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A weak criterion of absolute continuity for jump processes: application to the Boltzmann equation

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We first prove a general and quite simple criterion of absolute continuity, based on the use of almost sure derivatives, which is applicable even when integration by parts may not be used. We apply it to Poisson-driven stochastic differential equations. Next, using a typically probabilistic substitution in the Boltzmann equation, we extend Tanaka's probabilistic interpretation for spatially homogeneous Boltzmann equations with Maxwell molecules and without angular cut-off to much more general spatially homogeneous two-dimensional Boltzmann equations. We relate a measure-solution $\{Q_t\}_t$ of the equation to a solution V_t of a nonlinear Poisson-driven stochastic differential equation: for each t, Q_t is the law of V_t. We extend our absolute continuity criterion to these nonlinear Poisson functionals and prove that even in the case of degenerate initial distribution, the law of V_t admits a density $f(t, \cdot)$ for each $t > 0$, which is hence a solution to the Boltzmann equation. We thus obtain an original existence result.

Keywords: Boltzmann equations; stochastic calculus of variations; stochastic differential equations with jumps

1. Introduction

The Boltzmann equation we consider describes the evolution of the density $f(t, v)$ of particles with velocity $v \in \mathbb{R}^2$ at time t in a rarefied homogeneous gas:

$$\frac{\partial f}{\partial t}(t, v) = \int_{\mathbb{R}^2}\int_{\theta=-\pi}^{\pi} (f(t, v')f(t, v'_*) - f(t, v)f(t, v_*))B(|v - v_*|, \theta)\mathrm{d}\theta\mathrm{d}v_*. \tag{1.1}$$

The post-collisional velocities v' and v'_* are given by

$$v' = v + A(\theta)(v - v_*), \qquad v'_* = v_* - A(\theta)(v - v_*), \tag{1.2}$$

where

$$A(\theta) = \frac{1}{2}\begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix}. \tag{1.3}$$

The cross-section B is a positive function, and is even in the θ-variable. If the molecules in the gas interact according to an inverse power law in $1/r^s$ with $s \geqslant 2$, then the physical cross-sections $B(z, \theta)$ tend to infinity when θ goes to zero, but satisfy $\int_{-\pi}^{\pi} |\theta|^2 B(z, \theta) \mathrm{d}\theta < +\infty$ for each z. Physically, this explosion near 0 comes from the accumulation of grazing collisions.

In this general (spatially homogeneous) setting, the Boltzmann equation is very difficult to study. A large literature deals with the non-physical equation with angular cut-off, that is, under the assumption $\int_0^{\pi} B(z, \theta) \mathrm{d}\theta < \infty$. More recently, the case of Maxwell molecules, for which the cross-section $B(z, \theta) = \beta(\theta)$ only depends on θ, has been much studied without the cut-off assumption. In the Maxwell context, Tanaka (1978) considered the case where $\int_0^{\pi} \theta\beta(\theta) \mathrm{d}\theta < \infty$, and Horowitz and Karandikar (1990), Desvillettes (1997), Desvillettes *et al*., (1999) and Fournier (2000) worked under the physical assumption $\int_0^{\pi} \theta^2 \beta(\theta) \mathrm{d}\theta < +\infty$. The case in which B depends on z is much harder and there are few results on it (see Alexandre *et al*. 2000; Fournier and Méléard 2001).

In the present paper, we first prove a weak, general and quite simple criterion of absolute continuity which we apply to standard Poisson-driven stochastic differential equations, and which in some sense generalizes results of Bichteler and Jacod (1983) and Bichteler *et al*. (1987). In Section 2, we extend the probabilistic interpretation of Tanaka (1978), who was dealing with Maxwell molecules, to much more general spatially homogeneous Boltzmann equations, under the condition $\sup_z \int_0^{\pi} \theta B(z, \theta) \mathrm{d}\theta < \infty$. Indeed, using a typically probabilistic substitution in the Boltzmann equation, we relate the solution of the equation to the solution V_t of a Poisson-driven nonlinear stochastic differential equation: the law of V is a measure solution to the Boltzmann equation. Then we develop in Section 3 our weak approach to the stochastic calculus of variations for our nonlinear Poisson functionals, to prove that even when the initial distribution is degenerate, the law of V_t has a density when $t > 0$. This leads to a new existence result for the Boltzmann equation, generalizing the Maxwell case in Graham and Méléard (1999).

The reason why we consider equations in two dimensions is technical. However, we are far from able to prove such a result in the three-dimensional case; technical problems are outlined in Fournier and Méléard (2002). This limitation is not new: for example, Desvillettes (1997) had to consider equations in one or two dimensions to obtain regularization results.

Let us now comment on the probabilistic tools we develop. The stochastic calculus of variations for Poisson processes was first investigated by Bismut (1983). Bichteler and Jacod (1983) rewrote and developed Bismut's main ideas to prove the existence of densities for diffusion processes with jumps. Much work has since been done. In almost all cases, existence of densities was based on integration by parts, as in the standard Malliavin calculus for Wiener functionals. But it is now well known in the Wiener case that the use of integration by parts is not very efficient when one restricts oneself to studying absolute continuity: it is much easier to use the Bouleau–Hirsch approach; see Nualart (1995). However this sort of approach has not been investigated in the case of Poisson functionals.

Unfortunately, we cannot use an integration by parts formula in the present study, because our random variables V_t cannot be differentiated in an $L^2(\Omega)$ sense. Indeed, the 'Malliavin derivative' of V_t is not square-integrable. To overcome this limitation, we will use the following weak, general and quite simple criterion of absolute continuity.

Lemma 1.1. *Let $d \in \mathbb{N}^*$, and let X be an $\mathbb{R}^d$-valued random variable on a probability space $(\Omega, \mathcal{F}, P)$. Let Λ be a neighbourhood of 0 in $\mathbb{R}^d$. Assume that there exists a family $\{X^\lambda\}_{\lambda\in\Lambda}$ of $\mathbb{R}^d$-valued random variables such that the following conditions hold:*

(*i*) *For each $\lambda \in \Lambda$, the law of X^λ is absolutely continuous with respect to that of X. We denote by $G^\lambda = \mathrm{d}X/\mathrm{d}X^\lambda$ the associated Radon–Nikodym density. The family G^λ satisfies the integrability condition*

$$\sup_\lambda \mathrm{E}(|G^\lambda|^2) < \infty. \tag{1.4}$$

(*ii*) *For almost all ω, there exists a neighbourhood $\mathcal{V}(\omega)$ of 0 in $\mathbb{R}^d$ on which the map $\lambda \mapsto X^\lambda(\omega)$ is of class C^1.*

(*iii*) *For almost all ω, the derivative $(\partial X^\lambda/\partial\lambda)|_{\lambda=0}$ is invertible.*

Then the law of X is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

Proof. Let A be a negligible subset of $\mathbb{R}^d$. We have to prove that $P(X \in A) = 0$.

Applying the inverse local theorem, we deduce from (ii) and (iii) that, for almost all ω, there exists a neighbourhood $\overline{\mathcal{V}}(\omega)$ of 0 in $\mathbb{R}^d$ on which the map $\lambda \mapsto X^\lambda(\omega)$ is a C^1 diffeomorphism. We now set, for $n \in \mathbb{N}^*$, $\Omega_n = \{\omega \in \Omega | [-1/n, 1/n]^d \subset \overline{\mathcal{V}}(\omega)\}$. Then it is clear that Ω_n grows to some $\tilde{\Omega}$, with $P(\tilde{\Omega}) = 1$.

On the other hand, we know from (i) that for each $\lambda \in \Lambda$, $P(X \in A) = \mathrm{E}(1_A(X^\lambda)G^\lambda)$. Hence, for each n,

$$P(X \in A) = \mathrm{E}\left[\left(\frac{n}{2}\right)^d \int_{[-1/n,1/n]^d} 1_A(X^\lambda)G^\lambda \,\mathrm{d}\lambda\right]. \tag{1.5}$$

It is straightforward to conclude, using (1.4), the Cauchy–Schwarz inequality, and the fact that $\lim_n P[\Omega_n] = 1$, that

$$P(X \in A) = \lim_{n\to\infty} \mathrm{E}\left[\left(\frac{n}{2}\right)^d \left\{\int_{[-1/n,1/n]^d} 1_A(X^\lambda)G^\lambda \,\mathrm{d}\lambda\right\} \times 1_{\Omega_n}\right]. \tag{1.6}$$

To conclude that $P(X \in A) = 0$, it thus suffices to prove that, for each n and each $\omega \in \Omega_n$,

$$\int_{[-1/n,1/n]^d} 1_A(X^\lambda)G^\lambda \,\mathrm{d}\lambda = 0. \tag{1.7}$$

It is of course enough to show that, for each n and each $\omega \in \Omega_n$, $I_n(\omega) = 0$, where

$$I_n(\omega) = \int_{[-1/n,1/n]^d} 1_A(X^\lambda)\mathrm{d}\lambda = 0. \tag{1.8}$$

But ω belongs to Ω_n, thus $\lambda \mapsto X^\lambda(\omega)$ is a C^1 diffeomorphism from $[-1/n, 1/n]^d$ into some set $D_n(\omega)$. Substituting $y = X^\lambda(\omega)$ in (1.8), and denoting by $J_n(\omega, y)$ the associated Jacobian, we obtain

$$I_n(\omega) = \int_{D_n(\omega)} 1_A(y) J_n(\omega, y) \mathrm{d}y, \tag{1.9}$$

which of course vanishes since A is Lebesgue negligible. This concludes the proof. □

We can apply this absolute continuity criterion to standard Poisson-driven stochastic differential equations, and the theorem we obtain (Theorem 1.2 below) generalizes the result of Bichteler and Jacod (1983) and Bichteler *et al*. (1987) in the case of processes with finite variations. Bichteler and Jacod (1983) deal with the unidimensional case, while Bichteler *et al*. (1987) treat the multidimensional case; our result is stated for any dimension. The technical hypotheses on the coefficients are less stringent: instead of boundedness, we assume polynomial growth and the integrability assumption is also weaker. Furthermore, our proof (in the case of a process related to the Boltzmann equation) is technically simpler. Let us now state our result.

Theorem 1.2. *Consider, on a probability space* $(\Omega, \mathcal{F}, P)$*, a Poisson point measure* $N(\omega, \mathrm{d}t, \mathrm{d}z)$ *on* $[0, T] \times \mathbb{R}$ *with intensity measure* $m(\mathrm{d}t, \mathrm{d}z) = \mathrm{d}t\,\mathrm{d}z$*, and consider the* $\mathbb{R}^d$*-valued stochastic differential equation*

$$X_t = x_0 + \int_0^t \int_{\mathbb{R}} \gamma(X_{s-}, z) N(\mathrm{d}s, \mathrm{d}z) + \int_0^t b(X_{s-}) \mathrm{d}s, \tag{1.10}$$

where $x_0 \in \mathbb{R}^d$*, and where the coefficients* γ *and* b *satisfy the following hypotheses:*

(*i*) *The maps* $\gamma(X, z) \colon \mathbb{R}^d \times \mathbb{R}^* \mapsto \mathbb{R}^d$ *and* b *are of class* C^2*. There exist* $p \in \mathbb{N}$*,* $K \in \mathbb{R}^+$*, and a bounded positive function* $\eta : \mathbb{R}^* \mapsto \mathbb{R}^+$ *satisfying the integrability condition*

$$\bar{\eta}(z) = \sup_{|u-z| \leq |z|/2 \wedge 1/|z|} \eta(u) \in L^1(\mathbb{R}^*, \mathrm{d}z), \tag{1.11}$$

such that, for $X \in \mathbb{R}^d$ *and* $z \in \mathbb{R}$*,*

$$|\gamma(X, z)| \leq (1 + |X|)\eta(z), \qquad |b(X)| \leq K(1 + |X|); \tag{1.12}$$

$$|\gamma'_X(X, z)| + |\gamma''_{XX}(X, z)| \leq (1 + |X|^p)\eta(z), \qquad |b'(X)| + |b''(X)| \leq K(1 + |X|^p); \tag{1.13}$$

$$|\gamma'_z(X, z)| + |\gamma''_{zz}(X, z)| + |\gamma''_{Xz}(X, z)| \leq K(1 + |X|^p). \tag{1.14}$$

Notice that the integrability condition (1.11) *is not much more stringent than the simple condition* $\eta \in L^1(\mathbb{R}^*, \mathrm{d}z)$.

(*ii*) *The following non-degeneracy condition holds: for each* $X \in \mathbb{R}^d$ *and each* $Y \in \mathbb{R}^d \backslash \{0\}$,

$$\int_{\mathbb{R}} 1_{\{Y^{\mathrm{T}} \gamma'_z(X,z)(\gamma'_z(X,z))^{\mathrm{T}} Y \neq 0\}}(z) \mathrm{d}z = \infty.$$

Then there exists a solution $\{X_t\}_{t \in [0,T]}$ *to* (1.10)*, and for all* $t > 0$ *the law of* X_t *admits a density with respect to the Lebesgue measure on* $\mathbb{R}^d$.

We do not give the proof of this result here, since we do so later (Section 3) for the more complicated (because nonlinear) case of Boltzmann processes.

In Bichteler and Jacod (1983) and Bichteler *et al*. (1987), the assumptions equivalent to (1.13) and (1.14) are given by

$$|\gamma'_X(X, z)| + |\gamma''_{XX}(X, z)| \leqslant \eta(z), \qquad |b'(X)| + |b''(X)| \leqslant K; \tag{1.15}$$

$$|\gamma'_z(X, z)| + |\gamma''_z(X, z)| + |\gamma''_{Xz}(X, z)| \leqslant K. \tag{1.16}$$

Finally, notice that a localization procedure may be used to generalize directly the results of Bichteler and Jacod and of Bichteler *et al*. But this would probably not lead to such weak assumptions. Furthermore, localization cannot be used in nonlinear settings, such as that of the Boltzmann equation.

2. Transformation of the Boltzmann equation, and main results

First of all, we specify the family of cross-sections we will study.

Hypothesis 2.1. *For all* $x \in \mathbb{R}_+$, $B(x, \theta)$ *is an even, strictly positive function on* $[-\pi, \pi]\backslash\{0\}$ *satisfying*

$$\int_{-\pi}^{\pi} B(x, \theta)\mathrm{d}\theta = \infty \quad \text{and} \quad \sup_{x\in\mathbb{R}_+}\int_{-\pi}^{\pi} |\theta| B(x, \theta)\mathrm{d}\theta < \infty. \tag{2.1}$$

For $X \in \mathbb{R}^2$, *we will denote by* $B(X, \theta)$ *the quantity* $B(|X|, \theta)$.

Equation (1.1) has to be understood in a weak sense. Integrating (1.1) against test functions, and using standard integration by parts (see Desvillettes, 1997), we obtain the following weak formulation. First of all, we define, for each probability measure $q \in \mathcal{P}(\mathbb{R}^2)$ and each $\phi \in C_b^1(\mathbb{R}^2)$,

$$L_q\phi(v) = \int_{\mathbb{R}^2}\int_{-\pi}^{\pi} (\phi(v + A(\theta)(v - v_*)) - \phi(v))B(v - v_*, \theta)\mathrm{d}\theta q(\mathrm{d}v_*). \tag{2.2}$$

This kernel is well defined since $|A(\theta)| \leqslant K|\theta|$ and thanks to (2.1).

Definition 2.2. *Assume Hypothesis 2.1. Consider a probability measure* Q_0 *on* $\mathbb{R}^2$. *We say that a probability measure family* $\{Q_t\}_{t\in[0,T]}$ *is a measure solution of the Boltzmann equation* (1.1) *with initial data* Q_0 *if, for each* $\phi \in C_b^1(\mathbb{R}^2)$ *and all* $t \in [0, T]$,

$$\langle \phi, Q_t\rangle = \langle \phi, Q_0\rangle + \int_0^t \langle L_{Q_s}\phi(v), Q_s(\mathrm{d}v)\rangle\, \mathrm{d}s. \tag{2.3}$$

If, furthermore, for all $t \in]0, T]$, *the probability measure* Q_t *admits a density* $f(t, \cdot)$ *with respect to the Lebesgue measure on* $\mathbb{R}^2$, *the function* $f(t, v) :]0, T] \times \mathbb{R}^2 \mapsto \mathbb{R}_+$ *obtained is said to be a function solution of the Boltzmann equation* (1.1).

The probabilitistic approach will consist in considering (2.3) as the evolution equation of the flow of time-marginals of a Markov process.

This entire work is based on the following substitution in L_q. For each $X \in \mathbb{R}^2$, we consider the function h_X defined on $[-\pi, \pi]\backslash\{0\}$ by

$$h_X(\theta) = \int_\theta^\pi B(X, \varphi)\mathrm{d}\varphi \textit{ if } \theta > 0; \; h_X(\theta) = -\int_{-\pi}^\theta B(X, \varphi)\mathrm{d}\varphi \textit{ if } \theta < 0. \tag{2.4}$$

Thanks to Hypothesis 2.1, it is clear that for each X, $h_X(\theta)$ is strictly decreasing from 0 to $-\infty$ between $\theta = -\pi$ and $\theta = 0^-$, and from $+\infty$ to 0 between $\theta = 0^+$ and $\theta = \pi$. We thus can set, for each $X \in \mathbb{R}^2$ and each $z \in \mathbb{R}^*$,

$$g(X, z) = h_X^{-1}(z), \quad \text{i.e.} \quad h_X(g(X, z)) = z. \tag{2.5}$$

Notice that for each X, z, the derivative $g_z'(X, z) = -1/B(X, g(X, z)) < 0$, thanks to Hypothesis 2.1. The function $g(X, z)$ is thus strictly decreasing from 0 to $-\pi$ between $-\infty$ and 0^-, and from π to 0 between 0^+ and $+\infty$. Notice also that $g(X, \cdot)$ is odd and depends only on $|X|$. Finally, observe that (2.1) can be written as

$$\sup_{X \in \mathbb{R}^2} \int_{\mathbb{R}^*} |g(X, z)|\mathrm{d}z < +\infty. \tag{2.6}$$

For $X \in \mathbb{R}^2$ and $z \in \mathbb{R}^*$, we set

$$\gamma(X, z) = A(g(X, z)).X : \mathbb{R}^2 \times \mathbb{R}^* \mapsto \mathbb{R}^2. \tag{2.7}$$

In this way, we obtain a new expression for the operator L_q.

Proposition 2.3. *Assume Hypothesis 2.1. Then, for each $q \in \mathcal{P}(\mathbb{R}^2)$ and $\phi \in C_b^1(\mathbb{R}^2)$,*

$$L_q\phi(v) = \int_{\mathbb{R}^2}\int_{z \in \mathbb{R}^*} (\phi(v + \gamma(v - v_*, z)) - \phi(v))\mathrm{d}zq(\mathrm{d}v_*). \tag{2.8}$$

Proof. It suffices to use the substitution $\theta = g(v - v_*, z)$ in (2.2), which implies that $z = h_{v-v_*}(\theta)$ and thus $\mathrm{d}z = -B(v - v_*, \theta)\mathrm{d}\theta$. □

Let us now explain why this substitution is of interest. Tanaka (1978) dealt with the much simpler case of Maxwell molecules (i.e. $B(X, \theta) = \beta(\theta)$). In this case, the jump measure appearing in (2.2) is $\beta(\theta)\mathrm{d}\theta q(\mathrm{d}v_*)$, and does not depend on v. The main attraction of the substitution described above is in transforming the jump measure $B(v - v_*, \theta)\mathrm{d}\theta q(\mathrm{d}v_*)$ into a measure of the form $\mathrm{d}zq(\mathrm{d}v_*)$, independent of v. This will enable a probabilistic interpretation in terms of Poisson measure.

Finally, we wish to introduce a nonlinear stochastic differential equation associated with our Boltzmann equation. To this end, we follow the main ideas of Tanaka. We first consider two probability spaces: the first is the abstract space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$; and the second is the auxiliary space $([0, 1], \mathcal{B}([0, 1]), \mathrm{d}\alpha)$ introduced to model the nonlinearity. In order to avoid any confusion, the processes on $([0, 1], \mathcal{B}([0, 1]), \mathrm{d}\alpha)$ will be called α-processes, the expectation under $\mathrm{d}\alpha$ will be denoted by E_α, and the laws $\mathcal{L}_\alpha$.

Definition 2.4. *Assume Hypothesis* 2.1. *We will say that* (V, W, N, V_0) *is a solution of* (*SDE*) *if:*

(*i*) $\{V_t(\omega)\}_{t\in[0,T]}$ *is a* $\mathbb{R}^2$*-valued cadlag adapted process on* Ω *such that* $\mathrm{E}(\sup_{t\in[0,T]}|V_t|^2) < +\infty$*;*
(*ii*) $\{W_t(\alpha)\}_{t\in[0,T]}$ *is a* $\mathbb{R}^2$*-valued cadlag* α*-process on* $[0, 1]$*;*
(*iii*) $N(\omega, \mathrm{d}t, \mathrm{d}\alpha, \mathrm{d}z)$ *is a Poisson measure on* $[0, T] \times [0, 1] \times \mathbb{R}^*$ *with intensity measure*

$$m(\mathrm{d}t, \mathrm{d}\alpha, \mathrm{d}z) = \mathrm{d}t\,\mathrm{d}\alpha\,\mathrm{d}z; \tag{2.9}$$

(*iv*) $V_0(\omega)$ *is a square-integrable variable independent of* N;
(*v*) *the laws of* V *and* W *are the same, i.e.* $\mathcal{L}(V) = \mathcal{L}_\alpha(W)$;
(*vi*) *the following S.D.E. is satisfied:*

$$V_t = V_0 + \int_0^t\int_0^1\int_{\mathbb{R}^*}\gamma(V_{s-} - W_{s-}(\alpha), z)N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z) \tag{2.10}$$

The following remark shows the connection between (*SDE*) and the Boltzmann equation (1.1).

Remark 2.5. Let (V, W, N, V_0) be a solution of (SDE), and set $Q_t = \mathcal{L}(V_t) = \mathcal{L}_\alpha(W_t)$ for each $t \in [0, T]$. Then one can easily prove, using the Itô formula, that the family $\{Q_t\}_{t\in[0,T]}$ is a measure solution of (2.3) with initial data Q_0.

Let us now state a hypothesis, which, combined with Hypothesis 2.1, will be sufficient to obtain existence of a function solution to the Boltzmann equation.

Hypothesis 2.6. (*i*) *The initial distribution* $Q_0 = \mathcal{L}(V_0)$ *admits moments of all orders, and is not a Dirac mass.*

(*ii*) *The map* $\gamma(X, z) : \mathbb{R}^2 \times \mathbb{R}^* \mapsto \mathbb{R}^2$ *is of class* C^2. *There exist* $p \in \mathbb{N}$, $K \in \mathbb{R}^+$, *and a bounded positive function* $\eta : \mathbb{R}^* \mapsto \mathbb{R}^+$ *satisfying the integrability condition*

$$\bar{\eta}(z) = \sup_{|u-z|\leqslant(|z|/2)\wedge(1/|z|)} \eta(u) \in L^1(\mathbb{R}^*, \mathrm{d}z), \tag{2.11}$$

such that

$$|\gamma(X, z)| \leqslant (1 + |X|)\eta(z), \tag{2.12}$$

$$|\gamma'_X(X, z)| + |\gamma''_{XX}(X, z)| \leqslant (1 + |X|^p)\eta(z), \tag{2.13}$$

$$|\gamma'_z(X, z)| + |\gamma''_{zz}(X, z)| + |\gamma''_{Xz}(X, z)| \leqslant K(1 + |X|^p). \tag{2.14}$$

Notice that the integrability condition (2.11) *is not much more stringent than the simple condition* $\eta \in L^1(\mathbb{R}^*, \mathrm{d}z)$.

The following existence result for (SDE) is an easy but tedious exercise, and can be proved by following the main ideas of Fournier and Méléard (2001).

Proposition 2.7. *Assume Hypotheses 2.1 and 2.6. Then there is weak existence for* (SDE). *This means that there exists a probability space* $(\Omega, \mathcal{F}, P)$*, on which there exists a solution* (V, W, N, V_0) *to* (SDE). *Furthermore, we have for any* $q \in \mathbb{N}$,

$$\mathrm{E}\left(\sup_{[0,T]} |V_t|^q\right) = \mathrm{E}_\alpha\left(\sup_{[0,T]} |W_t|^q\right) < \infty. \tag{2.15}$$

Finally, momentum and kinetic energy are conserved, that is, for all $t \in [0, T]$, $\mathrm{E}(V_t) = \mathrm{E}(V_0)$ *and* $\mathrm{E}(|V_t|^2) = \mathrm{E}(|V_0|^2)$.

Then the following regularization result holds:

Theorem 2.8. *Assume Hypotheses 2.1 and 2.6. Consider a solution* (V, W, N, V_0) *of SDE. Then, for all* $t > 0$*, the law of* V_t *admits a density* $f(t, .)$ *with respect to the Lebesgue measure on* $\mathbb{R}^2$.

The next corollary, immediately deduced from Theorem 2.8, states the analytical version of our result.

Corollary 2.9. *Assume Hypotheses 2.1 and 2.6. Then there exists a function solution*

$$f \in L^\infty(]0, T], L^1((1 + |v|^2)\mathrm{d}v)) \tag{2.16}$$

to the Boltzmann equation without cut-off, for non-Maxwell molecules, with initial data Q_0, *and* $f(t, .)$ *is, for each* $t > 0$*, a probability density function.*

We now give an example where Hypotheses 2.1 and 2.8 can easily be verified.

Example 2.10. Assume that the cross-section is of the form $B(X, \theta) = \psi(X)/|\theta|^\alpha$, with ψ positive and $\alpha \in [1, 2[$. Then Hypotheses 2.1 and 2.6 are satisfied if ψ is bounded from above and from below, and is of class C^2 on $\mathbb{R}^2$, and if ψ' and ψ'' have at most polynomial growth. We can prove this by means of simple computations, observing that when $\alpha = 1$,

$$g(X, z) = \mathrm{sign}(z)\mathrm{e}^{-|z|/\psi(X)},$$

and when $\alpha > 1$,

$$g(X, z) = \mathrm{sign}(z)\left(\frac{\pi^{\alpha-1}\psi(X)}{(\alpha-1)|z|\pi^{\alpha-1} + \psi(X)}\right)^{1/(\alpha-1)}.$$

3. Existence of a function solution

In this section we will prove Theorem 2.8, assuming Hypotheses 2.1 and 2.6 throughout. We consider a fixed solution (V, W, N, V_0) of (*SDE*). Our aim is to prove that the law of V_T admits a density with respect to the Lebesgue measure on $\mathbb{R}^2$, for fixed terminal time $T > 0$, which of course suffices since $T > 0$ is fixed arbitrarily.

To prove such a result, we will apply Lemma 1.1 to the random variable $X = V_T$. First, we will construct in Section 3.1 some absolutely continuous changes of measure, on our Poisson space, which will allow us to define the perturbed processes V^λ. In fact, we will define a 'class' of changes of measure, depending on the 'direction' in which we want to perturb our process. In Section 3.2 we will study the almost sure differentiability of V_T^λ with respect to λ. In Section 3.3 we will choose a 'direction', and we prove that the associated $(\partial V_T^\lambda/\partial\lambda)_{\lambda=0}$ is almost surely invertible. In Section 3.4 we bring everything together to conclude our proof.

3.1. Absolutely continuous changes of measure

Following the ideas of Bichteler and Jacod (1983), we now construct a family of shifts S^λ on Ω, such that the family $V_T^\lambda = V_T \circ S^\lambda$ satisfies the assumptions of the criterion given in Lemma 1.1. We begin with a definition, which describes in which 'directions' we are authorized to perturb our process.

Definition 3.1. *We say that a predictable function* $v(\omega, s, \alpha, z) : \Omega \times [0, T] \times [0, 1] \times \mathbb{R}^* \mapsto \mathbb{R}^2$ *is a* direction *if it is of class* C^1 *in z, and if there exists a deterministic positive function* $\rho(z) : \mathbb{R}^* \mapsto \mathbb{R}^+$ *such that*

$$|v(\omega, s, \alpha, z)| + |v'(\omega, s, \alpha, z)| \leqslant \rho(z) \tag{3.1}$$

(*where* $v' = v'_z$), *and*

$$\rho \in L^1(\mathbb{R}^*, \mathrm{d}z), \qquad \forall z \in \mathbb{R}^*, \rho(z) \leqslant (|z|/2) \wedge (1/|z|), \qquad \rho(z) \leqslant 1/2. \tag{3.2}$$

Let v now be a fixed direction. We associate with v many objects.

We consider a neighbourhood Λ of 0 in $B(0, 1) \subset \mathbb{R}^2$. For $\lambda \in \Lambda$, we define the perturbation

$$\Gamma^\lambda(\omega, t, z, \alpha) = z + \lambda v(\omega, t, z, \alpha) = z + \lambda_x v_x(\omega, t, z, \alpha) + \lambda_y v_y(\omega, t, z, \alpha). \tag{3.3}$$

One can verify, using (3.1), that, for every $\lambda \in \Lambda$ and every ω, t, α, the map $z \mapsto \Gamma^\lambda(\omega, t, z, \alpha)$ is an increasing bijection from $\mathbb{R}^*$ into itself.

For $\lambda \in \Lambda$, we set $N^\lambda = \Gamma^\lambda(N)$: if A is a Borel set of $[0, T] \times [0, 1] \times \mathbb{R}^*$,

$$N^\lambda(A) = \int_0^T \int_0^1 \int_{\mathbb{R}^*} \mathbf{1}_A(s, \Gamma^\lambda(\omega, s, z, \alpha), \alpha) N(\omega, \mathrm{d}z, \mathrm{d}\alpha, \mathrm{d}s). \tag{3.4}$$

We consider the shift S^λ defined by

$$V_0 \circ S^\lambda(\omega) = V_0(\omega), \qquad N \circ S^\lambda(\omega) = N^\lambda(\omega). \tag{3.5}$$

We now look for a family of probability measures P^λ on Ω satisfying $P^\lambda \circ (S^\lambda)^{-1} = P$. To this end, we consider the predictable real-valued function on $\Omega \times [0, T] \times \mathbb{R}^* \times [0, 1]$ given by

$$Y^\lambda(\omega, s, z, \alpha) = 1 + \lambda_x v'_x(\omega, s, z, \alpha) + \lambda_y v'_y(\omega, s, z, \alpha). \tag{3.6}$$

We have

$$|Y^\lambda(\omega, s, z, \alpha) - 1| \leqslant |\lambda| \rho(z). \tag{3.7}$$

Then we consider the following square-integrable Doléans–Dade martingale:

$$G_t^\lambda = 1 + \int_0^t \int_0^1 \int_{\mathbb{R}^*} G_{s-}^\lambda (Y^\lambda(s, z, \alpha) - 1) \tilde{N}(\mathrm{d}z, \mathrm{d}\alpha, \mathrm{d}s). \tag{3.8}$$

Proposition 3.2. *G_t^λ is strictly positive for every $t \in [0, t]$. If P^λ is the probability measure defined by $P^\lambda = G_T^\lambda P$, then $P^\lambda \circ (S^\lambda)^{-1} = P$. Furthermore,*

$$\sup_\lambda \mathrm{E}[(G_T^\lambda)^2] < \infty. \tag{3.9}$$

Proof. Recall that if

$$M_t^\lambda = \int_0^t \int_0^1 \int_{\mathbb{R}^*} (Y^\lambda(\omega, s, z, \alpha) - 1) \tilde{N}(\mathrm{d}z, \mathrm{d}\alpha, \mathrm{d}s), \tag{3.10}$$

then (see Jacod and Shiryaev 1987, p. 59) $G_t^\lambda = \mathrm{e}^{M_t^\lambda} \Pi_{s \leqslant t} (1 + \Delta M_s^\lambda) \mathrm{e}^{-\Delta M \lambda_s}$. Since, by construction, $|Y^\lambda(\omega, s, z, \alpha) - 1| \leqslant \frac{1}{2}$ for $z \in \mathbb{R}_+^*$, the jumps of M^λ are greater than $-\frac{1}{2}$, and thus G_t^λ is strictly positive. Now, using the definition of the shift S^λ and the Girsanov theorem (see Jacod and Shiryaev 1987, p. 157), we see that the compensator of N under P^λ is $\Gamma^\lambda(Y^\lambda m)$. But Y^λ has been chosen such that $\Gamma^\lambda(Y^\lambda m) = m$. Indeed, considering a Borel set A of $[0, T] \times \mathbb{R}^* \times [0, 1]$, we have

$$\Gamma^\lambda(Y^\lambda . m)(A) = \int_0^t \int_0^1 \int_{\mathbb{R}^*} \mathbf{1}_A(s, \Gamma^\lambda(s, z, \alpha), \alpha) Y^\lambda(s, z, \alpha) \mathrm{d}z \, \mathrm{d}\alpha \, \mathrm{d}s. \tag{3.11}$$

The substitution $z' = \Gamma^\lambda(s, z, \alpha)$ implies that $\Gamma^\lambda(Y^\lambda m)(A) = m(A)$. Hence, since the law of a Poisson point measure is characterized by its intensity, we deduce that $\mathcal{L}(N^\lambda | P^\lambda) = \mathcal{L}(N | P)$. Finally, since V_0 is independent of G^λ, it is clear that $\mathcal{L}(V_0 | P^\lambda) = \mathcal{L}(V_0 | P)$. We have shown that $P^\lambda \circ (S^\lambda)^{-1} = P$.

We deduce (3.9) from (3.8), (3.7), the fact that $\rho \in L^1 \cap L^\infty(\mathbb{R}^*, \mathrm{d}z)$, and the Gronwall lemma. □

3.2. Perturbation and derivation of V_t

In this subsection, we consider a fixed direction υ, we use the notation of the previous subsection, and we study the smoothness of the map $\lambda \mapsto V_t^\lambda = V_t \circ S^\lambda$. Here the α-process W is fixed, deterministic (from the point of view of the probability space Ω), and thus behaves as a parameter.

Proposition 3.3. *Let $\lambda \in \Lambda$ be fixed. The perturbed process V^λ, defined by $V_t^\lambda = V_t \circ S^\lambda$, satisfies the equation*

$$V_t^\lambda = V_0 + \int_0^t \int_0^1 \int_{\mathbb{R}^*} \gamma(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) N(\mathrm{d}z, \mathrm{d}\alpha, \mathrm{d}s) \tag{3.12}$$

Proof. It suffices to replace ω by $S^\lambda(\omega)$ everywhere in equation (2.10). □

We will need the following lemma.

Lemma 3.4. *For each λ,* (3.12) *admits a unique solution that is almost surely cadlag from* $[0, T]$ *into* $\mathbb{R}^2$. *We furthermore have almost surely that*

$$\sup_\lambda \sup_{0 \leqslant t \leqslant T} |V_t^\lambda| < \infty. \tag{3.13}$$

We omit the proof of this lemma, because it can be done in the same way as that of the next one.

The following lemma deals with the possible derivative of V_t^λ, which should satisfy the equation obtained by formally differentiating (3.12) with respect to λ.

Lemma 3.5. *For each λ, the equation*

$$\begin{aligned} D_t^\lambda = & \int_0^t \int_0^1 \int_{\mathbb{R}^*} \gamma'_X(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) D_{s-}^\lambda N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z) \\ & + \int_0^t \int_0^1 \int_{\mathbb{R}^*} \gamma'_z(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) \upsilon(s, \alpha, z) N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z) \end{aligned} \tag{3.14}$$

admits a unique solution which is almost surely cadlag from $[0, T]$ *into* $\mathcal{M}_{2\times 2}(\mathbb{R})$). *We have, furthermore, almost surely that*

$$\sup_\lambda \sup_{0 \leqslant t \leqslant T} |D_t^\lambda| < \infty. \tag{3.15}$$

Observe that there is no reason why for some λ fixed, say for $\lambda = 0$, D_T^0 should belong to L^2 (or even L^1). The only assumption that ensures that D_T^0 belongs to L^2 is the Maxwell assumption $B(X, \theta) = \beta(\theta)$, which yields that $\gamma(X, z) = A(g(z))X$, with g no longer depending on X, and thus $\gamma'_X(X, z) = A(g(z))$. In any other case, D_T^0 behaves almost as the Doléans–Dade exponential of a pure jump process with finite variations, belonging to all the L^qs, but this does not imply that D_T^0 belongs to L^1. (One can easily construct Poisson-driven semimartingales which belong to all the L^qs, and whose Doléans–Dade exponential is not in L^1). This is the reason why we have to use the almost sure derivatives and the weak criterion given by Lemma 1.1.

Proof. We first prove the uniqueness. We will use Lemma A.1 in the Appendix, for λ and ω fixed. Thus, let λ be fixed, and let D and E be two cadlag solutions of (3.14). A simple computation shows that

$$|D_t - E_t| \leq \int_0^t \int_0^1 \int_{\mathbb{R}^*} |D_{s-} - E_{s-}| \times |\gamma'_X(V^\lambda_{s-} - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z))| N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z). \quad (3.16)$$

Since $\Gamma^\lambda(s, \alpha, z) = z + \lambda v(s, \alpha, z)$, we deduce from (3.1) and (3.2) that $|\Gamma^\lambda(s, \alpha, z) - z| \leq (|z|/2) \wedge (1/|z|)$. Hence, using (2.11) and (2.13) in Hypothesis 2.6, we obtain the existence of a constant C such that

$$|\gamma'_X(V^\lambda_{s-} - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z))| \leq C(1 + |V^\lambda_{s-}|^p + |W_{s-}(\alpha)|^p)\bar{\eta}(z). \quad (3.17)$$

We set $\tilde{\eta}(s, \alpha, z) = (1 + |W_{s-}(\alpha)|^p)\bar{\eta}(z)$. Then $\tilde{\eta}$ belongs to $L^1(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z)$, thanks to Hypothesis 2.6 and (2.15), and hence $\tilde{\eta}$ belongs almost surely to $L^1(N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z))$. We also set $c = 1 + \sup_\lambda \sup_{s\in[0,T]} |V^\lambda_{s-}|^p$, which is almost surely finite thanks to Lemma 3.4. We finally obtain

$$|D_t - E_t| \leq Kc \int_0^t \int_0^1 \int_{\mathbb{R}^*} |D_{s-} - E_{s-}| \times \tilde{\eta}(s, \alpha, z) N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z). \quad (3.18)$$

Applying Lemma A.1, we finally deduce that $\sup_{[0,T]} |D_t - E_t| = 0$ almost surely, as required.

We now prove the existence. We continue to fix λ. We first consider the simpler equation, for $n \in \mathbb{N}_*$ fixed,

$$\begin{aligned} \overline{D}^n_t = & \int_0^t \int_0^1 \int_{|z|\leq n} \gamma'_X(V^\lambda_{s-} - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) \overline{D}^n_{s-} N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z) \\ & + \int_0^t \int_0^1 \int_{\mathbb{R}^*} \gamma'_z(V^\lambda_{s-} - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) v(s, \alpha, z) N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z). \end{aligned} \quad (3.19)$$

We denote by U_t the last term of this equation. Notice that thanks to (2.14) in Hypothesis 2.6 and to (3.1), almost surely, $\sup_{[0,\mathrm{T}]} |U_t| \leq A$, where

$$A = K\left(1 + \sup_{\lambda,u} |V^\lambda_u|^p\right) \int_0^t \int_0^1 \int_{\mathbb{R}^*} (1 + |W_{s-}(\alpha)|^p)\rho(z) N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z), \quad (3.20)$$

which is almost surely finite thanks to (3.2), Lemma 3.4 and (2.15).

Since $N|_{[0,T]\times[0,1]\times\{|z|\leq n\}}$ is a finite counting measure, it can be written (for each ω) as a (finite) sum of n Dirac measures at some points (T_i, α_i, z_i), and one may assume that $0 < T_1 < T_2 < \ldots < T_n < T$. Thus equation (3.19) can be solved by working recursively on the time intervals $[T_i, T_{i+1}[$: for $t \in [0, T_1[$, we set

$$\overline{D}^n_t = U_t;$$

for $t \in [T_1, T_2[$, we set

$$\overline{D}^n_t = \gamma'_X(V^\lambda_{T_1-} - W_{T_1} - (\alpha_1), \Gamma^\lambda(T_1 s, \alpha_1, z_1))\overline{D}^n_{T_1-} + U_t;$$

and so on. Then we have to prove that, for almost all ω,

$$\sup_n \sup_{t\in[0,T]} |\overline{D}^n_t| < \infty. \tag{3.21}$$

Using (3.20) and the same arguments and notation as in the proof of uniqueness, we obtain

$$|\overline{D}^n_t| \leqslant A + Kc\int_0^t\int_0^1\int_{\mathbb{R}^*} |\overline{D}^n_{s-}|\tilde{\eta}(s, \alpha, z)N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z). \tag{3.22}$$

Lemma A.1 allows us to conclude that

$$\sup_{[0,T]} |\overline{D}^n_t| \leqslant A \exp\left(\int_0^T\int_0^1\int_{\mathbb{R}^*} \ln(1 + Kc\tilde{\eta}(s, \alpha, z))N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z)\right) \tag{3.23}$$

and (3.21) is proved. We finally check that the family $\overline{D}^n$ is Cauchy for the supremum norm on $[0, T]$ (for almost all ω fixed). Let $n < n'$ be fixed. Then

$$|\overline{D}^n_t - \overline{D}^{n'}_t| \leqslant \int_0^t\int_0^1\int_{\mathbb{R}^*} |\gamma'_X(V^\lambda_{s-} - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z))|\,|\overline{D}^n_{s-} - \overline{D}^{n'}_{s-}|N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z)$$
$$+ \sup_{l,u\in[0,T]} |\overline{D}^l_u| \times \int_0^T\int_0^1\int_{n<|z|<n'} |\gamma'_X(V^\lambda_{s-} - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z))| \, N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z). \tag{3.24}$$

Still using the same notation as in the proof of uniqueness, we obtain

$$|\overline{D}^n_t - \overline{D}^{n'}_t| \leqslant Kc\int_0^t\int_0^1\int_{\mathbb{R}^*} \tilde{\eta}(s, \alpha, z)|\overline{D}^n_{s-} - \overline{D}^{n'}_{s-}|N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z) + Z^{n,n'}, \tag{3.25}$$

where

$$Z^{n,n'} = \sup_{l,u\in[0,T]} |\overline{D}^l_u| \times cK\int_0^T\int_0^1\int_{n<|z|<n'} \tilde{\eta}(s, \alpha, z)N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z). \tag{3.26}$$

Since $\tilde{\eta}$ belongs (almost surely) to $L^1(N)$, it is clear that when n, n' go to infinity, $Z^{n,n'}$ goes to 0. Lemma A.1 immediately yields that

$$\sup_{[0,T]} |\overline{D}^n_t - \overline{D}^{n'}_t| \leqslant B \times Z^{n,n'}, \tag{3.27}$$

where B is an almost surely finite random variable. The family $\overline{D}^n_t$ is thus almost surely Cauchy for the supremum norm on $[0, T]$, and hence admits a limit $\overline{D}_t$. Making n tend to infinity in (3.19), using (3.21), we show that $\overline{D}$ satisfies (3.14). This concludes the proof of the existence.

We finally check (3.15). Still using the same arguments and notation, we obtain

$$|D_t^\lambda| \leqslant Kc\int_0^t\int_0^1\int_{\mathbb{R}^*}\tilde{\eta}(s, \alpha, z)|D_{s-}^\lambda|N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z) + A \tag{3.28}$$

and Lemma A.1 allows to conclude as usual that

$$\sup_\lambda \sup_{[0,T]} |D_t^\lambda| \leqslant A\exp\left(\int_0^T\int_0^1\int_{\mathbb{R}^*} \ln(1 + Kc\tilde{\eta}(s, \alpha, z))N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z)\right), \tag{3.29}$$

which implies (3.15). □

To check that $\lambda \mapsto V_T^\lambda$ is almost surely differentiable, we first need a Lipschitz estimate.

Lemma 3.6. *There exists an almost surely finite random variable A such that, for all $0 \leqslant t \leqslant T$ and all $\lambda, \mu \in \Lambda$,*

$$|V_t^\lambda - V_t^\mu| \leqslant A|\lambda - \mu|. \tag{3.30}$$

Proof. Let λ, μ be fixed. Notice that thanks to Hypothesis 2.6, to the definition of Γ^λ, and to the properties of the direction v, there exists a constant C such that

$$\begin{aligned}
&|\gamma(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) - \gamma(V_{s-}^\mu - W_{s-}(\alpha), \Gamma^\mu(s, \alpha, z))| \\
&\qquad \leqslant C(1 + |V_{s-}^\lambda|^p + |V_{s-}^\mu|^p + |W_{s-}(\alpha)|^p)\bar{\eta}(z)|V_{s-}^\lambda - V_{s-}^\mu| \\
&\qquad\quad + C(1 + |V_{s-}^\mu|^p + |W_{s-}(\alpha)|^p)|\Gamma^\lambda(s, \alpha, z) - \Gamma^\mu(s, \alpha, z)| \\
&\qquad \leqslant C\sup_{\lambda',u}(1 + |V_u^{\lambda'}|^p) \times (\bar{\eta}(z)(1 + |W_{s-}(\alpha)|^p) \times |V_{s-}^\lambda - V_{s-}^\mu| + |\lambda - \mu|\rho(z)) \\
&\qquad = Cc(\tilde{\eta}(s, \alpha, z)|V_{s-}^\lambda - V_{s-}^\mu| + |\lambda - \mu|\rho(z)),
\end{aligned} \tag{3.31}$$

where the last inequality defines some notation. As in the previous proofs, $\tilde{\eta} \in L^1(N)$ almost surely, and c is almost surely finite. We thus deduce that

$$\begin{aligned}
|V_t^\lambda - V_t^\mu| \leqslant{}& Cc\int_0^t\int_0^1\int_{\mathbb{R}^*}|V_{s-}^\lambda - V_{s-}^\mu|\tilde{\eta}(s, \alpha, z)N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z) \\
&+ Cc|\lambda - \mu|\int_0^T\int_0^1\int_{\mathbb{R}^*}\rho(z)N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z).
\end{aligned} \tag{3.32}$$

Lemma A.1 allows us to conclude once again that (3.30) holds with

$$A = Cc\int_0^T\int_0^1\int_{\mathbb{R}^*}\rho(z)N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z)\exp\left(\int_0^T\int_0^1\int_{\mathbb{R}^*}\ln[1 + Cc\tilde{\eta}(s, \alpha, z)]N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z)\right) \tag{3.33}$$

□

We finally can prove the differentiability of V_T^λ.

Proposition 3.7. *For almost all* ω, *the map* $\lambda \mapsto V_T^\lambda$ *is differentiable on* Λ, *and* $\partial V_T^\lambda \partial \lambda = D_T^\lambda$.

Proof. We will check the existence of an almost surely finite random variable B such that almost surely, for all $0 \leqslant s \leqslant T$ and all $\lambda, \mu \in \Lambda$,

$$|V_s^\lambda - V_s^\mu - D_s^\lambda(\lambda - \mu)| \leqslant B|\lambda - \mu|^2, \tag{3.34}$$

which will of course suffice. Thus, let $\lambda, \mu \in \Lambda$ be fixed. Set $\Delta_s(\lambda, \mu) = V_s^\lambda - V_s^\mu - D_s^\lambda(\lambda - \mu)$. Using Hypothesis 2.6, Lemma 3.6, the definition of Γ^λ, the properties of the direction υ, and the notation of the proof of Lemma 3.5, we deduce the existence of a constant K such that, for all $s \leqslant T$,

$$\begin{aligned}
&|\gamma(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) - \gamma(V_{s-}^\mu - W_{s-}(\alpha), \Gamma^\mu(s, \alpha, z)) - \gamma'_X(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) \\
&\qquad \times D_{s-}^\lambda(\lambda - \mu) - \gamma'_z(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z))\upsilon(s, \alpha, z)(\lambda - \mu)| \\
&\leqslant |\gamma(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) - \gamma(V_{s-}^\mu - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) \\
&\quad - \gamma'_X(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z))(V_{s-}^\lambda - V_{s-}^\mu)| \\
&\quad + |\gamma'_X(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z))\Delta_{s-}(\lambda, \mu)| \\
&\quad + |\gamma(V_{s-}^\mu - W_{s-}(\alpha), \Gamma^\mu(s, \alpha, z)) - \gamma(V_{s-}^\mu - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) \\
&\quad - \gamma'_z(V_{s-}^\mu - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z))\upsilon(s, \alpha, z)(\lambda - \mu)| \\
&\quad + |[\gamma'_z(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z)) - \gamma'_z(V_{s-}^\mu - W_{s-}(\alpha), \Gamma^\lambda(s, \alpha, z))]\upsilon(s, \alpha, z)(\lambda - \mu)| \\
&\leqslant K[c\tilde{\eta}(s, \alpha, z)|V_{s-}^\lambda - V_{s-}^\mu|^2 + c\tilde{\eta}(s, \alpha, z)|\Delta_{s-}(\lambda, \mu)| \\
&\qquad + c(1 + |W_{s-}(\alpha)|^p)\rho(z)|\lambda - \mu|\{|\lambda - \mu| + |V_{s-}^\lambda - V_{s-}^\mu|\}] \\
&\leqslant B_1|\lambda - \mu|^2[\tilde{\eta}(s, \alpha, z) + (1 + |W_{s-}(\alpha)|^p)\rho(z)] + B_1\tilde{\eta}(s, \alpha, z)|\Delta_{s-}(\lambda, \mu)|,
\end{aligned} \tag{3.35}$$

where B_1 is an almost surely finite random variable. Since $\tilde{\eta}(s, \alpha, z) + (1 + |W_{s-}(\alpha)|^p)\rho(z)$ belongs almost surely to $L^1(N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z))$, it is easily deduced from (2.15) and (3.30) that, for all $t \leqslant T$, almost surely,

$$|\Delta_t(\lambda, \mu)| \leqslant B_2|\lambda - \mu|^2 + B_1 \int_0^t \int_0^1 \int_{\mathbb{R}^*} \tilde{\eta}(s, \alpha, z)|\Delta_{s-}(\lambda, \mu)| N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z), \tag{3.36}$$

where B_2 is an almost surely finite random variable. Lemma A.1 allows us to deduce that (3.34) holds, which concludes the proof. □

3.3. Choice of v and invertibility of D_T^0

We still have to check that, for a good choice of v, D_T^0 is almost surely invertible (this will provide condition (iii) in Lemma 1.1). To this end, we adapt to our context the ideas of Bichteler and Jacod (1983). Thanks to (3.14), we may write

$$D_t^0 = \int_0^t \mathrm{d}X_s . D_{s-}^0 + H_t, \tag{3.37}$$

where

$$X_t = \int_0^t \int_0^1 \int_{\mathbb{R}^*} \gamma'_X(V_{s-} - W_{s-}(\alpha), z) N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z), \tag{3.38}$$

$$H_t = \int_0^t \int_0^1 \int_{\mathbb{R}^*} \gamma'_z(V_{s-} - W_{s-}(\alpha), z) v(s, \alpha, z) N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z). \tag{3.39}$$

Using Jacod (1982), we compute D_T^0 explicitly. First, we denote by K_t the Doléans–Dade exponential of X: for I the unit matrix of $\mathcal{M}_{2\times 2}(\mathbb{R})$,

$$K_t = \mathcal{E}(X)_t = I + \int_0^t dX_s . K_{s-} = \prod_{s \leqslant t} (I + \Delta X_s). \tag{3.40}$$

Then we consider the following sequence of stopping times:

$$S_0 = 0, \qquad S_{n+1} = \inf\{t \in]S_n, T] | \det(I + \Delta X_t) = 0\}, \tag{3.41}$$

with the convention $\inf \varnothing = \infty$. Then the sequence S_n is totally inaccessible, and we have almost surely, for all n, $T \neq S_n$. Furthermore, it is clear that, for all n and all $t \in]S_n, S_{n+1}[$, the Doléans–Dade exponential $\mathcal{E}(X - X^{S_n})_t = \Pi_{S_n < s \leqslant t}(I + \Delta X_s)$ is invertible. We thus know, again from Jacod (1982), that if ω satisfies $S_n < T < S_{n+1} = \infty$, then

$$D_T^0 = \mathcal{E}(X - X^{S_n})_T \left[\Delta H_{S_n} + \int_{]S_n, T]} \mathcal{E}(X - X^{S_n})_{s-}^{-1} (I + \Delta X_s)^{-1} \, \mathrm{d}H_s \right]. \tag{3.42}$$

We finally rewrite (3.42) explicitly:

Proposition 3.8. *For almost all ω, there exists n such that $S_n < T < S_{n+1}$, and*

$$\begin{aligned} D_T^0 = \mathcal{E}(X - X^{S_n})_T \Bigg[\Delta H_{S_n} & \\ + \int_{]S_n, T]} \int_0^1 \int_{\mathbb{R}^*} & \mathcal{E}(X - X^{S_n})_{s-}^{-1} (I + \gamma'_X(V_{s-} - W_{s-}(\alpha), z))^{-1} \gamma'_z(V_{s-} - W_{s-}(\alpha), z) \\ & \times v(s, \alpha, z) N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z) \Bigg] \end{aligned} \tag{3.43}$$

We now choose v. First of all, we denote by k a function from $\mathcal{M}_{2\times 2}(\mathbb{R})$ into [0, 1] such that

$$k(M) = 0 \quad \Leftrightarrow \quad \det M = 0, \tag{3.44}$$

and such that the map

$$M \mapsto \begin{cases} k(M)(M^{-1})^{\mathrm{T}} & \text{if } \det M \neq 0, \\ 0 & \text{if } \det M = 0, \end{cases} \tag{3.45}$$

is of class C_b^∞ from $\mathcal{M}_{2\times 2}(\mathbb{R})$ into itself.

We also consider a C^1 function f from $\mathbb{R}^*$ into $]0, 1]$ such that, for some $c \in]0, 1]$,

$$|f| + |f'| \leqslant c, \qquad |f(z)| + |f'(z)| \leqslant (|z|/2) \wedge (1/|z|), \qquad |f| + |f'| \in L^1(\mathbb{R}^*, \mathrm{d}z). \tag{3.46}$$

Definition 3.9. *We set*

$$\begin{aligned} \upsilon(s, \alpha, z) = {} & \frac{\gamma_z'(V_{s-} - W_{s-}(\alpha), z)^{\mathrm{T}}}{1 + |V_{s-}|^p + |W_{s-}(\alpha)|^p} \\ & \times \frac{(I + \gamma_X'(V_{s-} - W_{s-}(\alpha), z))^{-1,\mathrm{T}} \times k(I + \gamma_X'(V_{s-} - W_{s-}(\alpha), z))}{1 + |V_{s-}|^p + |W_{s-}(\alpha)|^p} \\ & \times \mathcal{E}(X - X^{S_n})_{s-}^{-1,\mathrm{T}} k(\mathcal{E}(X - X^{S_n})_{s-}) \times f(z). \end{aligned} \tag{3.47}$$

Then the following result is straightforward.

Lemma 3.10. *If c (see (3.46)) is small enough, which we assume, then the map υ defined in Definition 3.9 is a direction in the sense of Definition 3.1.*

In view of (3.43), our main interest in this direction is the following.

Lemma 3.11. *With our choice for υ, $\Delta H_{S_n} = 0$ for all n and almost all ω such that $S_n < T$.*

Proof. The stopping time S_n is a jump time of the Poisson measure. Let us denote by (α_{S_n}, z_{S_n}) the associated jump. We know, from the definition of S_n, that $\det(I + \Delta X_{S_n}) = 0$, which implies that $\det(I + \gamma_X'(V_{S_n} - W_{S_n}(\alpha_{S_n}), z_{S_n})) = 0$. Hence, thanks to the definition of υ and k, we deduce that $\upsilon(S_n, \alpha_{S_n}, z_{S_n}) = 0$, which clearly implies the result thanks to (3.39). □

Remark 3.12. (i) *We deduce from the lemma above that in order to prove that D_T^0 is almost surely invertible, it suffices to check that, for any n and for almost all ω satisfying $S_n < T < S_{n+1}$, Δ_T^n is invertible, where*

$$\begin{aligned} \Delta_T^n = \int_{]S_n, T]} \int_0^1 \int_{\mathbb{R}^*} & \mathcal{E}(X - X^{S_n})_{s-}^{-1} (I + \gamma_X'(V_{s-} - W_{s-}(\alpha), z))^{-1} \\ & \times \gamma_z'(V_{s-} - W_{s-}(\alpha), z) \upsilon(s, \alpha, z) N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z). \end{aligned} \tag{3.48}$$

(ii) *We can also write, using the explicit expression for υ,*

$$\Delta_T^n = \int_{]S_n,T]} \mathcal{E}(X - X^{S_n})_{s-}^{-1}\, \mathrm{d}R_s . \mathcal{E}(X - X^{S_n})_{s-}^{-1,\mathrm{T}}, \tag{3.49}$$

where

$$R_t = \int_{]S_n,T]} \int_0^1 \int_{\mathbb{R}^*} J(V_{s-} - W_{s-}(\alpha), z) \times h(s, \alpha, z) \times f(z) N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z) \tag{3.50}$$

with, for $X \in \mathbb{R}^2$,

$$J(X, z) = (I + \gamma'_X(X, z))^{-1} \gamma'_z(X, z) \gamma'_z(X, z)^{\mathrm{T}} (I + \gamma'_X(X, z))^{-1,\mathrm{T}} \tag{3.51}$$

and

$$h(s, \alpha, z) = \frac{1}{(1 + |V_{s-}|^p + |W_{s-}(\alpha)|^p)^2} \times k(I + \gamma'_X(V_{s-} - W_{s-}(\alpha), z)) k(\mathcal{E}(X - X^{S_n})_{s-}). \tag{3.52}$$

For all X, z, $J(X, z)$ *is a symmetric non-negative matrix. The function* h *is always non-negative. Hence,* R_t *is non-negative, symmetric, and increasing for the strong order. Since* h *never vanishes, and since* $\mathcal{E}(X - X^{S_n})_{s-}^{-1}$ *is invertible for all* $s \in]S_n, T]$, *it suffices to prove that, for all* $0 \leqslant s < t \leqslant T$, $\bar{R}_t - \bar{R}_s$ *is almost surely invertible, where*

$$\bar{R}_t = \int_{]S_n,T]} \int_0^1 \int_{\mathbb{R}^*} J(V_{s-} - W_{s-}(\alpha), z) \times f(z) N(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}z). \tag{3.53}$$

One may, for example, verify that, for all $0 \leqslant s < t \leqslant T$ *and all* $Y \in \mathbb{R}^2/\{0\}$, $Y^T(\bar{R}_t - \bar{R}_s)Y > 0$ *almost surely.*

Before concluding that D_T^0 is almost surely invertible, we state and prove a last lemma.

Lemma 3.13. *For all* $t \in]0, T]$, *the law of* V_t (*and thus that of* W_t) *is not a Dirac mass.*

Proof. Using the conservation of momentum and kinetic energy (see Proposition 2.7), one can easily prove that, for any a in $\mathbb{R}^2$, the quantity $\mathrm{E}(|V_t - a|^2)$ does not depend on t. Hence, if the law of V_t were a Dirac mass at some $a \in \mathbb{R}^2$, we would deduce that the law of V_0, i.e. Q_0, is also a Dirac mass at a. This contradicts Hypothesis 2.6. □

We finally prove that condition (iii) of Lemma 1.1 is satisfied by V_T.

Proposition 3.14. *With our choice of* υ, D_T^0 *is almost surely invertible.*

Proof. We of course use Remark 3.12. The proof necessitates several steps.

Step 1. Let X and Y be two non-zero vectors of $\mathbb{R}^2$. Then

$$\int_{\mathbb{R}^*} 1_{\{Y^{\mathrm{T}} \gamma'_z(X,z) \gamma'_z(X,z)^{\mathrm{T}} Y \neq 0\}}\, \mathrm{d}z = \infty. \tag{3.54}$$

To prove this, we first set $I(X, z) = \gamma'_z(X, z) \gamma'_z(X, z)^{\mathrm{T}}$. Notice that, by definition of γ,

$$I(X, z) = (g_z'(X, z))^2 A'(g(X, z))XX^{\mathrm{T}}A'(g(X, z))^{\mathrm{T}}. \tag{3.55}$$

But it is clear (see Section 2) that g_z' never vanishes. Hence, thanks to the substitution $\theta = g(X, z)$, we obtain (see Section 2 again)

$$\int_{\mathbb{R}^*} 1_{\{Y^{\mathrm{T}}I(X,z)Y\neq 0\}}\,\mathrm{d}z = \int_{-\pi}^{\pi} 1_{\{Y^{\mathrm{T}}A'(\theta)XX^{\mathrm{T}}A'(\theta)^{\mathrm{T}}Y\neq 0\}} B(X, \theta)\mathrm{d}\theta. \tag{3.56}$$

But a simple computation shows that $Y^{\mathrm{T}}A'(\theta)XX^{\mathrm{T}}A'(\theta)^{\mathrm{T}}Y$ will $1_{[-\pi,\pi]}(\theta)\mathrm{d}\theta$-almost never vanish (for $X \neq 0$ and $Y \neq 0$ fixed). Since $\int_{-\pi}^{\pi} B(X, \theta)\mathrm{d}\theta = \infty$, the proof of step 1 is complete.

Step 2. For all $s \in [0, T]$ and almost all ω,

$$\int_0^1 1_{\{V_{s-}-W_{s-}(\alpha)\neq 0\}}\,\mathrm{d}\alpha > 0. \tag{3.57}$$

Indeed, we know from Lemma 3.13 that $\mathcal{L}_\alpha(W_s)$ is not a Dirac mass. Hence, for any deterministic $X \in \mathbb{R}^2$,

$$\int_0^1 1_{\{X-W_{s-}(\alpha)\neq 0\}}\,\mathrm{d}\alpha = P_\alpha(W_s \neq X) > 0. \tag{3.58}$$

Since ω is fixed, $V_{s-}(\omega)$ is 'α-deterministic', and hence (3.58) holds for $X = V_{s-}(\omega)$, which leads immediately to (3.57).

Step 3. Putting together steps 1 and 2, we finally deduce that almost surely, for all non-zero vectors $Y \in \mathbb{R}^2$ and all $s \in [0, T]$,

$$\int_0^1\int_{\mathbb{R}^*} 1_{\{Y^{\mathrm{T}}\gamma_z'(V_{s-}-W_{s-}(\alpha),z)\gamma_z'(V_{s-}-W_{s-}(\alpha),z)^{\mathrm{T}}Y\neq 0\}}\,\mathrm{d}\alpha\,\mathrm{d}z = \infty. \tag{3.59}$$

Step 4. Let $s > 0$ and $Y \in \mathbb{R}^2\backslash\{0\}$ be fixed. We now prove that on the set $S_n < T < S_{n+1} = \infty$, for all $s > S_n$, almost surely, for all $t \in]s, T]$,

$$Y^{\mathrm{T}}(\overline{R}_t - \overline{R}_s)Y > 0. \tag{3.60}$$

To this end, it suffices to show that the stopping time defined by

$$\tau(Y) = \inf\left\{ u > s \middle| \int_s^u\int_{\mathbb{R}^*}\int_0^1 1_{\{Y^{\mathrm{T}}J(V_{s-}-W_{s-}(\alpha),\, z)Y>0\}} N(\mathrm{d}s,\, \mathrm{d}\alpha,\, \mathrm{d}z) > 0 \right\} \tag{3.61}$$

satisfies $\tau(Y) = s$ almost surely. We have, by construction,

$$\int_s^{\tau(Y)}\int_{\mathbb{R}^*}\int_0^1 1_{\{Y^{\mathrm{T}}J(V_{s-}-W_{s-}(\alpha),\, z)Y>0\}} N(\mathrm{d}s,\, \mathrm{d}\alpha,\, \mathrm{d}z) \leqslant 1. \tag{3.62}$$

Taking expectations, we obtain

$$\mathrm{E}\left(\int_s^{\tau(Y)}\int_{\mathbb{R}^*}\int_0^1 1_{\{Y^{\mathrm{T}}J(V_{s-}-W_{s-}(\alpha),\, z)Y>0\}}\,\mathrm{d}s\,\mathrm{d}\alpha\,\mathrm{d}z\right) \leqslant 1 \tag{3.63}$$

and we deduce that, almost surely,

$$\int_s^{\tau(Y)} \int_{\mathbb{R}^*} \int_0^1 1_{\{Y^\top J(V_{s-}-W_{s-}(\alpha),z)Y>0\}} \, \mathrm{d}s \, \mathrm{d}\alpha \, \mathrm{d}z < \infty. \tag{3.64}$$

Due to (3.59), this is impossible, unless $\tau(Y) = s$ almost surely.

Step 5. The previous step shows that on the set $S_n < T < S_{n+1} = \infty$, for all $s \in]S_n, T]$, almost surely, for all $u \in]s, T]$, $\bar{R}_u - \bar{R}_s$ is invertible. What we have to prove is that on the set $S_n < T < S_{n+1} = \infty$, almost surely, for all $s \in]S_n, T]$, for all $u \in]s, T]$, $\bar{R}_u - \bar{R}_s$ is invertible. This extension is straightforward, using the fact that $\bar{R}$ is increasing. The proof is complete. □

3.4. Conclusion

We are finally in a position to fulfil the aim of this section.

Proof of Theorem 2.8. Since $T > 0$ is arbitrarily fixed, it of course suffices to prove that the law of V_T admits a density. We thus apply Lemma 1.1 with $X = V_T$. The family X^λ is defined by $V_T^\lambda = V_T \circ S^\lambda$, the shift S^λ being defined by (3.5), relative to the direction υ chosen in Definition 3.9. Condition (i) of Lemma 1.1 is satisfied thanks to Proposition 3.2. Condition (ii) holds thanks to Proposition 3.7. Finally, Proposition 3.14 shows that condition (iii) is met. Hence the law of V_T admits a density, as required. □

Appendix

Our purpose is to prove the following Gronwall-type lemma.

Lemma A.1. *Let $\mathcal{X}$ be a measurable space. We consider a counting σ-finite measure $\mu(\mathrm{d}t, \mathrm{d}x)$ on $[0, T] \times \mathcal{X}$. Let $\eta(s, x)$ be a positive function belonging to $L^1(\mu)$. Then every bounded positive function φ_t on $[0, T]$, satisfying, for all $t > 0$,*

$$\varphi_t \leqslant a + \int_0^t \int_{\mathcal{X}} \varphi_{s-} \eta(s, x) \mu(\mathrm{d}s, \mathrm{d}x) \tag{A.1}$$

is bounded by

$$\sup_{[0,T]} \varphi_t \leqslant a \exp\left(\int_0^T \int_{\mathcal{X}} \ln(1 + \eta(s, x)) \mu(\mathrm{d}s, \mathrm{d}x) \right). \tag{A.2}$$

Proof. We divide the proof into two steps.

Step 1. We begin with the case where $\mu(\eta \neq 0) < \infty$. In this case, we can consider the support of μ to be finite, and thus that μ is of the form $\sum_{i=1}^n \delta_{(T_i, X_i)}$, with $0 < T_1 < T_2 < \ldots < T_n < T$. Then we use (A.1). First, for all $t \in [0, T_1[$, $\varphi_t \leqslant a$, from which we deduce, for all $t \in [T_1, T_2[$,

$$\varphi_t \leqslant a + a\eta(T_1, X_1) \leqslant a(1 + \eta(T_1, X_1)), \tag{A.3}$$

which clearly also holds for all $t \in [0, T_2[$. And so on. We finally obtain that, for all $t \in [0, T]$,

$$\begin{aligned}\varphi_t &\leqslant a(1+\eta(T_1, X_1)) \times \ldots \times (1+\eta(T_n, X_n)) \\ &\leqslant a \exp\left(\sum_{i=1}^{n} \ln(1+\eta(T_i, X_i))\right) \\ &\leqslant a \exp\left(\int_0^T \int_{\mathcal{X}} \ln(1+\eta(s, x))\mu(\mathrm{d}s, \mathrm{d}x)\right),\end{aligned} \tag{A.4}$$

as required.

Step 2. If $\mu(\eta \neq 0) = \infty$, then we split the space $\mathcal{X}$ into $\mathcal{X}_\epsilon \cup \mathcal{X}_\epsilon^c$, in such a way that for all ϵ, $\mu([0, T] \times \mathcal{X}_\epsilon) < \infty$, and such that $\mathcal{X}_\epsilon$ grows to $\mathcal{X}$ when ϵ goes to 0. Then we rewrite (A.1) as

$$\varphi_t \leqslant (a + u_\epsilon) + \int_0^t \int_{\mathcal{X}_\epsilon} \varphi_{s-}\eta(s, x)\mu(\mathrm{d}s, \mathrm{d}x), \tag{A.5}$$

where $u_\epsilon = \|\varphi\|_\infty \int_0^t \int_{\mathcal{X}_\epsilon^c} \eta(s, x)\mu(\mathrm{d}s, \mathrm{d}x)$ clearly goes to 0 since $\eta \in L^1(\mu)$. Applying step 1, we obtain, for each ϵ,

$$\sup_{[0,T]} \varphi_t \leqslant (a + u_\epsilon)\exp\left(\int_0^T \int_{\mathcal{X}_\epsilon} \ln(1+\eta(s, x))\mu(\mathrm{d}s, \mathrm{d}x)\right) \tag{A.6}$$

Making ϵ tend to 0 immediately leads to (A.2). □

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