g \circ h) - \sum_{l=1}^d (g'_l \circ h) \Delta^- h_l \right], \tag{3.2}
 \end{aligned}$$

where the two sums in (3.2) converge absolutely.

Remark. Since the functions $g'_l \circ h$ and h_l have bounded p -variation with $p < 2$, the existence of the integrals in (3.1) and (3.2) follows from Theorem 2.8. A more general chain rule can be proved so that the existence of the integrals in (3.1) and (3.2) cannot be derived from general existence theorems such as Theorem 2.8. The special form of the integrals involved in the chain rule then plays an important role. In other words, the chain rule becomes an existence theorem for integrals as in (3.1) and (3.2); see Norvaiša (1998) for details. However, it is also shown there that the assumption $p < 2$ cannot be replaced by $p = 2$.

For sample paths of stochastic processes for which the quadratic variation is defined, a chain rule was proved by Föllmer (1981) using a left-Cauchy type integral defined for a fixed sequence of subdivisions.

Proof. First we prove the left Young part, then we indicate the necessary changes needed for the right Young case. We start by showing that the LY integrals in (3.1) exist. Since a function of bounded p -variation is bounded there exists a finite constant M such that $|h_l(x)| \leq M$ for all $x \in [a, b]$ and every l . The partial derivatives g'_l of g are locally Lipschitz, and therefore we can find another finite constant K such that, for all $u = (u_l), v = (v_l) \in [-M, M]^d$,

$$\max_{1 \leq l \leq d} |g'_l(u) - g'_l(v)| \leq K \sum_{l=1}^d |u_l - v_l|. \tag{3.3}$$

It follows that the functions $g'_l \circ h$, hence $(g'_l \circ h)_a^-$, as well as $(h_l)_b^+$, are of bounded p -variation. We conclude from Theorem 2.4 that $(g'_l \circ h)_a^-$ is MPS integrable with respect to $(h_l)_b^+$ for every l . First using the Lipschitz property (3.3) and then applying the Cauchy–Schwarz inequality, we obtain

$$\sum_{(a,b)} |\Delta^-(g'_l \circ h) \Delta^+ h_l| \leq K \max_{1 \leq l \leq d} \left(\sum_{(a,b)} [\Delta^+ h_l]^2 \right)^{1/2} \sum_{k=1}^d \left(\sum_{(a,b)} [\Delta^- h_k]^2 \right)^{1/2}.$$

Since $\mathcal{O}_2(h_l) < \infty$ (see (2.1)) the sum on the left-hand side converges absolutely. In view of Definition 2.5 we may conclude that the LY integrals in (3.1) exist.

Next we show that the sums in (3.1) converge absolutely. Consider $x \in (a, b]$ with $\Delta^- h_l(x) \neq 0$ for some l . By the mean value theorem, there exist $\theta_l \in [h_l(x-) \wedge h_l(x), h_l(x-) \vee h_l(x)]$, $l = 1, \dots, d$, such that

$$\begin{aligned} \phi(x) &= \left| \Delta^-(g \circ h)(x) - \sum_{l=1}^d (g'_l \circ h)^-(x) \Delta^- h_l(x) \right| \\ &= \left| \sum_{l=1}^d [g'_l(h^{\theta_l}(x-)) - g'_l(h(x-))] [\Delta^- h_l(x)] \right| \leq K \sum_{l=1}^d |\Delta^- h_l(x)|^2, \end{aligned}$$

where

$$h^{\theta_l}(x-) = (h_l(x-), \dots, \theta_l, \dots, h_d(x-))$$

and K is the Lipschitz constant from (3.3). Therefore we have

$$\sum_{x \in (a,b)} \phi(x) \leq K \sum_{l=1}^d \sum_{x \in (a,b)} [\Delta^- h_l(x)]^2.$$

Since $\mathcal{O}_2(h_l) < \infty$, the second sum in (3.1) converges absolutely. The first sum can be dealt with analogously.

We showed that the right-hand side in (3.1) is well defined. Now we turn to the proof of (3.1). Consider any subdivision $\kappa = \{a = x_0 < \dots < x_n = b\}$. For each $l = 1, \dots, d$, define the quantities

$$I_l(\kappa) = \sum_{i=1}^n g'_l(h(x_{i-})) [(h_l)_b^+(x_i) - (h_l)_b^+(x_{i-1})],$$

$$T_l(\kappa) = (g'_l \circ h)(a) \Delta^+ h_l(a) + \sum_{i=1}^{n-1} \Delta^-(g'_l \circ h)(x_i) \Delta^+ h_l(x_i),$$

$$S^-(\kappa) = \sum_{i=1}^n \left[\Delta^-(g \circ h)(x_i) - \sum_{l=1}^d g'_l(h(x_{i-})) \Delta^- h_l(x_i) \right],$$

$$S^+(\kappa) = \sum_{i=1}^n \left[\Delta^+(g \circ h)(x_{i-1}) - \sum_{l=1}^d g'_l(h(x_{i-1})) \Delta^+ h_l(x_{i-1}) \right],$$

$$R(\kappa) = \sum_{i=1}^n \left[g(h(x_{i-})) - g(h(x_{i-1}+)) - \sum_{l=1}^d g'_l(h(x_{i-})) [h_l(x_{i-}) - h_l(x_{i-1}+)] \right].$$

We obtain the telescoping sum representation

$$(g \circ h)(b) - (g \circ h)(a) = \sum_{l=1}^d I_l(\kappa) + \sum_{l=1}^d T_l(\kappa) + S^-(\kappa) + S^+(\kappa) + R(\kappa). \tag{3.4}$$

We intend to show that the right-hand sides of (3.1) and (3.4) can be made arbitrarily close to each other by choosing appropriate subdivisions κ .

Choose an $\varepsilon > 0$. Each $I_l(\kappa)$ is an RS sum for $(g'_l \circ h)_a^-$ and $(h_l)_b^+$ based on κ and its

intermediate subdivision $\sigma = \{x_1, \dots, x_n\}$. Since the corresponding MPS integrals exist one can find a subdivision μ of $[a, b]$ such that for all refinements κ of μ and each l , we have

$$\left| I_l(\kappa) - (\text{MPS}) \int_a^b (g'_l \circ h)_a^- d(h_l)_b^+ \right| < \varepsilon. \tag{3.5}$$

Moreover, since the sums corresponding to the LY integrals in (3.1) converge absolutely, there exists a finite subset λ of (a, b) such that for each $\nu \supset \lambda$ and all l ,

$$\left| \sum_{(a,b) \setminus \nu} [\Delta^-(g'_l \circ h) \Delta^+ h_l] \right| < \varepsilon. \tag{3.6}$$

We showed above that the other two sums in (3.1) converge absolutely. Thus we can find finite subsets λ_- of (a, b) and λ_+ of $[a, b)$ such that, for each $\nu \supset \lambda_- \cup \lambda_+$,

$$\left| \sum_{(a,b) \setminus \nu} \left[\Delta^-(g \circ h) - \sum_{l=1}^d (g'_l \circ h)^- \Delta^- h_l \right] \right| < \varepsilon, \tag{3.7}$$

$$\left| \sum_{[a,b) \setminus \nu} \left[\Delta^+(g \circ h) - \sum_{l=1}^d (g'_l \circ h) \Delta^+ h_l \right] \right| < \varepsilon. \tag{3.8}$$

Finally, by virtue of Lemma 2.1 one can find a subdivision $\chi = \{y_j : j = 0, 1, \dots, m\}$ of $[a, b]$ such that for each j and all l , $\text{Osc}(h_l; [y_{j-1}+, y_j-]) \leq \varepsilon$, where the oscillation of a function is defined in (2.2). We use this property to estimate $R(\kappa)$ for any $\kappa \supset \chi$. Using the mean value theorem, we can find vectors $(\theta_{1,i}, \dots, \theta_{d,i})$ such that $\theta_{l,i} \in [h_l(x_{i-1}+) \wedge h_l(x_i-), h_l(x_{i-1}+) \vee h_l(x_i-)]$ and

$$\begin{aligned} |R(\kappa)| &\leq \sum_{l=1}^d \sum_{i=1}^n |(g'_l(h^{\theta_{l,i}}(x_i-)) - g'_l(h(x_i-)))(h_l(x_i-) - h_l(x_{i-1}+))| \\ &\leq K \sum_{l=1}^d \sum_{i=1}^n [h_l(x_i-) - h_l(x_{i-1}+)]^2 \\ &\leq K \sum_{l=1}^d \left(v_p(h_l) \max_{1 \leq i \leq n} |h_l(x_i-) - h_l(x_{i-1}+)|^{2-p} \right) \\ &\leq \varepsilon^{2-p} K \sum_{l=1}^d v_p(h_l), \end{aligned} \tag{3.9}$$

where $h^{\theta_{l,i}}(x_i-)$ is defined analogously to $h^\theta(x-)$ above. The last inequality follows from $\kappa \supset \chi$. Now define the subdivision

$$\kappa(\varepsilon) = \mu \cup \lambda \cup \lambda_- \cup \lambda_+ \cup \chi.$$

By virtue of (3.5)–(3.9) we obtain, for every $\kappa(\varepsilon) \subset \kappa$,

$$\begin{aligned} & \left| \sum_{l=1}^d \left[I_l(\kappa) + T_l(\kappa) - (\text{LY}) \int_a^b (g'_l \circ h) dh_l \right] \right| + \left| S^+(\kappa) - \sum_{[a,b]} \left[\Delta^+(g \circ h) - \sum_{l=1}^d (g'_l \circ h) \Delta^+ h_l \right] \right| \\ & + \left| S^-(\kappa) - \sum_{(a,b]} \left[\Delta^-(g \circ h) - \sum_{l=1}^d (g'_l \circ h)^- \Delta^- h_l \right] \right| + |R(\kappa)| \\ & \leq 2(d+1)\varepsilon + \varepsilon^{2-p} K \sum_{l=1}^d v_p(h_l). \end{aligned}$$

By virtue of (3.4) and since $\varepsilon > 0$ is arbitrary, this completes the proof of (3.1).

The proof of (3.2) is analogous. Instead of the telescoping sum representation (3.4), we now have

$$\begin{aligned} & (g \circ h)(b) - (g \circ h)(a) \\ & = \sum_{l=1}^d \sum_{i=1}^n g'_l(h(x_{i-1}+)) [(h_l)^-(x_i) - (h_l)_a^-(x_{i-1})] \\ & + \sum_{l=1}^d \left[(g'_l \circ h)(b) \Delta^- h_l(b) - \sum_{i=2}^n \Delta^+(g'_l \circ h)(x_{i-1}) \Delta^- h_l(x_{i-1}) \right] \\ & + \sum_{i=1}^n \left[\Delta^-(g \circ h)(x_i) - \sum_{l=1}^d g'_l(h(x_i)) \Delta^- h_l(x_i) \right] \\ & + \sum_{i=1}^n \left[\Delta^+(g \circ h)(x_{i-1}) - \sum_{l=1}^d g'_l(h(x_{i-1}+)) \Delta^+ h_l(x_{i-1}) \right] \\ & + \sum_{i=1}^n \left[g(h(x_i-)) - g(h(x_{i-1}+)) - \sum_{l=1}^d g'_l(h(x_{i-1}+)) [h_l(x_i-) - h_l(x_{i-1}+)] \right]. \end{aligned}$$

This concludes the proof of Theorem 3.1. □

4. Linear integral equations

In what follows, we solve linear left and right Young integral equations, using the chain rule from the previous section.

We say that a function F , LY integrable with respect to f on $[a, b]$, satisfies the *homogenous forward linear integral equation with respect to f* if, for all $y \in [a, b]$,

$$F(y) = 1 + (\text{LY}) \int_a^y F df. \tag{4.1}$$

Analogously, a function G , RY integrable with respect to f on $[a, b]$, satisfies the homogeneous backward linear integral equation with respect to f if, for all $y \in [a, b]$,

$$G(y) = 1 + (\text{RY}) \int_y^b G \, df. \tag{4.2}$$

Let f be regulated and $\tau \subset [a, b]$ a non-degenerate interval, open or closed at either end. For each $\delta > 0$, set $I_{\pm}(\delta; \tau) = \{x \in \tau : |(\Delta_{\tau}^{\pm} f)(x)| > \delta\}$ and

$$\prod_{I(\delta; \tau)} (1 + \Delta f) e^{-\Delta f} = \prod_{I^-(\delta; \tau)} (1 + \Delta_{\tau}^- f) e^{-\Delta_{\tau}^- f} \prod_{I_+(\delta; \tau)} (1 + \Delta_{\tau}^+ f) e^{-\Delta_{\tau}^+ f}.$$

Define the infinite product

$$\prod_{\tau} (1 + \Delta f) e^{-\Delta f} = \lim_{\delta \downarrow 0} \prod_{I(\delta; \tau)} (1 + \Delta f) e^{-\Delta f},$$

whenever it converges absolutely. Finally, consider the real-valued functions

$$E_a(f)(y) = \begin{cases} e^{f(y)-f(a)} \prod_{[a,y]} (1 + \Delta f) e^{-\Delta f} & \text{for } y \in (a, b), \\ 1 & \text{for } y = a, \end{cases}$$

and

$$E^b(f)(y) = \begin{cases} e^{f(b)-f(y)} \prod_{[y,b]} (1 + \Delta f) e^{-\Delta f} & \text{for } y \in [a, b), \\ 1 & \text{for } y = b, \end{cases}$$

provided each of the infinite products involved in these expressions converges absolutely. Notice that $E_a(f)$ has the form of the Doléans-Dade stochastic exponential when f is a purely discontinuous semimartingale (cf. (1.7) above).

Now we are well prepared to solve the linear forward and backward equations (4.1) and (4.2).

Theorem 4.1. *Assume $f \in \mathscr{W}_p$ for some $p \in (0, 2)$. Then the functions $E_a(f)$ and $E^b(f)$ are well defined. Moreover, in \mathscr{W}_r , for any $r \geq p$ with $p^{-1} + r^{-1} > 1$, the equations (4.1) and (4.2) have unique solutions $E_a(f)$ and $E^b(f)$, respectively.*

Remarks. (1) A glance at the structure of $E_a(f)$ and $E^b(f)$ reveals the special role of a jump size -1 of f at $y_0 \in (a, b)$, say. Then both functions vanish for $y > y_0$ and $y < y_0$, respectively. Now consider a solution F of (4.1). Suppose first $\Delta^- f(y_0) = -1$. By Lemma 2.6,

$$F(y_0) = 1 + (\text{LY}) \int_a^{y_0} F \, df = 1 + (\text{LY}) \int_a^{y_0^-} F \, df - F(y_0^-) = 0.$$

If $y \in (y_0, b]$, by additivity of the LY integral (Proposition 3.25 of Dudley and Norvaiša 1999a),

$$F(y) = 1 + (\text{LY}) \int_a^{y_0} F \, df + (\text{LY}) \int_{y_0}^y F \, df = (\text{LY}) \int_{y_0}^y F \, df.$$

Thus $F(y) = 0$ for $y \in [y_0, b]$. Now suppose $\Delta^+ f(y_0) = -1$. By Proposition 3.25 of Dudley and Norvaiša (1999a) and Definition 2.5, we have, for all $y \in (y_0, b]$,

$$\begin{aligned} F(y) &= 1 + (\text{LY}) \int_a^{y_0} F \, df + (\text{MPS}) \int_{y_0}^y F_{y_0}^- \, df_y^+ - F(y_0) + \sum_{(y_0, y)} \Delta^- F \Delta^+ f \\ &= (\text{MPS}) \int_{y_0}^y F_{y_0}^- \, df_y^+ + \sum_{(y_0, y)} \Delta^- F \Delta^+ f. \end{aligned}$$

Since f_y^+ is right-continuous at y_0 , $F(y)$ vanishes for each $y \in (y_0, b]$.

(2) Notice that the form of the solution F at jump points of f depends on the definition of the integral involved in (4.1). For example, Hildebrandt (1959) using W.H. Young's integral and assuming f of bounded variation, obtains for a discontinuous function f a solution F different from ours. A similar remark applies to (4.2) and the right Young integral.

For the proof of Theorem 4.1 we need the following auxiliary result.

Lemma 4.2. *Let $f \in \mathscr{W}_p$ for some $p \in (0, 2)$. Then the function*

$$\phi(y) = \phi(f)(y) = \begin{cases} \prod_{[a, y]} (1 + \Delta f) e^{-\Delta f} & \text{for } y \in (a, b], \\ 1 & \text{for } y = a, \end{cases} \tag{4.3}$$

is well defined, $\phi \in \mathscr{W}_{p/2}$ and satisfies the relations

$$\phi(y) = \phi(y-)[1 + \Delta^- f(y)] e^{-\Delta^- f(y)} \quad \text{for } y \in (a, b], \tag{4.4}$$

$$\phi(y+) = \phi(y)[1 + \Delta^+ f(y)] e^{-\Delta^+ f(y)} \quad \text{for } y \in [a, b). \tag{4.5}$$

Proof. First we show that (4.3) is well defined. A Taylor series expansion with remainder yields

$$\xi(u) := (1 + u)e^{-u} = 1 - \theta(u)u^2,$$

where $1/(4\sqrt{e}) \leq \theta(u) \leq 3\sqrt{e}/4$ for $|u| \leq \frac{1}{2}$. It follows that

$$\sum_{[a, b] \setminus I} |1 - \xi(\Delta f)| \leq \frac{3\sqrt{e}}{4} \mathscr{O}_2^2(f) < \infty,$$

where $I = I_-(0.5; [a, b]) \cup I_+(0.5; [a, b])$. Therefore the products in (4.3) converge absolutely for every $y \in (a, b]$. Hence the function ϕ is well defined.

Next we show that

$$\phi(x-) = \prod_{(a,x)} (1 + \Delta f)e^{-\Delta f} \quad \text{for } x \in (a, b]. \tag{4.6}$$

Define, for $a \leq z < y \leq b$,

$$U(z, y) = \sup_{\delta > 0} \left| \prod_{I(\delta; [z, y])} (1 + \Delta f)e^{-\Delta f} \right|$$

and

$$\|U\|_\infty = \sup\{U(z, y) : a \leq z < y \leq b\}.$$

A Taylor series expansion with remainder gives

$$\log(1 + u) = u - \theta(u)u^2, \tag{4.7}$$

where $2/9 \leq \theta(u) \leq 2$ for $|u| \leq \frac{1}{2}$. Write

$$I^*(\delta; [z, y]) = [I_+(\delta; [z, y]) \setminus I_+(0.5; [z, y])] \cup [I_-(\delta; [z, y]) \setminus I_-(0.5; [z, y])].$$

For $a \leq z < y \leq b$ and $\delta \in (0, 0.5)$,

$$\begin{aligned} \left| \prod_{I(\delta; [z, y])} (1 + \Delta f)e^{-\Delta f} \right| &= \left| \prod_{I(0.5; [z, y])} (1 + \Delta f)e^{-\Delta f} \right| \exp \left\{ \sum_{I^*(\delta; [z, y])} [\log(1 + \Delta f) - \Delta f] \right\} \\ &\leq \sup_{a \leq z < y \leq b} \left\{ \left| \prod_{I(0.5; [z, y])} (1 + \Delta f)e^{-\Delta f} \right| \right\} \exp \left\{ -\frac{2}{9} \mathcal{O}_2^2(f) \right\} < \infty. \end{aligned}$$

Here we used the fact that $\mathcal{O}_2(f) < \infty$. We conclude that $\|U\|_\infty < \infty$.

Now we turn to the proof of (4.6). Assume $x \in (a, b]$. An application of the inequality $|e^u - 1| \leq |u|e^{|u|}$ for $u \in \mathbb{R}$ and (4.7) implies that

$$\begin{aligned} &\left| \prod_{I(\delta; [a, x])} (1 + \Delta f)e^{-\Delta f} - \prod_{I(\delta; [a, y])} (1 + \Delta f)e^{-\Delta f} \right| \\ &= \prod_{I(\delta; [a, y])} (1 + \Delta f)e^{-\Delta f} \left| \exp \left\{ \sum_{I(\delta; [y, x])} [\log(1 + \Delta f) - \Delta f] \right\} - 1 \right| \\ &\leq 2e^{2\mathcal{O}_2^2(f)} \|U\|_\infty \sum_{[y, x]} (\Delta f)^2 \tag{4.8} \end{aligned}$$

for every $\delta \in (0, 0.5)$ and $y \in (a, x)$ such that absolute values of all jumps on $[y, x)$ do not exceed 0.5. Letting $\delta \downarrow 0$, we obtain

$$\left| \prod_{[a, x]} (1 + \Delta f)e^{-\Delta f} - \phi(y) \right| \leq 2e^{2\mathcal{O}_2^2(f)} \|U\|_\infty \sum_{[y, x]} (\Delta f)^2.$$

Since $\mathcal{O}_2(f) < \infty$, choosing y close enough to x , we obtain that $\phi(x-)$ exists and (4.6) holds.

Notice that, for every $\delta > 0$,

$$\prod_{I(\delta; [a,x])} (1 + \Delta f)e^{-\Delta f} = (1 + \Delta^- f(x))e^{-\Delta^- f(x)} \prod_{I(\delta; [a,x])} (1 + \Delta f)e^{-\Delta f}.$$

Letting $\delta \downarrow 0$ and using (4.6), we obtain (4.4). Similarly, we can show that $\phi(x+)$ exists for $x \in [a, b)$ and (4.5) holds.

Next we show $\phi \in \mathcal{W}_{p/2}$. Since ϕ is regulated we may assume without loss of generality that jumps of size greater than 0.5 only appear at the end-points a and b . For any subdivision $\kappa = \{a = x_0 < x_1 < \dots < x_n = b\}$, we have

$$\sum_{i=1}^n |\phi(x_i) - \phi(x_{i-1})|^{p/2} = A + B + \sum_{i=2}^{n-1} |\phi(x_i) - \phi(x_{i-1})|^{p/2}, \tag{4.9}$$

where

$$A = |\phi(x_1) - \phi(a)|^{p/2} \leq |\phi(x_1) - \phi(a+)|^{p/2} + |\Delta^+ \phi(a)|^{p/2},$$

$$B = |\phi(b) - \phi(x_{n-1})|^{p/2} \leq |\Delta^- \phi(b)|^{p/2} + |\phi(b-) - \phi(x_{n-1})|^{p/2}.$$

Since $I(0.5; (a, b)) = \emptyset$, for each $i = 2, \dots, n - 1$ it follows as in (4.8) that

$$|\phi(x_i) - \phi(x_{i-1})| = \phi(x_{i-1}) \left| \prod_{[x_{i-1}, x_i]} (1 + \Delta f)e^{-\Delta f} - 1 \right|$$

$$\leq 2e^{2\mathcal{C}_2^2(f)} \|U\|_\infty \mathcal{C}_2^2(f; [x_{i-1}, x_i]).$$

We also have the bounds

$$|\phi(x_1) - \phi(a+)| = (1 + \Delta^+ f(a))e^{-\Delta^+ f(a)} \left| \prod_{(a, x_1]} (1 + \Delta f)e^{-\Delta f} - 1 \right|$$

$$\leq 2e^{2\mathcal{C}_2^2(f)} \|U\|_\infty \mathcal{C}_2^2(f; (a, x_1]),$$

$$|\phi(b-) - \phi(x_{n-1})| = \phi(x_{n-1}) \left| \prod_{[x_{n-1}, b)} (1 + \Delta f)e^{-\Delta f} - 1 \right|$$

$$\leq 2e^{2\mathcal{C}_2^2(f)} \|U\|_\infty \mathcal{C}_2^2(f; [x_{n-1}, b)).$$

By virtue of (4.9), it follows that

$$\sum_{i=1}^n |\phi(x_i) - \phi(x_{i-1})|^{p/2} \leq |\Delta^+ \phi(a)|^{p/2} + (2\|U\|_\infty)^{p/2} e^{p\mathcal{C}_2^2(f)} \sum_{(a,b)} |\Delta f|^p + |\Delta^- \phi(b)|^{p/2}.$$

Since κ is arbitrary and $f \in \mathcal{W}_p$, the bound implies that $v_{p/2}(\phi) < \infty$. The proof of Lemma 4.2 is now complete. □

Proof of Theorem 4.1. We only prove that part of the theorem which concerns (4.1). The other part is analogous. Bearing in mind what was said above about jumps of size -1 of f (Remark 1 following Theorem 4.1) and redefining the function f at the point b if necessary, we may and do assume that f has no jumps of size -1 on $[a, b]$.

The existence of $E_a(f)$ follows from Lemma 4.2. We only prove that $E_a(f)$ satisfies (4.1) by an application of the chain rule (3.1). The uniqueness of this solution follows from Theorem 5.21 in Dudley and Norvaiša (1999a).

Define the functions $g(u, v) = ve^u$ for $u, v \in \mathbb{R}$ and $h = (\psi, \phi)$, where $\psi(y) = f(y) - f(a)$, $y \in [a, b]$, and ϕ is defined by (4.3). By assumption on f , $\psi \in \mathscr{W}_p$. By Lemma 4.2, $\phi \in \mathscr{W}_{p/2}$. Thus $E_a(f) \in \mathscr{W}_p$, as a product of two functions from \mathscr{W}_p , and the conditions of Theorem 3.1 are satisfied for $g \circ f$ with $d = 2$. Notice that

$$g \circ h = g'_1 \circ h = E_a(f) \quad \text{and} \quad g'_2 \circ h = e^{f-f(a)} = e^\psi.$$

An application of the chain rule (3.1) yields

$$\begin{aligned} (g \circ h)(y) - (g \circ h)(a) &= E_a(f)(y) - 1 \\ &= (\text{LY}) \int_a^y E_a(f) \, df + (\text{LY}) \int_a^y e^\psi \, d\phi \\ &\quad + \sum_{(a,y)} [\Delta^- E_a(f) - (E_a(f))^- \Delta^- f - e^{\psi^-} \Delta^- \phi] \\ &\quad + \sum_{[a,y)} [\Delta^+ E_a(f) - E_a(f) \Delta^+ f - e^\psi \Delta^+ \phi], \end{aligned}$$

where the LY integrals are well defined and the two sums converge absolutely. By (4.4) and (4.5), it follows that

$$\Delta^- E_a(f)(x) = (E_a(f))^- \Delta^- f(x), \quad x \in (a, b], \tag{4.10}$$

$$\Delta^+ E_a(f)(x) = E_a(f)(x) \Delta^+ f(x), \quad x \in [a, b). \tag{4.11}$$

Thus we have to show that

$$(\text{LY}) \int_a^y e^\psi \, d\phi = \sum_{(a,y)} e^{\psi^-} \Delta^- \phi + \sum_{[a,y)} e^\psi \Delta^+ \phi \tag{4.12}$$

for each $y \in (a, b]$. For notational convenience, we proceed only for $y = b$. By the definition of the LY integral, (4.12) with $y = b$ is equivalent to

$$(\text{MPS}) \int_a^b e^{\psi^-} \, d\phi_b^+ = \sum_{(a,b]} e^{\psi^-} \Delta^- \phi_b^+, \tag{4.13}$$

where

$$(\Delta^- \phi_b^+)(x) = \begin{cases} \phi(x+) - \phi(x-) & \text{if } x \in (a, b), \\ (\Delta^- \phi)(b) & \text{if } x = b. \end{cases}$$

Choose an $\varepsilon > 0$. Since ϕ is regulated, by Lemma 2.1, one can find a subdivision $\chi = \{y_j: j = 0, 1, \dots, m\}$ of $[a, b]$ such that $\text{Osc}(\phi; [y_{j-1}+, y_j-]) < \varepsilon$ for all j . Since the MPS integral in (4.13) exists and the sum on the right-hand side converges absolutely, one can assume χ so refined that

$$\left| \sum_{i=1}^n e^{\psi^-(x_i)} [\phi_b^+(x_i) - \phi^+(x_{i-1})] - (\text{MPS}) \int_a^b e^{\psi_a^-} d\phi_b^+ \right| < \varepsilon$$

and

$$\left| \sum_{i=1}^n e^{\psi^-(x_i)} [\phi_b^+(x_i) - \phi(x_{i-1})] - \sum_{(a,b)} e^{\psi^-} \Delta^- \phi_b^+ \right| < \varepsilon$$

for all $\kappa = \{x_i: i = 1, \dots, n\} \supset \chi$. Therefore

$$\begin{aligned} & \left| \sum_{i=1}^n e^{\psi^-(x_i)} [\phi_b^+(x_i) - \phi^+(x_{i-1})] - \sum_{i=1}^n e^{\psi^-(x_i)} [\phi_b^+(x_i) - \phi^-(x_i)] \right| \\ &= \left| \sum_{i=1}^n e^{\psi^-(x_i)} [\phi(x_{i-1}) - \phi(x_{i-1}+)] \right| \\ &\leq e^{\|\psi\|_\infty} \max_{1 \leq i \leq n} (\text{Osc}(\phi; [x_{i-1}+, x_i-]))^{1-p/2} \sum_{i=1}^n |\phi(x_{i-1}) - \phi(x_{i-1}+)|^{p/2} \\ &\leq \varepsilon^{1-p/2} e^{\|\psi\|_\infty} v_{p/2}(\phi). \end{aligned}$$

Since $p < 2$, $v_{p/2}(\phi) < \infty$ and ε is arbitrary we conclude that (4.13) holds. This completes the proof of Theorem 4.1. \square

Remarks. (1) For a real-valued function f on $[a, b]$ the *product integral with respect to f* on $[a, b]$ is defined as the limit

$$\lim_{(\kappa)} \prod_{i=1}^n [1 + f(x_i) - f(x_{i-1})], \tag{4.14}$$

if it exists under refinements of subdivisions $\kappa = \{a = x_0 < \dots < x_n = b\}$. By Theorem 4.4 and Lemma 2.14 of Dudley and Norvaiša (1999a), if $f \in \mathscr{W}_p$ for some $p \in (0, 2)$, then the product integral (4.14) exists and is equal to $E_a(f)(b)$. Moreover, by Theorem 4.26 of Dudley and Norvaiša (1999a), if f is in addition either right- or left-continuous, then the limit (4.14) exists if and only if there exists the limit

$$\lim_{|\kappa| \downarrow 0} \prod_{i=1}^n [1 + f(x_i) - f(x_{i-1})],$$

where the mesh $|\kappa| = \max_i(x_i - x_{i-1})$, and both are equal. The latter relation allows one to calculate numerically the solutions of homogeneous linear integral equations in an intuitively appealing way.

(2) Dudley and Norvaiša solve (4.1) when f and F take values in a (possibly non-

commutative) Banach algebra. Their method of proof is based on establishing a Duhamel identity from which the solution of a linear integral equation is derived as a special case.

Next we consider *non-homogeneous linear integral equations*. As before, assume that the functions F, G, f and g are regulated on $[a, b]$. We say that the function F , LY integrable with respect to f on $[a, b]$, satisfies the *non-homogeneous forward linear integral equation* if, for all $y \in [a, b]$,

$$F(y) = F(a) + (\text{LY}) \int_a^y F \, df + g(y) - g(a). \tag{4.15}$$

Similarly, we say that the function G , RY integrable with respect to f on $[a, b]$ satisfies the *non-homogeneous backward linear integral equation* if, for all $y \in [a, b]$,

$$G(y) = G(b) + (\text{RY}) \int_y^b G \, df + g(b) - g(y). \tag{4.16}$$

We will solve these non-homogeneous linear integral equations assuming $f, g \in \mathscr{W}_p$ for some $p \in (0, 2)$ and that the solutions of the corresponding homogeneous linear integral equations do not vanish.

Theorem 4.3. *Assume $f, g \in \mathscr{W}_p$ for some $0 < p < 2$ and $\Delta^+ f(x) \neq -1 \neq \Delta^- f(y)$ for $a \leq x < b$ and $a < y \leq b$. Then the following hold:*

(i) *The function $E_a(f)$ exists and does not vanish at any point of $[a, b]$, the integral $(\text{LY}) \int_a^b (E_a(f))^{-1} \, dg$ exists, the sums*

$$\sum_{(a,b]} \frac{\Delta^- f \Delta^- g}{E_a(f)}, \quad \sum_{[a,b)} \frac{\Delta^+ f \Delta^+ g}{(E_a(f))^+} \tag{4.17}$$

converge absolutely and the function F given on $[a, b]$ by

$$F(y) = E_a(f)(y) \left[F(a) + (\text{LY}) \int_a^y \frac{dg}{E_a(f)} - \sum_{(a,y]} \frac{\Delta^- f \Delta^- g}{E_a(f)} - \sum_{[a,y)} \frac{\Delta^+ f \Delta^+ g}{(E_a(f))^+} \right]$$

is the unique solution of (4.15) in \mathscr{W}_r for any $r \geq p$ with $p^{-1} + r^{-1} > 1$.

(ii) *The function $E^b(f)$ exists and does not vanish at any point of $[a, b]$, the integral $(\text{RY}) \int_a^b (E^b(f))^{-1} \, dg$ exists, the sums*

$$\sum_{(a,b]} \frac{\Delta^- f \Delta^- g}{(E^b(f))^-}, \quad \sum_{[a,b)} \frac{\Delta^+ f \Delta^+ g}{E^b(f)}$$

converge absolutely and the function G given on $[a, b]$ by

$$G(y) = E^b(f)(y) \left[G(b) + (\text{RY}) \int_y^b \frac{dg}{E^b(f)} + \sum_{(y,b]} \frac{\Delta^- f \Delta^- g}{(E^b(f))^-} + \sum_{[y,b)} \frac{\Delta^+ f \Delta^+ g}{E^b(f)} \right]$$

is the unique solution of (4.16) in \mathscr{W}_r for any $r \geq p$ with $p^{-1} + r^{-1} > 1$.

Remark. The solution of a non-homogeneous linear stochastic differential equation analogous to (4.15) is given in Jacod (1979, p. 194).

Proof. We only prove part (i) of the theorem because the proof of part (ii) is similar. The existence of $E_a(f)$ in \mathscr{W}_p follows from Theorem 4.1. The function $E_a(f)^{-1}$ is bounded since, for each $y \in [a, b]$,

$$|E_a(f)(y)| \geq \exp\{f(y) - f(a) - 2\mathcal{O}_2^2(f)\} \left| \prod_{I(0.5;[a,y])} (1 + \Delta f) e^{-\Delta f} \right| \geq C,$$

for some $C > 0$, where we have made use of (4.7) and the fact that f has no jumps of size -1 . Then $(E_a(f))^{-1} \in \mathscr{W}_p$ and $(E_a(f))^{-1}$ is LY integrable with respect to g by Theorem 2.8. The absolute convergence of the sums (4.17) follows from Hölders inequality in the form of Young (1936, p. 252).

The proof of the uniqueness of the solution (4.15) follows as in the proof of Theorem 5.21 in Dudley and Norvaiša (1999a). Thus it remains to show that F is indeed a solution. We again apply the chain rule. Consider the composition $g \circ h$ of the functions $g(u, v) = uv$ for $u, v \in \mathbb{R}$ and $h(y) = (h_1(y), h_2(y))$ for $y \in [a, b]$, where $h_1(y) = E_a(f)(y)$ and

$$h_2(y) = F(a) + (\text{LY}) \int_a^y \frac{dg}{E_a(f)} - \sum_{(a,y]} \frac{\Delta^- f \Delta^- g}{E_a(f)} - \sum_{[a,y)} \frac{\Delta^+ f \Delta^+ g}{(E_a(f))^+}.$$

For each $y \in [a, b]$, let

$$U_1(y) = \sum_{(a,b]} \frac{\Delta^- f \Delta^- g}{E_a(f)} \quad \text{and} \quad U_2(y) = \sum_{[a,y)} \frac{\Delta^+ f \Delta^+ g}{(E_a(f))^+}.$$

Note that $U_1(a) = U_2(a) = 0$. Using Hölder's inequality as above, we can show that $U_1, U_2 \in \mathscr{W}_p$. Thus $h_2 \in \mathscr{W}_p$ because the indefinite left Young integral has bounded p -variation by Proposition 3.32 of Dudley and Norvaiša (1999a). Note that $F = g \circ h = h_1 h_2$. Since $E_a(f)$ is the solution of the homogeneous equation (4.1), the substitution rule for the left Young integrals (see Theorem 9 in Norvaiša 1998) yields

$$(\text{LY}) \int_a^y (g'_1 \circ h) dh_1 = (\text{LY}) \int_a^y h_2 dE_a(f) = (\text{LY}) \int_a^y F df.$$

Another application of the substitution rule implies that

$$\begin{aligned} (\text{LY}) \int_a^y (g'_2 \circ h) dh_2 &= (\text{LY}) \int_a^y E_a(f) dh_2 \\ &= g(y) - g(a) - (\text{LY}) \int_a^y E_a(f) dU_1 - (\text{LY}) \int_a^y E_a(f) dU_2. \end{aligned}$$

We show next that, for each $y \in [a, b]$,

$$(LY) \int_a^y E_a(f) dU_1 = (MPS) \int_a^y (E_a(f))_a^- dU_1 = \sum_{(a,y]} \frac{(E_a(f))^-}{E_a(f)} \Delta^- f \Delta^- g. \tag{4.18}$$

The first equality holds because U_1 is right-continuous. To show the second one, fix $y \in (a, b]$. Given a finite set $\mu = \{z_j : j = 1, \dots, m\} \subset (a, y]$ and a subdivision $\kappa = \{x_i : i = 0, \dots, n\}$ of $[a, b]$ such that $\mu \subset \kappa$, for each $j = 1, \dots, m$, let $i(j) \in \{1, \dots, n\}$ be an integer such that $x_{i(j)} = z_j$. Then, for a Riemann–Stieltjes sum based on κ and on an intermediate subdivision $\sigma = \{y_i : i = 1, \dots, n\}$ of κ , we have

$$\begin{aligned} & \left| S((E_a(f))_a^-, U_1, \kappa, \sigma) - \sum_{(a,y]} \frac{(E_a(f))^-}{E_a(f)} \Delta^- f \Delta^- g \right| \\ & \leq R(\mu) + \left| \sum_{i=1}^n (E_a(f))_a^-(y_i) \sum_{(x_{i-1}, x_i]} \frac{\Delta^- f \Delta^- g}{E_a(f)} - \sum_{\mu} \frac{(E_a(f))^-}{E_a(f)} \Delta^- f \Delta^- g \right| \\ & \leq 2R(\mu) + \sum_{j=1}^m |(E_a(f))_a^-(y_{i(j)}) - (E_a(f))_a^-(x_{i(j)})| \frac{|\Delta^- f(z_j) \Delta^- g(z_j)|}{|E_a(f)(z_j)|}, \end{aligned}$$

where

$$R(\mu) = C^{-1} \|E_a(f)\|_{\infty} \sum_{(a,y] \setminus \mu} |\Delta^- f \Delta^- g|.$$

We can make the right-hand side of the last bound arbitrarily small by taking first μ so that $R(\mu)$ is small and then taking $\kappa \supset \mu$ so that $x_{i(j)} - x_{i(j)-1}$ is small. This will also make the sum on the right-hand side small because $(E_a(f))_a^-$ is left-continuous. Therefore (4.18) holds for each $y \in [a, b]$. Since $\Delta^+ U_2 = (\Delta^+ f \Delta^+ g)/(E_a(f))^+$, by Definition 2.5, for each $y \in [a, b]$, we have

$$\begin{aligned} (LY) \int_a^y E_a(f) dU_2 &= (MPS) \int_a^y (E_a(f))_a^- d(U_2)_y^+ + \left[\frac{E_a(f)}{(E_a(f))^+} \Delta^+ f \Delta^+ g \right] (a) \\ & \quad + \sum_{(a,y)} \frac{\Delta^- E_a(f)}{(E_a(f))^+} \Delta^+ f \Delta^+ g. \end{aligned}$$

Using the left-continuity of $(E_a(f))_a^-$, we can show in the same way as for the second equality in (4.18) that

$$(MPS) \int_a^y (E_a(f))_a^- d(U_2)_y^+ = \sum_{(a,y)} \frac{(E_a(f))^-}{(E_a(f))^+} \Delta^+ f \Delta^+ g.$$

Therefore, for each $y \in [a, b]$, we have

$$(LY) \int_a^y E_a(f) dU_2 = \sum_{[a,y)} \frac{E_a(f)}{(E_a(f))^+} \Delta^+ f \Delta^+ g. \tag{4.19}$$

By (4.10), (4.11) and Lemma 2.6, because U_1 is right-continuous and U_2 is left-continuous, it follows that

$$\begin{aligned} \Delta^-(g \circ h) - (g'_1 \circ h)^- \Delta^- h_1 - (g'_2 \circ h)^- \Delta^- h_2 &= \Delta^- h_1 \Delta^- h_2 \\ &= \Delta^- f \Delta^- g - \Delta^- E_a(f) \Delta^- U_1 \\ &= \frac{E_a(f)^-}{E_a(f)} \Delta^- f \Delta^- g \end{aligned}$$

and

$$\begin{aligned} \Delta^+(g \circ h) - (g'_1 \circ h) \Delta^+ h_1 - (g'_2 \circ h) \Delta^+ h_2 &= \Delta^+ h_1 \Delta^+ h_2 \\ &= \Delta^+ f \Delta^+ g - \Delta^+ E_a(f) \Delta^+ U_2 \\ &= \frac{E_a(f)}{E_a(f)^+} \Delta^+ f \Delta^+ g, \end{aligned}$$

on $(a, b]$ and $[a, b)$, respectively. It finally follows from the chain rule for $g \circ f$ that, for each $y \in [a, b]$,

$$\begin{aligned} F(y) - F(a) &= (g \circ h)(y) - (g \circ h)(a) \\ &= (\text{LY}) \int_a^y (g'_1 \circ h) dh_1 + (\text{LY}) \int_a^y (g'_2 \circ h) dh_2 \\ &\quad + \sum_{(a,y]} [\Delta^-(g \circ h) - (g'_1 \circ h)^- \Delta^- h_1 - (g'_2 \circ h)^- \Delta^- h_2] \\ &\quad + \sum_{[a,y)} [\Delta^+(g \circ h) - (g'_1 \circ h) \Delta^+ h_1 - (g'_2 \circ h) \Delta^+ h_2] \\ &= (\text{LY}) \int_a^y F df + g(y) - g(a) - (\text{LY}) \int_a^y E_a(f) dU_1 - (\text{LY}) \int_a^y E_a(f) dU_2 \\ &\quad + \sum_{(a,y]} \frac{(E_a(f))^-}{E_a(f)} \Delta^- f \Delta^- g + \sum_{[a,y)} \frac{E_a(f)}{(E_a(f))^+} \Delta^+ f \Delta^+ g \\ &= (\text{LY}) \int_a^y F df + g(y) - g(a). \end{aligned}$$

The last equality holds by (4.18) and (4.19). This concludes the proof of Theorem 4.3. \square

We next show that a solution of the non-homogeneous linear equation (4.15) need not vanish at points following jumps of f of size -1 as in the case of homogeneous linear equations (see Remark 1 following Theorem 4.1). Due to the regularity of f there are at most finitely many jumps of size -1 . Suppose first that $\Delta^- f(x) = -1$ for some $x \in (a, b)$. Consider any $y \in [x, b]$ such that $\Delta^-_{[x,y]} f$ and $\Delta^+_{[x,y]} f$ do not assume the value -1 on $[x, y]$, so that $f, g \in \mathcal{W}_p$ satisfy the conditions of Theorem 4.3 on the interval $[x, y]$. If F is a solution of (4.15) then by adding and subtracting $F(x)$ and $F(x-)$ we obtain

$$\begin{aligned}
 F(y) &= F(y) - F(x) - \Delta^- F(x) + F(x-) = F(y) - F(x) + \Delta^- g(x) \\
 &= \Delta^- g(x) + (\text{LY}) \int_x^y F \, df + g(y) - g(x) \\
 &= E_x(f)(y) \left[\Delta^- g(x) + (\text{LY}) \int_x^y \frac{dg}{E_x(f)} - \sum_{(x,y]} \frac{\Delta^- f \Delta^- g}{E_x(f)} - \sum_{[x,y)} \frac{\Delta^+ f \Delta^+ g}{(E_x(f))^+} \right].
 \end{aligned}$$

The second equality follows from Lemma 2.6. The third follows by additivity of the left Young integral (cf. Proposition 3.25 of Dudley and Norvaiša 1999a), while the last equality is a consequence of Theorem 4.3 with $a = x$ and $F(a) = \Delta^- g(x)$.

Suppose now that $\Delta^+ f(x) = -1$ for some $x \in [a, b)$. Consider any $y \in (x, b]$ such that $\Delta_{(x,y]}^- f$ and $\Delta_{(x,y]}^+ f$ do not assume the value -1 on $(x, y]$. To provide the solution of (4.15) in this case we need auxiliary functions \tilde{f}, \tilde{g} on $[x, y]$ defined by $\tilde{f} = f, \tilde{g} = g$ on $(x, y]$ and $\tilde{f}(x) = f(x+), \tilde{g}(x) = g(x+)$. Notice that $\tilde{f}_y^+ = f_y^+$ on $[x, y]$ and \tilde{f}, \tilde{g} satisfy the conditions of Theorem 4.3 on the interval $[x, y]$. If F is the solution of (4.15) then, as in the previous case, it follows that

$$\begin{aligned}
 F(y) &= F(y) - F(x+) - \Delta^+ F(x) + F(x) = F(y) - F(x+) + \Delta^+ g(x) \\
 &= \Delta^+ g(x) + (\text{LY}) \int_x^y F \, d\tilde{f} + \tilde{g}(y) - \tilde{g}(x) \\
 &= E_x(\tilde{f})(y) \left[\Delta^+ g(x) + (\text{LY}) \int_x^y \frac{d\tilde{g}}{E_x(\tilde{f})} - \sum_{(x,y]} \frac{\Delta^- \tilde{f} \Delta^- \tilde{g}}{(E_x(\tilde{f}))^-} - \sum_{[x,y)} \frac{\tilde{f} \Delta^+ \tilde{g}}{(E_x(\tilde{f}))^+} \right].
 \end{aligned}$$

We can give another form to the solution F by letting

$$E_{x+}(f) := E_x(\tilde{f}) \quad \text{and} \quad (\text{LY}) \int_{x+}^y h \, df := (\text{LY}) \int_a^y h \, d\tilde{f}.$$

The first definition is a natural one for the product over the interval $[x+, y]$, while the second definition can be justified as an analogue to the central Young integral given by Lemma 3.24 in Dudley and Norvaiša (1999a). Then the above expression for the solution $F(y)$ is equal to

$$E_{x+}(f)(y) \left[\Delta^+ g(x) + (\text{LY}) \int_{x+}^y \frac{dg}{E_{x+}(f)} - \sum_{(x,y]} \frac{\Delta^- f \Delta^- g}{E_{x+}(f)} - \sum_{(x,y)} \frac{\Delta^+ f \Delta^+ g}{(E_{x+}(f))^+} \right].$$

We finish this section by illustrating Theorem 4.3 in a simple situation.

Example. Assume A, C Riemann integrable and $D, h \in \mathscr{W}_p$ for some $0 < p < 2$. To solve the equation

$$F(y) = F(a) + \int_a^y [A(x)F(x) + C(x)] \, dx + (\text{LY}) \int_a^y D \, dh, \quad y \in [a, b],$$

we apply Theorem 4.3 with

$$g(y) = \int_a^y C(x) dx + (LY) \int_a^y D dh \quad \text{and} \quad f(y) = \int_a^y A(x) dx.$$

Notice that f is continuous and has bounded variation. By Proposition 3.32 of Dudley and Norvaiša (1999a), g has bounded $(1 \vee p)$ -variation. Theorem 4.3 then yields the solution

$$F(y) = e^{\int_a^y A(z) dz} \left[F(a) + \int_a^y C(x) e^{-\int_a^x A(z) dz} dx + (LY) \int_a^y D(x) e^{-\int_a^x A(z) dz} dh(x) \right]$$

which is unique in \mathscr{H}_r for any $r \in [1 \vee p, p^*)$, where

$$p^* = \begin{cases} p/(p-1) & \text{if } 1 < p < \infty, \\ \infty & \text{if } 0 < p \leq 1. \end{cases} \tag{4.20}$$

5. Applications to stochastic integral equations

5.1. The Langevin equation

Consider the *Langevin equation*

$$\dot{u}(t) = -\beta u(t) + L(t) \tag{5.1}$$

describing the velocity $u(t) = \dot{x}(t)$ of a particle with x -coordinate $x(t)$ at time t , while L represents the random force acting on the particle. This equation is symbolic in so far as u has no time derivative if one assumes that L exhibits highly erratic behaviour. For a Lévy process X , Doob (1942) wrote the Langevin equation in the form

$$du(t) = -\beta u(t) dt + dX(t) \tag{5.2}$$

meaning that, for continuous f and $a < b$,

$$\int_a^b f(t) du(t) = -\beta \int_a^b f(t) u(t) dt + \int_a^b f(t) dX(t)$$

with probability 1. The first two integrals are defined as limits in probability of the corresponding RS sums, and the third (stochastic) integral is defined as proposed by Wiener and Paley (1934, pp. 151–157), and Doob (1937, pp. 131–134). Then (5.2) has solution (cf. Doob 1942, p. 360)

$$u(t) = e^{-\beta t} \left[u(0) + \int_0^t e^{\beta s} dX(s) \right], \quad t \geq 0,$$

with probability 1. Doob (1942) gave a detailed description of the properties of u when X is symmetric α -stable Lévy motion with $0 < \alpha \leq 2$. He called u an $OU(\alpha)$ process if $0 < \alpha \leq 2$ or simply an OU process if $\alpha = 2$. According to Doob, the description of the OU process goes back at least to Smoluchowski, although it was first derived by Ornstein and Uhlenbeck as the process describing the velocity of a particle in Brownian motion.

In stochastic analysis Itô's integral is used to model the random force L in (5.1). The Langevin equation then takes on the form

$$u(t) = u(0) - \beta \int_0^t u(s) ds + (\text{I}) \int_0^t \sigma(s) dB(s), \quad (5.3)$$

where B is standard Brownian motion and σ is a deterministic measurable, locally bounded function; cf. Section 5.6 of Karatzas and Shreve (1991). The unique solution of (5.3) is given by

$$u(t) = e^{-\beta t} \left[u(0) + (\text{I}) \int_0^t e^{\beta s} \sigma(s) dB(s) \right], \quad t \geq 0. \quad (5.4)$$

The process

$$g(t) = (\text{I}) \int_0^t \sigma(s) dB(s), \quad 0 \leq t \leq T,$$

is a semimartingale and, by Theorem 1 in Lépingle (1976), $g \in \mathscr{W}'_p = \mathscr{W}'_p([0, T])$ for $2 < p < \infty$ with probability 1. Thus, by Theorem 2.4(i), the integral (RS) $\int_0^t \exp\{\beta s\} dg(s)$ exists on $[0, T]$ path by path. By associativity of the Itô integral, and since the Itô integral is the limit in probability of certain RS sums, we also have on $[0, T]$,

$$(\text{I}) \int_0^t e^{\beta s} \sigma(s) dB(s) = (\text{I}) \int_0^t e^{\beta s} dg(s) = (\text{RS}) \int_0^t e^{\beta s} dg(s).$$

So the function (5.4) has RS integral representation

$$u(t) = e^{-\beta t} \left[u(0) + (\text{RS}) \int_0^t e^{\beta s} dg(s) \right], \quad 0 \leq t \leq T. \quad (5.5)$$

In an insurance context and using different arguments, this was observed by Harrison (1977, Proposition 2.1).

Given a (deterministic) function $g \in \mathscr{W}'_p$, we say that u satisfies the *Langevin equation* if

$$u(t) = u(0) - \beta \int_0^t u(s) ds + g(t) - g(0), \quad 0 \leq t \leq T. \quad (5.6)$$

If g is a stochastic process, we apply (5.6) path by path. Define p^* by (4.20).

Theorem 5.1. *Let $g \in \mathscr{W}'_p$ for some $0 < p < 2$. Then (5.5) is the unique solution in \mathscr{W}'_r of (5.6) for any $r \in [1 \vee p, p^*]$.*

Proof. We apply Theorem 4.3 with $f(t) = -\beta t$, so that $f, g \in \mathscr{W}'_{1 \vee p}$. The value of a Riemann integral is the limit of Riemann sums when the mesh of the subdivisions converges to zero. Alternatively, this value is the limit of the Riemann sums under refinements of the subdivisions. Bearing this fact in mind and utilizing the fact that Riemann integrable functions are bounded and continuous almost everywhere, we obtain

$$(LY) \int_0^t u(s) d(-\beta s) = -\beta \int_0^t u_0^-(s) ds = -\beta \int_0^t u(s) ds.$$

Thus (5.6) is a particular case of (4.15). By Theorem 4.3, $\exp\{\beta t\}$ is LY integrable with respect to g , and

$$u(t) = e^{-\beta t} \left[u(0) + (LY) \int_0^t e^{\beta s} dg(s) \right] \tag{5.7}$$

is the solution in \mathscr{W}_r , $r \in [1 \vee p, p^*]$, of (5.6). By Definition 2.5 above and Lemma 3.2 of Dudley and Norvaiša (1999a), we have

$$(LY) \int_0^t e^{\beta s} dg(s) = (MPS) \int_0^t e^{\beta s} dg_t^+(s) + (\Delta^+ g)(0) = (MPS) \int_0^t e^{\beta s} dg_{0,t}^+(s),$$

where

$$g_{0,t}^+(s) = g(s+) \quad \text{for } s \in (0, t), \quad g_{0,g}^+(0) = g(0), \quad g_{0,t}^+(t) = g(t).$$

By Theorem 3.9 and Corollary 3.18 of Dudley and Norvaiša (1999a) and by Theorem II.10.9 in Hildebrandt (1963), for every $t \in [0, T]$,

$$(MPS) \int_0^t e^{\beta s} dg_{0,t}^+(s) = (MPS) \int_0^t e^{\beta s} dg(s) = (RS) \int_0^t e^{\beta s} dg(s).$$

By virtue of (5.7), (5.5) is the solution of (5.6) as stated. □

Next we give a solution to the Langevin equation when the random force L is modelled by a particular non-semimartingale. The following equation and its solution should be compared with the corresponding expression from Itô calculus; cf. (5.3) and (5.4).

Proposition 5.2. *Let B_H be fractional Brownian motion on $[0, T]$ with index $H \in (0.5, 1)$ and $\sigma \in \mathscr{W}_{p^*}$ for some $p \in (H^{-1}, \infty)$. Then σ is RS integrable with respect to almost all sample paths of B_H , and the equation*

$$u(t) = u(0) - \beta \int_0^t u(s) ds + (RS) \int_0^t \sigma(s) dB_H(s), \quad 0 \leq t \leq T, \tag{5.8}$$

has the unique solution

$$u(t) = e^{-\beta t} \left[u(0) + (RS) \int_0^t e^{\beta s} \sigma(s) dB_H(s) \right], \quad 0 \leq t \leq T, \tag{5.9}$$

in \mathscr{W}_r for any $r \in (H^{-1}, (1 - H)^{-1})$.

Proof. Let $p' \in (H^{-1}, p)$. Then $(p^*)^{-1} + (p')^{-1} > (p^*)^{-1} + p^{-1} = 1$. Since almost all sample paths of B_H are continuous and have bounded p' -variation (cf. Proposition 2.2), the RS integral in (5.8) exists by Theorem 2.4(i) path by path. By Proposition 3.32 of Dudley and Norvaiša (1999a), the indefinite integral $g(t) = (RS) \int_0^t \sigma dB_H$ is in \mathscr{W}_q with probability 1 for any $q \in (H^{-1}, 2)$. Thus, by Theorem 5.1 and the substitution rule for RS integrals, (5.9) is

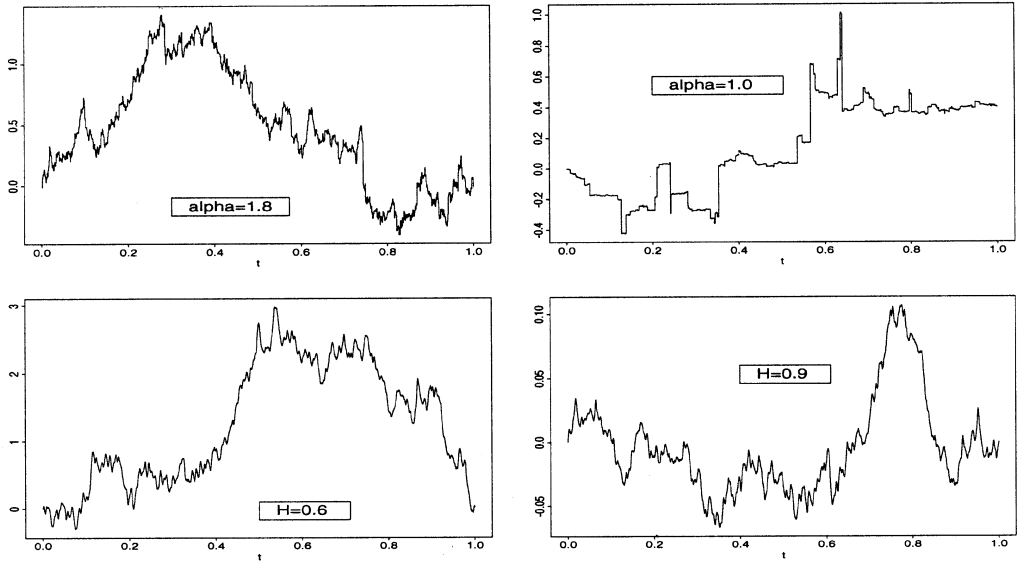


Figure 1. Solution to (5.10) (top) and (5.8) (bottom) with $\beta = -0.01$ and $\sigma \equiv 0.01$.

the unique solution of (5.8) in \mathscr{W}_r for any $r \in [q, q^*) \subset (H^{-1}, (1 - H)^{-1})$. This implies Proposition 5.2. \square

Fractional Brownian motion B_H in (5.8) may be considered as the driving process in the Langevin equation. If the sample paths of the driving process are discontinuous, then extended RS integrals can replace the ordinary RS integral. We illustrate this approach for symmetric α -stable Lévy motion as the driving process. It can be extended to larger classes of Lévy processes; see Section 2.2.

Proposition 5.3. *Let X_α be symmetric α -stable Lévy motion with $\alpha \in (0, 2)$ and $\sigma \in \mathscr{W}_{p^*}$ for some $p \in (\alpha, \infty)$. Then the integral (MPS) $\int_0^T \sigma(s-) dX_\alpha(s)$ with $\sigma(0-) = \sigma(0)$ exists for almost all sample paths of X_α and the equation*

$$u(t) = u(0) - \beta \int_0^t u(s) ds + (\text{MPS}) \int_0^t \sigma(s-) dX_\alpha(s), \quad 0 \leq t \leq T, \quad (5.10)$$

has the unique solution

$$u(t) = e^{-\beta t} \left[u(0) + (\text{MPS}) \int_0^t e^{\beta s} \sigma(s-) dX_\alpha(s) \right], \quad 0 \leq t \leq T,$$

in \mathscr{W}_r for any $r \in (1 \vee \alpha, \alpha^*)$.

Proof. One can follow the lines of the proof of Proposition 5.2; instead of part (i) use part (ii) of Theorem 2.4 and instead of Proposition 2.2 use Proposition 2.3. \square

5.2. Equations with multiplicative noise

The random force L in the Langevin equation is called additive because its contribution to the solution is additive. Next we consider integral equations with multiplicative noise.

Proposition 5.4. *Let B_H be fractional Brownian motion, $H \in (0.5, 1)$. For almost all sample paths of B_H , the equation*

$$F(t) = 1 + c \int_0^t F(s) ds + (RS) \int_0^t \gamma F(s) dB_H(s), \quad 0 \leq t \leq T, \tag{5.11}$$

has the unique solution $F_{c,\gamma}(t) = e^{ct + \gamma B_H(t)}$ in \mathscr{W}_r for any $r \in (H^{-1}, (1 - H)^{-1})$.

Proof. Let $f(t) = ct + \gamma B_H(t)$ and $p \in (H^{-1}, 2)$. By Proposition 2.2, f is continuous with probability 1 and in \mathscr{W}_p . By Definition 2.5, Lemma 2.9 above and Theorem II.10.9 in Hildebrandt (1963), the following integrals exist and satisfy the relation

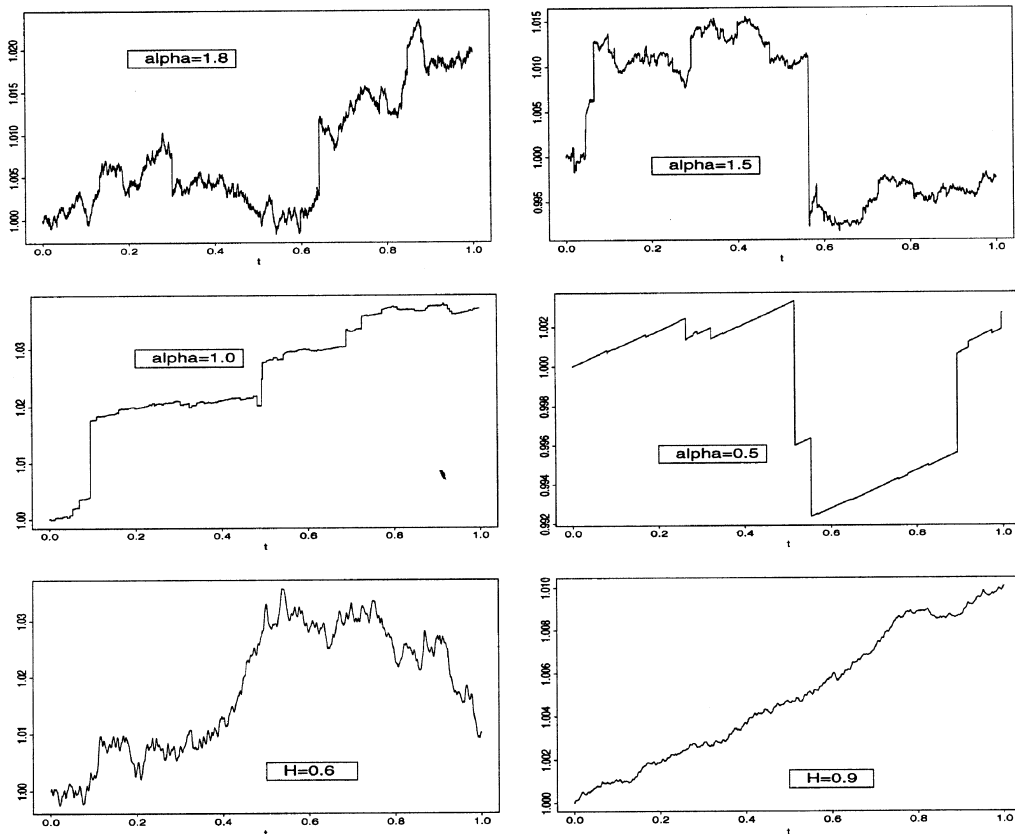


Figure 2. Solution to (5.12) (top and middle) and (5.11) (bottom) with $\gamma = 0.01$ and $c = 0.01$.

$$(LY) \int_0^t F df = (MPS) \int_0^t F_0^- df = c \int_0^t F(s) ds + (RS) \int_0^t \gamma F dB_H$$

with probability 1 for $F \in \mathscr{W}_p$. Thus the statement follows from Theorem 4.1. □

Proposition 5.5. *Let X_α be α -stable Lévy motion with $\alpha \in (0, 2)$. For almost all sample paths of X_α the equation*

$$F(t) = 1 + c \int_0^t F(s) ds + (MPS) \int_0^t \gamma F(s-) dX_\alpha(s), \quad 0 \leq t \leq T, \tag{5.12}$$

with $F(0-) = F(0)$, has the unique solution

$$F_{c,\gamma}(t) = e^{ct + \gamma X_\alpha(t)} \prod_{[0,t]} (1 + \gamma \Delta^- X_\alpha) e^{-\gamma \Delta^- X_\alpha}, \quad 0 \leq t \leq T, \tag{5.13}$$

in \mathscr{W}_r for any $r \in (1 \vee \alpha, \alpha^*)$. If X_α is symmetric, $F_{0,1}$ has representation

$$F_{0,1}(t) = \lim_{\delta \downarrow 0} \prod_{I(\delta, [0,t])} (1 + \Delta^- X_\alpha), \quad 0 \leq t \leq T.$$

Proof. Let $f(t) = ct + \gamma X_\alpha(t)$ and $p \in (1 \vee \alpha, 2)$. By Proposition 2.3 and the discussion preceding it, f is right-continuous with probability 1 and in \mathscr{W}_p . By Definition 2.5, Lemma 2.9 above and Theorem II.10.9 in Hildebrandt (1963), the following integrals exist and satisfy the relation

$$(LY) \int_0^t F df = (MPS) \int_0^t F_0^- df = c \int_0^t F(s) ds + (MPS) \int_0^t \gamma F(s-) dX_\alpha(s)$$

with probability 1 for $F \in \mathscr{W}_p$. By Theorem 4.1, $F_{c,\gamma}$ in (5.13) is a solution to (5.12) as stated.

If X_α is symmetric, we have, for every $\delta > 0$ and $0 \leq t \leq T$,

$$\prod_{I(\delta, [0,t])} (1 + \Delta^- X_\alpha) = e^{X_\alpha(t)} \prod_{I(\delta, [0,t])} (1 + \Delta^- X_\alpha) e^{-\Delta^- X_\alpha} \exp \left\{ \sum_{I(\delta, [0,t])} \Delta^- X_\alpha - X_\alpha(t) \right\}.$$

Letting $\delta \downarrow 0$ and using the Lévy–Itô representation (2.3), we arrive at the desired relation for $F_{0,1}$.

6. Concluding remarks

The referees of this paper were so kind as to point out some related literature which we included above. According to one of the referees, Klungenhöfer and Zähle (1999) deal with nonlinear equations where the driving process is Hölder continuous of order greater than 0.5.

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References

- Barndorff-Nielsen, O.E. (1978) Hyperbolic distributions and distributions on hyperbolae. *Scand. J. Statist.*, **5**, 151–157.
- Barndorff-Nielsen, O.E. (1986) Sand, wind and statistics. *Acta Mech.*, **64**, 1–18.
- Barndorff-Nielsen, O.E. (1995) Normal/inverse Gaussian processes and the modelling of stock returns. Research Report No. 300, Department of Theoretical Statistics, Aarhus University.
- Barndorff-Nielsen, O.E. (1997) Normal inverse Gaussian distributions and stochastic volatility modelling. *Scand. J. Statist.*, **24**, 1–13.
- Bertoin, J. (1996) *Lévy Processes*. Cambridge: Cambridge University Press.
- Bretagnolle, J. (1972) p -variation de fonctions aléatoires, 2ème partie: processus à accroissements indépendants. In *Séminaire de Probabilités VI*, Lecture Notes in Math. 258, pp. 64–71. Berlin: Springer-Verlag.
- Cutland, N.J., Kopp, P.E. and Willinger, W. (1995) Stock price returns and the Joseph effect: a fractional version of the Black–Scholes model. In E. Bolthausen, M. Dozzi and F. Russo (eds), *Seminar on Stochastic Analysis, Random Fields and Applications*, Progress in Probability 36, pp. 327–351. Basel: Birkhäuser.
- Dai, W. and Heyde, C.C. (1996) Itô's formula with respect to fractional Brownian motion and its application. *J. Appl. Math. Stochastic Anal.*, **9**, 439–448.
- Doléans-Dade, C. (1970) Quelques applications de la formule de changement de variables pour les semimartingales. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **16**, 181–194.
- Doob, J.L. (1937) Stochastic processes depending on a continuous parameter. *Trans. Amer. Math. Soc.*, **42**, 107–140.
- Doob, J.L. (1942) The Brownian movement and stochastic equations. *Ann. Math.*, **43**, 351–369.
- Dudley, R.M. (1992) Fréchet differentiability, p -variation and uniform Donsker classes. *Ann. Probab.*, **20**, 1968–1982.
- Dudley, R.M. and Norvaiša, R. (1999a) Product integrals, Young integrals and p -variation. In R.M. Dudley and R. Norvaiša, *Differentiability of Six Operators on Nonsmooth Functions and p -Variation*, Lecture Notes in Math. 1703. New York: Springer-Verlag.
- Dudley, R.M. and Norvaiša, R. (1999b) A survey on differentiability of six operators in relation to probability and statistics. In R.M. Dudley and R. Norvaiša, *Differentiability of Six Operators on Nonsmooth Functions and p -Variation*, Lecture Notes in Math. 1703. New York: Springer-Verlag.
- Dudley, R.M., Norvaiša, R. and Jinghau Qian (1999) Bibliographies on p -variations and ϕ -variation. In R.M. Dudley and R. Norvaiša, *Differentiability of Six Operators on Nonsmooth Functions and p -Variation*, Lecture Notes in Math. 1703. New York: Springer-Verlag.
- Eberlein, E. and Keller, U. (1995) Hyperbolic distributions in finance. *Bernoulli*, **1**, 281–299.
- Fernique, X. (1964) Continuité des processus gaussiens. *C. R. Acad. Sci. Paris*, **258**, 6058–6060.
- Föllmer, H. (1981) Calcul d'Itô sans probabilités. In J. Azéma and M. Yor (eds), *Séminaire de Probabilités XV*, Lecture Notes in Math. 850, pp. 144–150. Berlin: Springer-Verlag.
- Freedman, M.A. (1983) Operators of p -variation and the evolution representation problem. *Trans.*

- Amer. Math. Soc.*, **279**, 95–112.
- Fristedt, B. and Taylor, S.J. (1973) Strong variation for the sample functions of a stable process. *Duke Math. J.*, **40**, 259–278.
- Harrison, J.M. (1977) Ruin problems with compounding assets. *Stochastic Process. Appl.*, **5**, 67–79.
- Hildebrandt, T.H. (1938) Definitions of Stieltjes integrals of the Riemann type. *Amer. Math. Monthly*, **45**, 265–278.
- Hildebrandt, T.H. (1959) On systems of linear differentio-Stieltjes-integral equations. *Illinois J. Math.*, **3**, 352–373.
- Hildebrandt, T.H. (1963) *Introduction to the Theory of Integration*. New York: Academic Press.
- Itô, K. (1969) *Stochastic Processes*, Aarhus University Lecture Notes No. 16. Aarhus: Matematisk Institut, Aarhus Universitet.
- Jacod, J. (1979) *Calcul Stochastique et Problèmes de Martingales*, Lecture Notes in Math. 714. Berlin: Springer-Verlag.
- Jain, N.C. and Monrad, D. (1983) Gaussian measures in B_p . *Ann. Probab.*, **11**, 46–57.
- Janicki, A. and Weron, A. (1993) *Simulation and Chaotic Behaviour of α -Stable Stochastic Processes*. New York: Marcel Dekker.
- Karatzas, I. and Shreve, S.E. (1991) *Brownian Motion and Stochastic Calculus*, 2nd edition. New York: Springer-Verlag.
- Kawada, T. and Kôno, N. (1973) On the variation of Gaussian processes. In G. Maruyama and Y.V. Prokhorov (eds), *Proceedings of the Second Japan–USSR Symposium on Probability Theory*, Lecture Notes in Math. 330, pp. 176–192. Berlin: Springer-Verlag.
- Klingenhof, F. and Zähle, M. (1999) Ordinary differential equations with fractal noise. *Proc. Amer. Math. Soc.*, **127**, 1021–1028.
- Küchler, U., Neumann, K., Sørensen, M. and Streller, A. (1994) Stock returns and hyperbolic distributions. Discussion paper 23, Humboldt Universität Berlin.
- Lebesgue, H. (1973) *Leçons sur l'Intégration et la Recherche des Fonctions Primitives*, 3rd edition. New York: Chelsea.
- Lépingle, D. (1976) La variation d'ordre p des semi-martingales. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **36**, 295–316.
- Lin, S.J. (1995) Stochastic analysis of fractional Brownian motions. *Stochastics Stochastics Rep.*, **55**, 121–140.
- Liptser, R.S. and Shiryaev, A.N. (1986) *Theory of Martingales*. Moscow: Nauka (In Russian). English translation (1989), Dordrecht: Kluwer.
- Lyons, T. (1994) Differential equations driven by rough signals. I. An extension of an inequality of L.C. Young. *Math. Res. Lett.*, **1**, 451–464.
- Norvaiša, R. (1998) p -variation and integration of sample functions of stochastic processes. Preprint.
- Samorodnitsky, G. and Taqqu, M.S. (1994) *Stable Non-Gaussian Random Processes. Stochastic Models with Infinite Variance*. London: Chapman & Hall.
- Taylor, S.J. (1972) Exact asymptotic estimates of Brownian path variation. *Duke Math. J.*, **39**, 219–241.
- Wiener, N. and Paley, R.E.A.C. (1934) *Fourier Transforms in the Complex Domain*, Amer. Math. Soc. Colloq. Publ. 19. Providence, RI: American Mathematical Society.
- Young, L.C. (1936) An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.*, **67**, 251–282.

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