6. THE RUSSO-SEYMOUR-WELSH THEOREM.

The object of this chapter is a result which states that if the crossing probabilities of certain rectangles in both the horizontal and vertical direction are bounded away from zero, then so are the crossing probabilities for larger rectangles. This result will then be used to prove the existence of occupied circuits surrounding the origin. The idea is to connect an occupied horizontal crossing of $[0,n_1] \times [0,n_2]$ and an occupied horizontal crossing of $[m,n_1 + m] \times [0,n_2]$ by means of a suitable occupied vertical crossing, in order to obtain a horizontal crossing of $[0,n_1 + m] \times [0,n_2]$. This would be quite simple (compare the proof of Lemma 6.2) if one had a lower bound for the probability of an occupied vertical crossing of $[m,n_1] \times [0,n_2]$, but in the applications one only has estimates for the existence of occupied vertical crossings of rectangles which are wider and/or lower. One therefore has to use some trickery, based on symmetry to obtain the desired connections. Such tricks were developed independently by Russo (1978) and Seymour and Welsh (1978). (See also Smythe and Wierman (1978), Ch. 3 and Russo (1981).) These papers dealt with the one-parameter problems on the graphs $\mathcal{G}_0$ or $\mathcal{G}_1$ (see Ex. 2.1(i) and (ii)) and therefore had at their disposal symmetry with respect to both coordinate axes, as well as invariance of the problem under interchange of the horizontal and vertical direction. We believe that neither of these properties is necessary, but so far we still need at least one axis of symmetry. We also have to restrict ourselves to a planar modification $\mathcal{G}_{p\ell}$ of a graph $\mathcal{G}$ which is one of a matching pair of graphs in $\mathbb{R}^2$.

Throughout this chapter we deal with the following setup:

(6.1) $(\mathcal{G},\mathcal{G}^*)$ is a matching pair based on $(\mathcal{M},\mathcal{F})$ for some mosaic $\mathcal{M}$ satisfying (2.1)-(2.5) and subset $\mathcal{F}$ of its collection of faces (see Sect. 2.2). $\mathcal{G}_{p\ell}$ is the planar modification of $\mathcal{G}$ (see Sect. 2.3).

(6.2) $\mathcal{G}$ and $\mathcal{G}_{p\ell}$ are periodic and the second coordinate axis $L_0 : x(1) = 0$ is an axis of symmetry for $\mathcal{G}$ and for $\mathcal{G}_{p\ell}$ (Note that we can construct $\mathcal{G}_{p\ell}$ symmetrically with respect to L_0 as soon as $\mathcal{G}$ is symmetric with respect to this axis, by virtue of Comment 2.4(iii).)

(6.3) P is a product measure on $(\Omega_{\mathcal{V}}, \mathcal{B}_{\mathcal{V}})$, where $\mathcal{V}$ is the vertex set of $\mathcal{G}_{p\ell}$ (compare Sect. 3.1). P is symmetric with respect to L_0, i.e. if $v = (v(1), v(2))$ is any vertex of $\mathcal{G}_{p\ell}$, then $P\{v = (v(1), v(2))$ is occupied$\}$ $= P\{(-v(1), v(2))$ is occupied$\}$. (It is not required that (2.15), (2.16) be satisfied).

Finally Λ is a constant such that

(6.4) diameter of any edge of $\mathcal{G}$ or of $\mathcal{G}_{p\ell}$ is $\leq \Lambda$.

<u>Theorem 6.1</u>. <u>Assume</u> (6.1) - (6.4). <u>Let</u> $\pi \geq 1$ <u>be an integer and assume that</u> $\bar{n} = (n_1, n_2)$ <u>and</u> $\bar{m} = (m_1, m_2)$ <u>are integral vectors for which</u>

(6.5) $$\sigma(\bar{n};1,p,\mathcal{G}_{p\ell}) = P_p\{\exists \text{ an occupied horizontal crossing on } \mathcal{G}_{p\ell} \text{ of } [0,n_1] \times [0,n_2]\} \geq \delta_1 > 0 ,$$

(6.6) $$\sigma(\bar{m};2,p,\mathcal{G}_{p\ell}) = P_p\{\exists \text{ an occupied vertical crossing on } \mathcal{G}_{p\ell} \text{ of } [0,m_1] \times [0,m_2]\} \geq \delta_2 > 0 ,$$

<u>and</u>

(6.7) $$\frac{1}{\pi} \leq \frac{m_i}{n_i} \leq \pi \qquad i = 1,2$$

<u>Then there exist</u> $n_0 = n_0(\mathcal{G},\pi)$, <u>and for each integer</u> $k \geq 1$ <u>an</u> $f = f(\delta_1,\delta_2,\pi,k) > 0$ <u>depending on the indicated parameters only</u>, <u>such that for</u>

(6.8) $$n_i \geq n_0 = n_0(\mathcal{G},\pi)$$

<u>one has</u>

(6.9) $\sigma(kn_1, 2n_2); 1, p, \mathcal{G}_{p\ell}) = P_p\{\exists$ occupied horizontal crossing

on $\mathcal{G}_{p\ell}$ of $[0, kn_1] \times [0, 2n_2]\} \geq f(\delta_1, \delta_2, \pi, k) > 0$

<u>and</u>

(6.10) $\sigma((\pi + 3)n_1, kn_2); 2, p, \mathcal{G}_{p\ell}) = P_p\{\exists$ occupied vertical crossing

on $\mathcal{G}_{p\ell}$ of $[0, (\pi + 3)n_1] \times [0, kn_2]\} \geq f(\delta_1, \delta_2, \pi, k) > 0$.

<u>Moreover, for fixed</u> π, k

(6.11) $$\lim_{\substack{\delta_1 \to 1 \\ \delta_2 \to 2}} f(\delta_1, \delta_2, \pi, k) = 1 .$$

<u>Corollary 6.1</u>. <u>Under the hypotheses of Theorem 6.1 (including (6.8))</u>

(6.12) $P_p\{\exists$ occupied circuit on $\mathcal{G}_{p\ell}$ surrounding 0, and inside

the annulus $[-2(\pi + 3)n_1, 2(\pi + 3)n_1] \times [-3n_2, 3n_2]$

$(-(\pi + 3)n_1, (\pi + 3)n_1) \times (-n_2, n_2)\} \geq f^4(\delta_1, \delta_2, \pi, 4\pi + 12)$.

The very long proof will be broken down into several lemmas. If one is content with proving the theorem only for the case $m_1 = n_1$, $m_2 = n_2 (\pi = 1)$ and under the additional hypothesis that <u>both</u> the $x(1)$ and $x(2)$-axis are symmetry-axes, then Lemma 6.1 suffices. Since these extra hypotheses hold for most examples the reader is strongly urged to stop with Lemma 6.1 at first reading, or to read the original proofs of Russo (1978) or Seymour and Welsh (1978). The proof of Theorem 6.1 in its full generality is only included for readers interested in technical details, with the hope that it will lead someone to a proof which does not use symmetry.

The principal ideas appear already in the first lemma. These ideas are due to Russo (1978), (1981) and Seymour and Welsh (1978). A very important role is played by an analogue of the strong Markov property, not with respect to a stopping time, but with respect to a lowest occupied horizontal crossing (see step (b) of Lemma 6.1). Harris (1960) seems to have been the first person to use this property.

In each of the lemmas we construct an occupied crossing of a large rectangle by connecting several occupied horizontal and vertical crossings. The existence of suitable crossings will come from (6.5) - (6.7). The difficulty is to make sure that the vertical crossings really intersect the horizontal ones, so that they can all be connected. To do this we shall repeatedly use the FKG inequality (and symmetry considerations) to restrict the locations of the crossings. In other words, if we know that with high probability there exists an occupied crossing of some rectangle, we shall deduce that there is also a high probability for the existence of an occupied crossing with additional restrictions on its location. Lemma 6.3 and the proofs of Lemmas 6.6 - 6.8 exemplify this kind of argument.

Since we only consider paths and crossings on $\mathcal{G}_{p\ell}$ we shall drop the specification "on $\mathcal{G}_{p\ell}$" for paths for the remainder of this chapter. We remind the reader that $\mathcal{G}_{p\ell}$ is planar, and that a path in our terminology has therefore no self intersections (see beginning of Sect. 2.3). We shall suppress the subscript p in P_p. (6.1) - (6.4) will be in force throughout this chapter.

Lemma 6.1. Assume

(6.13) $$\sigma((\ell_1,\ell_2);\ 1,p,\mathcal{G}_{p\ell}) \geq \delta_3 > 0$$

and

(6.14) $$\sigma((\ell_3,\ell_2);\ 2,p,\mathcal{G}_{p\ell}) \geq \delta_4 > 0$$

for some integers $\ell_1,\ell_2,\ell_3 \geq 1$ with[1)]

(6.15) $$\ell_3 \leq \frac{3}{2}\ell_1, \quad \ell_1 \geq 32 + 16\Lambda, \quad \ell_2 > \Lambda.$$

Then for each k there exists an $f_1(\delta_3,\delta_4,k) > 0$ such that

(6.16) $$\sigma((k\ell_1,\ell_2);\ 1,p,\mathcal{G}_{p\ell}) \geq f_1(\delta_3,\delta_4,k) > 0$$

and

(6.17) $$\lim_{\substack{\delta_3 \to 1 \\ \delta_4 \to 1}} f_1(\delta_3,\delta_4,k) = 1 \quad .$$

1) The requirement $\ell_3 \leq \frac{3}{2}\ell_1$ can be replaced by $\ell_3 \leq (2-\delta)\ell_1$ for any $\delta > 0$.

Proof: The proof is somewhat lengthy and will be broken down into three steps.

Step (a). Consider a fixed horizontal crossing $r = (v_0, e_1, \dots, e_\nu, v_\nu)$ of $[0,\ell_1 - 1] \times [0,\ell_2]$ such that e_ν intersects the right edge, $\{\ell_1 - 1\} \times [0,\ell_2]$, in its interior only. In view of Def. 3.1 and $\ell_1 > 1 + \Lambda$, this implies that r intersects the vertical line $L: x(1) = \ell_1 - 1$ only in the open segment $\{\ell_1 - 1\} \times (0,\ell_2)$. The rather trivial technical reasons for insisting that the intersection of r with L is in this open segment rather than the closed segment $\{\ell_1 - 1\} \times [0,\ell_2]$ will become clear below. For the moment we merely observe that any horizontal crossing r_1 of $[0,\ell_1] \times [0,\ell_2]$ contains a path r with the above properties. Indeed we can simply take for r the initial piece of r_1 up till and including the first edge e of r_1 which intersects L. (Note that L is an axis of symmetry of $\mathcal{G}_{p\ell}$ because $L_0: x(1) = 0$ is an axis of symmetry and $\mathcal{G}_{p\ell}$ is periodic with period $\xi_1 = (1,0)$. As explained in Comment 2.4(ii) this implies that e intersects L in exactly one point; e cannot be in case (a) or (b) of that Comment because it has one endpoint strictly to the left of L.) Therefore

(6.18) $$\begin{aligned} &P\{\exists \text{ occupied crossing } r = (v_0, e_1, \dots, e_\nu, v_\nu) \text{ of } \\ &[0,\ell_1 - 1] \times [0,\ell_2] \text{ which intersects } L \text{ only in} \\ &\{\ell_1 - 1\} \times (0,\ell_2)\} \geq \sigma((\ell_1,\ell_2); 1,p,\mathcal{G}_{p\ell}) \geq \delta_3 . \end{aligned}$$

We shall write $\tilde{e}(\tilde{v})$ for the reflection of an edge e (a vertex v) in L. $\tilde{r}$ will denote the reflection of r in L. Then for r as above $r \cup \tilde{r}$ is a horizontal crossing of $[0,2\ell_1 - 2] \times [0,\ell_2]$, provided we interpret this statement with a little care. If v_ν lies on $\{\ell_1 - 1\} \times (0,\ell_2)$ then $r \cup \tilde{r}$ is simply the path $(v_0, e_1, \dots, e_\nu, v_\nu = \tilde{v}_\nu, \tilde{e}_\nu, \tilde{v}_{\nu-1}, \dots, \tilde{v}_0)$. As observed above, by Comment 2.4(ii) the only other possibility is that the intersection of e_ν and L is the midpoint of e_ν. Then $e_\nu = \tilde{e}_\nu$ and $r \cup \tilde{r}$ should be interpreted as the path $(v_0, e_1, \dots, e_\nu, v_\nu = \tilde{v}_{\nu-1}, \tilde{e}_{\nu-1}, \tilde{v}_{\nu-2}, \dots, \tilde{v}_0)$. Note that we insisted on e_ν intersecting L in the open segment $\{\ell_1 - 1\} \times (0,\ell_2)$ precisely to make $r \cup \tilde{r}$ a horizontal crossing of $[0, 2\ell_1 - 2] \times [0,\ell_2]$.

Now we take for J_2 the perimeter of $[0,2\ell_1 - 2] \times [0,\ell_2]$ viewed as a Jordan curve. We further take $B_1 = \{0\} \times [0,\ell_2]$, $B_2 = \{2\ell_1 - 2\} \times [0,\ell_2]$, $A_2 = [0,2\ell_1 - 2] \times \{0\}$ and $C_2 = [0,2\ell_1 - 2] \times \{\ell_2\}$. These are the left, right, bottom and top edge of $[0,2\ell_1 - 2] \times [0,\ell_2]$, respectively. These four edges make up J_2 and $r \cup \tilde{r}$ satisfies the analogues of (2.23) - (2.25), i.e., all its edges and vertices except for $v_0, e_1, \tilde{e}_1$ and $\tilde{v}_0$ lie in $\mathrm{int}(J_2)$, while $e_1(\tilde{e}_1)$ has exactly one point in common with $B_1(B_2)$. We can therefore define $J_2^+(r \cup \tilde{r})$ $J_2^-(r \cup \tilde{r}))$ as the component of $\mathrm{int}(J_2) \setminus r \cup \tilde{r}$ which contains $C_2(A_2)$ in its boundary, exactly as in Def. 2.11. We also introduce the events[1)]

(6.19) $\quad D(r) := \{ \exists$ path $s = (w_0, f_1, \ldots, f_\rho, w_\rho)$ such that $w_1, \ldots, w_{\rho-1}$ are occupied, $w_0 = v_i$ for some $v_i \in r$, f_ρ intersects C_2 in some point ζ, and

$$(f_1 \setminus \{w_0\}, w_1, f_2, \ldots, f_{\rho-1}, w_{\rho-1}, [w_{\rho-1}, \zeta))$$
$$\subset \{J_2^+(r \cup \tilde{r}) \cap [\lfloor \tfrac{\ell_1}{8} \rfloor, 2\ell_1 - 2 - \lfloor \tfrac{\ell_1}{8} \rfloor] \times (0,\ell_2)\}\}$$

and $D(\tilde{r})$, defined as $D(r)$, except that one now requires $w_0 = \tilde{v}_i$ for some $v_i \in r$, or equivalently that w_0 is a vertex on $\tilde{r}$. We shall prove in this step that

$$P\{D(r)\} \geq 1 - \sqrt{1-\delta_4} \quad . \tag{6.20}$$

Note that (6.19) estimates the probability of the existence of an "occupied connection from r to the upper edge C_2 of $[0,2\ell_1 - 2] \times [0,\ell_2]$ above $r \cup \tilde{r}$" and in the rectangle $[\lfloor \frac{\ell_1}{8} \rfloor, 2\ell_1 - 2 - \lfloor \frac{\ell_1}{8} \rfloor] \times [0,\ell_2]$ (see Fig. 6.1).

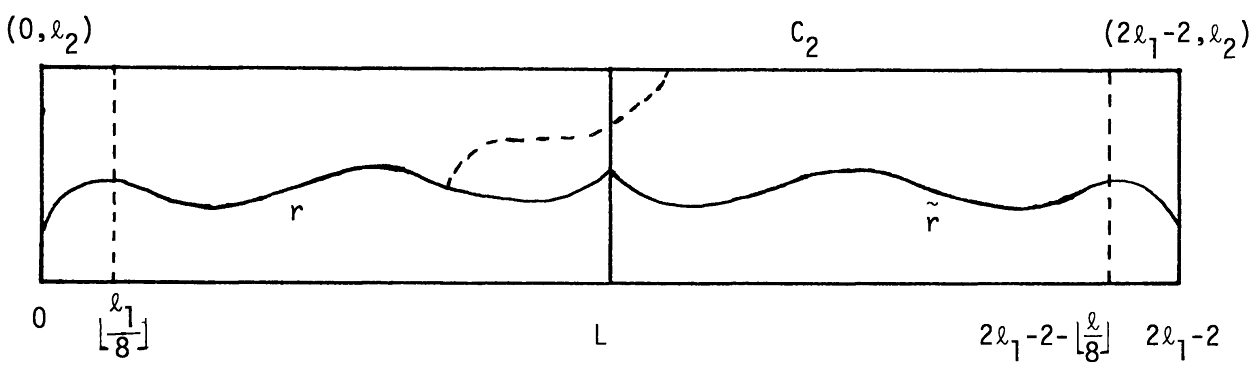

Figure 6.1

---

1) Recall that $\lfloor a \rfloor$ denotes the largest integer $\leq a$.

Before starting on the proof of (6.20) proper we first observe that for any two increasing events E_1, E_2 one obtains from the FKG inequality

$$P\{E_1 \cup E_2\} = 1 - P\{E_1^c \cap E_2^c\} \leq 1 - P\{E_1^c\}\, P\{E_2^c\}$$

or

$$(6.21) \qquad (1 - P\{E_1\})(1 - P\{E_2\}) \leq 1 - P\{E_1 \cup E_2\} \quad .$$

We apply this with $E_1 = D(r)$, $E_2 = D(\tilde{r})$. Since $D(\tilde{r})$ is obtained by "reflecting $D(r)$ in L" and L is an axis of symmetry we have $P\{D(\tilde{r})\} = P\{D(r)\}$ and (6.21) becomes

$$P\{D(r)\} \geq 1 - (1 - P\{D(r) \cup D(\tilde{r})\})^{1/2} \quad .$$

For (6.20) it therefore suffices to prove

$$(6.22) \qquad \begin{aligned} &P\{D(r) \cup D(\tilde{r})\} = P\{\exists \text{ path } s = (w_0, f_1, \dots, f_\rho, w_\rho) \text{ such} \\ &\text{that } w_1, \dots, w_{\rho-1} \text{ are occupied, } w_0 = v_i \text{ or } \tilde{v}_i \text{ for} \\ &\text{some } i,\ f_\rho \text{ intersects } C_2 \text{ in some point } \zeta \text{ and} \\ &(f_1 \setminus \{w_0\}, w_1, f_2, \dots, f_{\rho-1}, w_{\rho-1}, [w_{\rho-1}, \xi)) \\ &\subset \{J_2^+(r \cup \tilde{r}) \cap [\lfloor \tfrac{\ell_1}{8} \rfloor, 2\ell_1 - 2 - \lfloor \tfrac{\ell_1}{8} \rfloor] \times (0, \ell_2)\}\} \geq \delta_4 \; . \end{aligned}$$

To prove (6.22) assume for the moment that there exists an occupied vertical crossing $t = (u_0, g_1, \dots, g_\tau, u_\tau)$ of $[\lfloor \frac{\ell_1}{8} \rfloor, 2\ell_1 - 2 - \lfloor \frac{\ell_1}{8} \rfloor] \times [0, \ell_2]$. Then t contains a continuous curve from the bottom to the top of this rectangle, while $r \cup \tilde{r}$ contains a continuous curve from the left to the right edge of this rectangle. Both these curves are contained in the rectangle and must therefore intersect. Thus $r \cup \tilde{r}$ and t intersect, and since both are paths on the planar graph $\mathcal{G}_{p\ell}$ they intersect in a vertex. Let u_α be the last point of t on $r \cup \tilde{r}$ and let u_α equal v_i or $\tilde{v}_i$, $v_i \in r$. Since t is a vertical crossing of $[\lfloor \frac{\ell_1}{8} \rfloor, 2\ell_1 - 2 - \lfloor \frac{\ell_1}{8} \rfloor] \times [0, \ell_2]$ and $\ell_2 > \Lambda$ g_τ is the only edge of t which intersects C_2. Let ζ be the first intersection of g_τ with C_2, so that the segment from $u_{\tau-1}$ to ζ (excluding ζ) is disjoint from C_2. Since C_2 is part of $Fr(J_2^+(r \cup \tilde{r}))$, and $J_2^+(r \cup \tilde{r})$ as well as $u_{\tau-1}$ lie below C_2, it follows that near ζ the segment of g_τ from $u_{\tau-1}$ to ζ lies in $J_2^+(r \cup \tilde{r})$. Moreover, the connected set $g_{\alpha+1} \setminus \{u_\alpha\} \cup g_{\alpha+2} \cup \ldots \cup g_{\tau-1} \cup$ (the segment of g_τ from $u_{\tau-1}$ to ζ) does not intersect $\mathrm{Fr}(J_2^+(r \cup \tilde{r}))$. Consequently $g_{\alpha+1} \setminus \{u_\alpha\}, \ldots, g_{\tau-1}$ and the vertices $u_{\alpha+1}, \ldots, u_{\tau-1}$ on these edges also lie in $J_2^+(r \cup \tilde{r})$. These observations show that the path $(u_\alpha, g_{\alpha+1}, \ldots, g_\tau, u_\tau)$ satisfies the requirements for s in (6.22) (if we take $w_j = u_{\alpha+j}$, $f_j = g_{\alpha+j}$, $\rho = \tau - \alpha$). We have therefore proved that the event in (6.22) occurs whenever there exists an occupied vertical crossing of $[\lfloor \frac{\ell_1}{8} \rfloor, 2\ell_1 - 2 - \lfloor \frac{\ell_1}{8} \rfloor] \times [0, \ell_2]$ and consequently

$$P\{D(r) \cup D(\tilde{r})\} \geq P\{\exists \text{ an occupied vertical crossing of } [\lfloor \tfrac{\ell_1}{8} \rfloor, 2\ell_1 - 2 - \lfloor \tfrac{\ell_1}{8} \rfloor] \times [0,\ell_2]\}$$

$$\geq \sigma((\ell_3, \ell_2); 2, p, \mathcal{G}_{p\ell}) \geq \delta_4 .$$

For the second inequality we used periodicity, $\ell_3 < 2\ell_1 - 2 - 2\lfloor \frac{\ell_1}{8} \rfloor$ (see 6.15)) and the monotonicity property of Comment 3.3(v). This proves (6.22) and (6.20).

Step (b). We apply Prop. 2.3 with J equal to the perimeter of $[0, \ell_1 - 1] \times [0, \ell_2]$ and $B_1 = \{0\} \times [0, \ell_2]$, $B_2 = \{\ell_1 - 1\} \times [0, \ell_2]$, $A = [0, \ell_1 - 1] \times \{0\}$ and $C = [0, \ell_1 - 1] \times \{\ell_2\}$, and with $S = \mathbb{R}^2 \setminus \{(\ell_1 - 1, 0), (\ell_1 - 1, \ell_2)\}$. Note that B_2 here differs from B_2 in step (a); in any case $B_1 \cap B_2 = \emptyset$ so that (2.26) holds. Moreover the lines $x(1) = 0$ and $x(1) = \ell_1 - 1$ containing B_1 and B_2 are axes of symmetry. Prop. 2.3 now tells us that if there exists an occupied horizontal crossing of $[0, \ell_1 - 1] \times [0, \ell_2]$ in S, then there exists a lowest such crossing, i.e., an occupied crossing with minimal $J^-(r)$. As in Prop. 2.3 we denote the lowest such crossing by R if it exists. Note that a crossing in S is precisely one which intersects L in the open segment $\{\ell_1 - 1\} \times (0, \ell_2)$. Therefore, by Prop. 2.3 and (6.18), the probability that R exists is at least $\sigma((\ell_1, \ell_2); 1, p, \mathcal{G}_{p\ell}) \geq \delta_3$. For any fixed horizontal crossing $r = (v_0, e_1, \ldots, e_\nu, v_\nu)$ of $[0, \ell_1 - 1] \times [0, \ell_2]$ denote by $Y(r)$ the second coordinate of the last intersection of r with the vertical line $L_1 : x(1) = \lfloor \frac{\ell_1}{8} \rfloor$. Formally, if e_j intersects L_1

in $y = (y(1), y(2))$ and the segment of e_j from y to v_j as well as $e_{j+1},\dots,e_\nu$ do not intersect L_1 anymore, then $Y(r) = y(2)$. Note that $Y(r)$ is well-defined since r - which goes from $\{x(1) \le 0\}$ to $\{x(1) \ge \ell_1 - 1\}$ - must intersect L_1. Finally, we choose m as the conditional $(1-\varepsilon)$ - quantile of $Y(R)$, given that R exists, where

$$\varepsilon = \frac{1}{\delta_3}\{\sqrt{1-\delta_3} - (1-\delta_3)\} \ .$$

More formally, we choose m such that

$$(6.23) \qquad \begin{aligned} &P\{R \text{ exists and } Y(R) \le m\} \ge (1-\varepsilon)\, P\{R \text{ exists}\} \\ &\quad = (1-\varepsilon) \times (\text{left hand side of (6.18)}) \\ &\quad \ge (1-\varepsilon) \quad \sigma((\ell_1,\ell_2);\, 1,p,\mathcal{G}_{p\ell}) \end{aligned}$$

and

$$(6.24) \qquad \begin{aligned} &P\{R \text{ exists and } Y(R) < m\} \le (1-\varepsilon) P\,\{R \text{ exists}\} \\ &\quad = (1-\varepsilon) \times (\text{left hand side of (6.18)}). \end{aligned}$$

Finally, we take the segments $A_2 = [0,\, 2\ell_1 - 2] \times \{0\}$ and $C_2 = [0,\, 2\ell_1 - 2] \times \{\ell_2\}$ as in step (a) and define the horizontal semi-infinite strip H by

$$H = [\lfloor \frac{\ell_1}{8} \rfloor \ , \infty) \times (0,\ell_2) \ .$$

In this step we shall prove

$$(6.25) \qquad \begin{aligned} &P\{\ \exists \text{ an occupied horizontal crossing } r' \text{ of } [0,\ell_1 - 1] \\ &\times [0,\ell_2] \quad \text{with } Y(r') \le m \text{ and } r' \cap L \subset \{\ell_1 - 1\} \times (0,\ell_2) \\ &\text{and } \exists \text{ path } s' = (w_0,f_1,\dots,f_\rho,\, w_\rho) \text{ such that} \\ &w_0,\dots,w_{\rho-1} \text{ are occupied, } w_0 \text{ is a vertex of } r',\ f_\rho \\ &\text{intersects } C_2 \text{ in some point } \zeta, \text{ while } (w_0,f_1,\dots,f_{\rho-1}, \\ &w_{\rho-1},\, [w_\rho,\zeta)) \subset H\} \\ &\ge (1 - \sqrt{1-\delta_3})(1-\sqrt{1-\delta_4}) \end{aligned}$$

and

(6.26) $P\{\exists$ an occupied horizontal crossing r'' of $[0,\ell_1 - 1] \times [0,\ell_2]$ with $Y(r'') \geq m$ and $r'' \cap L \subset \{\ell_1 - 1\} \times (0,\ell_2)$ and $\exists$ a path $s'' = (u_0,g_1,\ldots,g_\tau, u_\tau)$ such that $u_0,\ldots,u_{\tau-1}$ are occupied, u_0 is a vertex of r'', g_τ intersects A_2 in some point ζ while $(u_0,g_1,\ldots,g_{\tau-1},u_{\tau-1}, [u_\tau,\zeta)) \subset H\}$

$\geq (1 - \sqrt{1-\delta_3})\ (1 - \sqrt{1-\delta_4}\ ).$

To prove (6.25) we observe that the event in the left hand side contains the union

(6.27) $\bigcup_{r'} \{R = r'$ and $\exists$ a path $s' = (w_0,f_1,\ldots,f_\rho,w_\rho)$ such that $w_0,\ldots,w_{\rho-1}$ are occupied, w_0 is a vertex on r', f_ρ intersects C_2 in some point ζ and $(w_0,f_1,\ldots,f_{\rho-1},w_{\rho-1},[w_\rho,\zeta)) \subset H\}$,

where the union in (6.27) is over all horizontal crossings r' of $[0,\ell_1 - 1] \times [0,\ell_2]$ with $Y(r') \leq m$ and which intersect L in $\{\ell_1 - 1\} \times (0,\ell_2)$. The events in (6.27) are clearly disjoint. In addition, if $R = r'$ and $D(r')$ occurs (see (6.19)) , then the event in (6.27) corresponding to r' occurs. Indeed, $D(r')$ implies the existence of a path $s = (w_0,f_1,\ldots,f_\rho,w_\rho)$ with $w_1,\ldots,w_{\rho-1}$ occupied, w_0 a vertex of r', f_ρ intersecting C_2 in a point ζ and $(f_1 \setminus \{w_0\}, w_1,\ldots,f_{\rho-1},w_{\rho-1},[w_{\rho-1},\zeta)) \subset [\lfloor \frac{\ell_1}{8} \rfloor\ , \infty) \times (0,\ell_2) = H$. In addition w_0 is occupied since it belongs to $r' = R$, and w_0 lies on $f_1 \cap r' \subset H$ (since r' lies strictly between the horizontal lines $x(2) = 0$ and $x(2) = \ell_2$ to the right of L_1) . Therefore s satisfies all requirements for s'. It follows from these observations that the left hand side of (6.25) is no less than

$$(6.28) \qquad \sum_{\substack{Y(r') \leq m \\ r' \cap L \subset \{\ell_1 - 1\} \times (0,\ell_2)}} P\{R = r'\}\ \ P\{D(r') \mid R = r'\}\quad .$$

The "strong Markov property" to which we referred earlier is that $\{R = r'\}$ and $D(r')$ are independent. This is true, because by Prop. 2.3 $\{R = r'\}$ depends only on the occupancies of vertices in

$\bar{J}^-(r') \cup \{v: v$ is a vertex of $\mathcal{G}_{p\ell}$ with its reflection $\tilde{v}$ in L_0: $x(1) = 0$ or $L: x(1) = \ell_1 - 1$ belonging to $\bar{J}^-(r')$ and such that $e \cap J \subset \bar{J}^-(r')$ for some edge e of $\mathcal{G}_{p\ell}$ between v and $\tilde{v}\}$. Here J is still the perimeter of $[0,\ell_1 - 1] \times [0,\ell_2]$ and $\bar{J}$ is the closure of $\mathrm{int}(J)$, i.e., $[0,\ell_1 - 1] \times [0,\ell_2]$. One easily sees that all these vertices lie in $\bar{J}_2^-(r' \cup \tilde{r}')$ plus possibly a collection of points in the half plane $x(1) < 0$ (note that the endpoint of r' lies on $\tilde{r}'$ in all cases; the notation here is as in step (a)). On the other hand the definition (6.19) shows that $D(r')$ depends only on vertices in $J_2^+(r' \cup \tilde{r}')$. Thus $\{R = r'\}$ and $D(r')$ depend on disjoint sets of vertices so that they are indeed independent. It now follows from (6.20), (6.23) and (6.13) that (6.28) is at least

$$\sum_{\substack{Y(r') \leq m \\ r' \cap L \subset \{\ell_1 - 1\} \times (0,\ell_2)}} P\{R = r'\}\, P\{D(r')\}$$

$$\geq (1 - \sqrt{1-\delta_4})\, P\{R \text{ exists and } Y(R) \leq m\}$$

$$\geq (1 - \sqrt{1-\delta_4})\,(1-\varepsilon)\,\sigma((\ell_1,\ell_2);1,p,\mathcal{G}_{p\ell})$$

$$\geq (1-\varepsilon)\delta_3\,(1 - \sqrt{1-\delta_4}) = (1 - \sqrt{1-\delta_3})\,(1 - \sqrt{1-\delta_4}) .$$

This proves (6.25).

The proof of (6.26) is essentially obtained from (6.25) by interchanging the role of "top and bottom" or rather the role of the positive and negative second coordinate axis. The lowest occupied horizontal crossing now has to be replaced by the highest occupied horizontal crossing, i.e., the roles of A and C have to be interchanged. We are not using symmetry with respect to the first coordinate axis, but merely saying that the same proof works when we make the above change, except for one step. The analogue of (6.23) which we need is the following: Let R^+ be the highest occupied horizontal crossing of $[0,\ell_1 - 1] \times [0,\ell_2]$ which intersects L in $\{\ell_1 - 1\} \times (0,\ell_2)$. In other words, R^+ is the occupied horizontal crossing r of the above type with minimal $J^+(r)$. R^+ exists by Prop. 2.3 as soon as there exists an occupied horizontal crossing of $[0,\ell_1 - 1] \times [0,\ell_2]$ in $S = \mathbb{R}^2 \setminus \{(\ell_1 - 1,0), (\ell_1 - 1,\ell_2)\}$ (Just interchange A and C). We want

$$(6.29) \qquad P\{R^+ \text{ exists and } Y(R^+) \geq m\} \geq (1-\varepsilon)\delta_3 .$$

Once one has (6.29) to replace (6.23), the proof of (6.26) becomes a copy of that of (6.25).

We now deduce (6.29) from (6.24). First observe that R^+ exists iff R exists iff there exists any occupied horizontal crossing of $[0,\ell_1 - 1] \times [0,\ell_2]$ in S. Second, if such crossings exist, then

$$(6.30) \qquad Y(R^+) \geq Y(r) \geq Y(R)$$

for any occupied horizontal crossing r of $[0,\ell_1 - 1] \times [0,\ell_2]$ in S. We only have to prove the right hand inequality in (6.30); the left hand inequality will then follow by interchanging the role of A and C. To obtain this right hand inequality note that the piece of R from its last intersection $\zeta_1 := (\lfloor \frac{\ell_1}{8} \rfloor , Y(R))$ with L_1 to its unique intersection, ζ_2 say, with the line $L: x(1) = \ell_1 - 1$ forms a crosscut of the rectangle $F := (\lfloor \frac{\ell_1}{8} \rfloor, \ell_1 - 1) \times (0,\ell_2)$ (see Fig. 6.2).

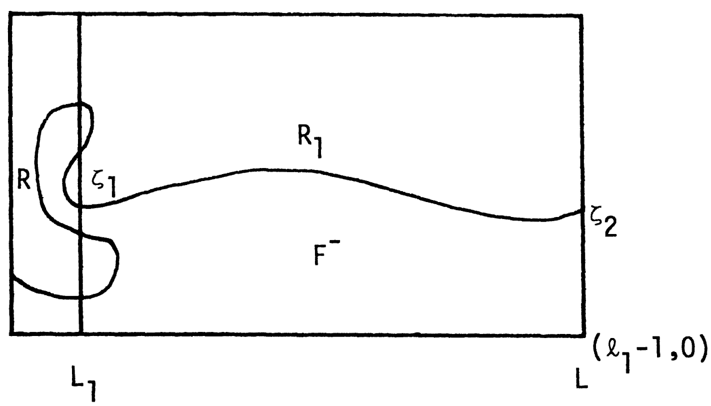

Figure 6.2

Let us write R_1 for the piece of R between ζ_1 and ζ_2. Thus R_1 divides F into two Jordan domains. The lower one, which we denote by F^- is bounded by R_1, the segment of L from ζ_2 to $(\ell_1 - 1,0)$, the horizontal segment at the bottom from $(\ell_1 - 1,0)$ to $(\lfloor \frac{\ell_1}{8} \rfloor , 0)$ and the segment of L_1 from $(\lfloor \frac{\ell_1}{8} \rfloor ,0)$ to ζ_1. Any point in F^- which is close enough to R_1 can be connected by a continuous curve in $F^- \setminus R$ to the segment of L below ζ_2, i.e., the segment from ζ_2 to $(\ell_1 - 1,0)$. This is obvious if R is a polygonal path. In general one can obtain this from the fact that $\overline{F^-}$ can be mapped homeomorphically onto the closed unit disc (see Newman (1951), Theorem VI. 17.1 or use conformal mapping as in Hille (1962), Theorem 17.5.3). Since the segment of L from ζ_2 to $(\ell_1 - 1, 0)$ belongs to $Fr(J^-(R))$ and not to $Fr(J^+(R))$ it follows that all points of F^- close to R_1 belong to $J^-(R)$. Consequently for any occupied horizontal crossing r of $[0,\ell_1 - 1] \times [0,\ell_2]$ in S, the piece between the last intersection of r with L_1 and the first intersection with L cannot enter F^-, because such a crossing r satisfies $r \cap \bar{J} \subset \bar{J}^+(R)$ (see (2.27)). In particular, the last intersection of r with L_1, $(\lfloor \frac{\ell_1}{8} \rfloor, Y(r))$, cannot lie strictly below ζ_1 on L_1. This just says $Y(r) \geq Y(R)$, and therefore proves (6.30).

Now we apply (6.21) with $E_1(E_2)$ the event that there exists an occupied horizontal crossing r of $[0,\ell_1 - 1] \times [0,\ell_2]$ in S with $Y(r) < m$ $(Y(r) \geq m)$. $E_1 \cup E_2$ is the event that there is some occupied horizontal crossing of $[0,\ell_1 - 1] \times [0,\ell_2]$ in S and this has probability at least δ_3 by (6.18). Also, by (6.30) $P\{E_1\}$ is given by the left hand side of (6.24), and hence is at most $(1 - \varepsilon)\, P\{E_1 \cup E_2\}$. Thus, by (6.30) and (6.21)

$$
\begin{aligned}
&P\{R^+ \text{ exists and } Y(R^+) \geq m\} \geq P\{\exists \text{ an occupied horizontal crossing } \\
&\quad r \text{ of } [0,\ell_1-1] \times [0,\ell_2] \text{ in } S \text{ with } Y(r) \geq m\} \\
&= P\{E_2\} \geq 1 - \frac{1-P\{E_1 \cup E_2\}}{1-P\{E_1\}} \geq 1 - \frac{1-P\{E_1 \cup E_2\}}{1-(1-\varepsilon)P\{E_1 \cup E_2\}} \\
&\quad 1 - \frac{1-\delta_3}{1-(1-\varepsilon)\delta_3} = 1 - \sqrt{1-\delta_3} = (1-\varepsilon)\delta_3 .
\end{aligned}
$$

This is precisely (6.29), and as stated above, implies (6.26).

<u>Step (c)</u>. In this step, we complete the proof of the lemma from (6.25) and (6.26). Assume that the events in braces in the left hand sides of (6.25) and (6.26) both occur. Then $r' \cup s'$ contains in H a continuous curve from $(\lfloor \frac{\ell_1}{8} \rfloor, Y(r'))$ = last intersection of r' with L_1 to the upper edge of H, $[\lfloor \frac{\ell_1}{8} \rfloor, \infty) \times \{\ell_2\}$. Also $r'' \cup s''$ contains in H a continuous curve from $(\lfloor \frac{\ell_1}{8} \rfloor, Y(r''))$ to the lower edge of H, $[\lfloor \frac{\ell_1}{8} \rfloor, \infty) \times \{0\}$. Moreover, $Y(r'') \geq m \geq Y(r')$, so that the second curve begins above the first curve on L_1 and ends

below the first curve. Thus these curves intersect, necessarily in a vertex and in H. Since all vertices of $s' \cup s''$ in H are occupied it follows that r' r'', $s' \cap H$ and $s'' \cap H$ all belong to one occupied component and $(r' \cup s' \cup r'' \cup s'') \cap H$ contains a continuous curve,

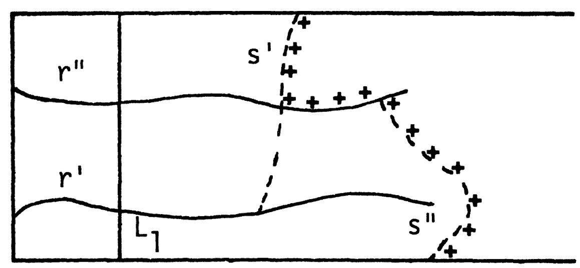

Figure 6.3 r' and r'' are solidly drawn, s' and s'' are dashed. The curve ψ is indicated by + signs.

ψ say, in H which connects the upper and lower edge of H. If ψ contains any point on or to the right of the vertical line $L_2 : x(1) = \lfloor \frac{\ell_1}{8} \rfloor + M$ for a given integer M (to be specified later) then $r' \cup s' \cup r'' \cup s''$ contains an occupied horizontal crossing of

$$(6.31) \qquad [0, \lfloor \tfrac{\ell_1}{8} \rfloor + M - \Lambda] \times [0, \ell_2]$$

If, on the other hand, ψ lies strictly to the left of L_2, then we must bring in a further path. Assume in this case that there also exists an occupied horizontal crossing r''' of

$$(6.32) \qquad [\lfloor \tfrac{\ell_1}{8} \rfloor - 1, \lfloor \tfrac{\ell_1}{8} \rfloor + M] \times [0, \ell_2]$$

If ψ lies entirely to the left of L_2, then ψ lies in the rectangle

$$[\lfloor \tfrac{\ell_1}{8} \rfloor , \lfloor \tfrac{\ell_1}{8} \rfloor + M] \times [0, \ell_2]$$

and connects the top and bottom edges of this rectangle. Thus ψ intersects r''' to the right of L_1 and r', r'', r''', $s' \cap H$ and $s'' \cap H$ all belong to one occupied component in this situation. Since r' begins on or to the left of $x(1) = 0$ and r''' ends on or to the right of $x(1) = \lfloor \frac{\ell_1}{8} \rfloor + M$, we see that now $r' \cup r'' \cup r''' \cup s' \cup s''$ contains an occupied horizontal crossing of the rectangle (6.31). Consequently

$$\sigma((\lfloor \frac{\ell_1}{8} \rfloor + M - \Lambda,\ \ell_2);\ 1,p,\mathcal{G}_{p\ell}) \geq P\{\text{the events in (6.25) and (6.26) both occur and there exists an occupied horizontal crossing } r''' \text{ of the rectangle (6.32)}\}\ .$$

By the FKG inequality, (6.25), (6.26) and periodicity we finally obtain from this

$$\sigma((\lfloor \frac{\ell_1}{8} \rfloor + M - \Lambda,\ \ell_2);\ 1,p,\mathcal{G}_{p\ell}) \geq (1 - \sqrt{1-\delta_3})^2\ (1 - \sqrt{1-\delta_4})^2\ \sigma((M + 1,\ell_2);\ 1,p,\mathcal{G}_{p\ell}) \quad . \tag{6.33}$$

We apply this first with $M = M_0 := \ell_1 - 1$. Then by (6.13)

$$\sigma((\lfloor \frac{\ell_1}{8} \rfloor + \ell_1 - \Lambda - 1,\ \ell_2);1,p,\mathcal{G}_{p\ell}) \geq \delta_3(1 - \sqrt{1-\delta_3})^2\ (1 - \sqrt{1-\delta_4})^2 \quad . \tag{6.34}$$

We now use (6.33) with $M = M_1 := M_0 + \lfloor \frac{\ell_1}{8} \rfloor - \Lambda - 2$, and use the estimate (6.34) for the last factor in the right hand side of (6.33). We can repeat this procedure and successively obtain lower bounds for $\sigma((M_{j+1} + 1,\ \ell_2);1,p,\mathcal{G}_{p\ell})$ in terms of $\sigma((M_j + 1,\ \ell_2);1,p,\mathcal{G}_{p\ell})$, where

$$M_j = \ell_1 - 1 + j(\lfloor \frac{\ell_1}{8} \rfloor - \Lambda - 1)\ .$$

By induction on j one sees that these lower bounds tend to one when $\delta_3 \uparrow 1$ and $\delta_4 \uparrow 1$. Since $M_{16k} \geq k\ell_1$ this implies (6.16) and (6.17) for a suitable f_1(cf Comment 3.3 (v)). $\square$

Lemma 6.2. Assume (6.13) holds as well as

$$\sigma((\ell_1,\ell_4);\ 2,p,\mathcal{G}_{p\ell}) \geq \delta_5 > 0 \tag{6.35}$$

for some integers $\ell_1,\ell_2,\ell_4 \geq 1$ with[1)]

$$\ell_2 \leq \frac{98}{100}\ell_4\ ,\ \ell_4 \geq 300\ . \tag{6.36}$$

---

1) The requirement $\ell_2 \leq \frac{98}{100}\ell_4$ can be replaced by $\ell_2 \leq (1-\delta)\ell_4$ for any $\delta > 0$.

Then for each k there exists an $f_2(\delta_3,\delta_5,k) > 0$ such that

$$(6.37) \qquad \sigma((\ell_1,k\ell_4);\, 2,p,\mathcal{G}_{p\ell}) \geq f_2(\delta_3,\delta_5,k) > 0$$

and

$$(6.38) \qquad \lim_{\substack{\delta_3 \to 1 \\ \delta_5 \to 1}} f_2(\delta_3,\delta_5,k) = 1 \ .$$

Remark.

The reader should note that the crossing probabilities in (6.13) and (6.35) are for rectangles of the same horizontal size ℓ_1, while in (6.13) and (6.14) they are for rectangles of the same vertical size. Also, this lemma estimates the probability of "long" vertical crossings, while Lemma 6.1 deals with "long" horizontal crossings. This lemma is much simpler than the last one and does not rely on symmetry. The simplification comes from the assumption that ℓ_4 is greater than ℓ_2, by a fixed fraction. In contrast to this, (6.15) allowed $\ell_1 \leq \ell_3$.

Proof: To prove (6.37), we observe that if there exist occupied vertical crossings of r' and r'' of $[0,\ell_1] \times [0, M + 1]$ and $[0,\ell_1] \times [M - \ell_2 - 1, 2M - \ell_2]$, for some integer M, and an occupied horizontal crossing t of $[0,\ell_1] \times [M - \ell_2,M]$, then t must intersect r' as well as r' in the open rectangle $(0,\ell_1) \times (M - \ell_2,M)$ (see Fig. 6.4). It follows that in this situation $r' \cup r'' \cup t$ contains a vertical crossing of $[0,\ell_1] \times [0,2M - \ell_2]$. Thus, again from the FKG inequality, periodicity and (6.13), we obtain

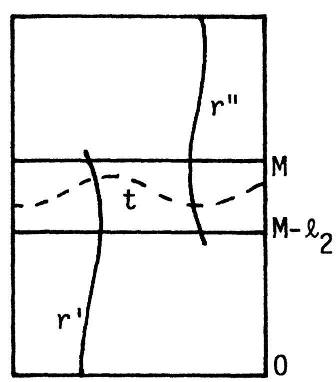

Figure 6.4.

$$(6.39) \qquad \sigma((\ell_1, 2M - \ell_2); 2,p,\mathcal{G}_{p\ell})$$
$$\geq \sigma((\ell_1, M + 1); 2,p,\mathcal{G}_{p\ell})$$
$$P\{ \exists \text{ occupied vertical crossing of } [0,\ell_1] \times [M-\ell_2-1,2M-\ell_2]\}$$
$$P\{ \exists \text{ occupied horizontal crossing of } [0,\ell_1] \times [M-\ell_2,M]\}$$
$$\geq \{\sigma((\ell_1, M + 1);2,p,\mathcal{G}_{p\ell})\}^2 \ \delta_3 \quad .$$

We use this in the same way as (6.33). We first take $M = M_0 := \ell_4 - 1$. Then by (6.35) the right hand side of (6.39) is at least $\delta_3 \delta_5^2$. This is also a lower bound for

$$(6.40) \qquad \sigma((\ell_1, M_j + 1); 2,p,\mathcal{G}_{p\ell})$$

when $j = 1$ and $M_j := \lfloor (1.01)^j \ell_4 \rfloor$ (use (6.36)). Once we have a lower bound for a given j we substitute it into the right hand side of (6.39) to obtain a lower bound for (6.40) with $M_j + 1$ replaced by $2M_j - \ell_2$. Since $2M_j - \ell_2 \geq M_{j+1} + 1$ this is also a lower bound for (6.40) with M_j replaced by M_{j+1}. Again we see by induction on j that the lower bound for (6.40) obtained after j iterations of this procedure tends to one when δ_3 and $\delta_5 \to 1$. (6.37) and (6.38) follow from this. □

<u>Lemma 6.3</u>. <u>Assume (6.13) holds. Let</u> $s > 0$ <u>be an integer. Then one has</u>

$$(6.41) \qquad \sigma((\ell_1, \frac{302}{s} \ell_2 + 2); 1,p,\mathcal{G}_{p\ell}) = P\{ \exists \text{ an occupied horizontal crossing of } [0,\ell_1] \times [0, \frac{302}{s} \ell_2 + 2]\}$$
$$\geq \delta_6 := 1 - (1-\delta_3)^{(s+2)^{-2}}$$

<u>or for some</u> $300 \leq j \leq s$ <u>the following estimate holds:</u>[1)]

---

1) $\lceil a \rceil$ denotes the smallest integer $\geq$ a.

(6.42) $\quad P\{\ \exists$ occupied horizontal crossing of

$[0,\ell_1] \times [0, \lceil \frac{j+2}{s} \ell_2 \rceil + 2]$ and $\exists$ occupied vertical

crossing of $[0,\ell_1] \times [\lceil \frac{\ell_2}{s} \rceil + 1, \lfloor \frac{j+1}{s} \ell_2 \rfloor]\}$

$\geq \delta_6 = 1 - (1-\delta_3)^{(s+2)^{-2}}$.

<u>This lemma does not depend on symmetry and the role of the horizontal and vertical direction may be interchanged.</u>

<u>Proof</u>: Let $r = (v_0, e_1, \ldots, e_\nu, v_\nu)$ be an occupied horizontal crossing of $[0,\ell_1] \times [0,\ell_2]$. Let ζ_1 be the last intersection of e_1 with the left edge, $\{0\} \times [0,\ell_2]$, of this rectangle, and ζ_ν the first intersection of e_ν with the right edge, $\{\ell_1\} \times [0,\ell_2]$. Then the segment $[\zeta_1, v_1]$ of e_1, together with the edges $e_2, \ldots, e_{\nu-1}$ and the segment $[v_{\nu-1}, \zeta_\nu]$ of e_ν form a continuous curve inside $[0,\ell_1] \times [0,\ell_2]$, connecting the left and right edge. Let $y_\ell(r)$ and $y_h(r)$ be the minimum and maximum value, respectively, of the second coordinates of the points on this curve. Also, let $E(j_1,j_2)$ for $0 \leq j_1, j_2 \leq s$ be the event

$$\{\ \exists \text{ occupied horizontal crossing } r \text{ of } [0,\ell_1] \times [0,\ell_2]$$
$$\text{with } \lfloor \frac{j_1}{s} \ell_2 \rfloor < y_\ell(r) \leq \lfloor \frac{(j_1+1)}{s} \ell_2 \rfloor \quad \text{and}$$
$$\lceil \frac{j_2}{s} \ell_2 \rceil \leq y_h(r) \leq \lceil \frac{(j_2+1)}{s} \ell_2 \rceil\} .$$

Any horizontal crossing r of $[0,\ell_1] \times [0,\ell_2]$ has $0 \leq y_\ell(r) \leq y_h(r) \leq \ell_2$, so that if there exists an occupied horizontal crossing of $[0,\ell_1] \times [0,\ell_2]$, then one of the events $E(j_1,j_2)$, $-1 \leq j_1, j_2 \leq s$ must occur. Exactly as in (6.21) we obtain from the FKG inequality and (6.13)

$$(6.43) \qquad 1-\delta_3 \geq P\{(\cup E(j_1,j_2))^c\} \geq \Pi\,(1-P\{E(j_1,j_2)\})$$

The union and product in (6.43) run over $-1 \leq j_1, j_2 \leq s$ and hence contain at most $(s+2)^2$ elements. Therefore, for some $-1 \leq j_1, j_2 \leq s$

$$(6.44) \qquad P\{E(j_1,j_2)\} \geq \delta_6 := 1 - (1-\delta_3)^{(s+2)^{-2}} .$$

Assume now that (6.44) holds for some $j_1 < j_2 - 300$. If $E(j_1,j_2)$ occurs for these j_1,j_2 then there exists an occupied horizontal crossing $r = (v_0,e_1,\dots,e_\nu,v_\nu)$ of $[0,\ell_1] \times [0,\ell_2]$ with

$\lfloor j_1 s^{-1}\ell_2 \rfloor < y_\ell(r) \le \lfloor (j_1+1)s^{-1}\ell_2 \rfloor$ and

$\lceil j_2 s^{-1}\ell_2 \rceil \le y_h(r) < \lceil (j_2+1)s^{-1} \ell_2 \rceil$. By Def. 3.1 of a crossing, r is then also an occupied horizontal crossing of

$[0,\ell_1] \times [\lfloor \frac{j_1}{s} \ell_2 \rfloor, \lceil \frac{j_{2+1}}{s} \ell_2 \rceil]$. But also

$y_\ell(r) \le \lfloor (j_1+1)s^{-1}\ell_2 \rfloor < \lceil j_2 s^{-1}\ell_2 \rceil \le y_h(r)$ implies that some edge e_α of r intersects the segment $[0,\ell_1] \times \{\lfloor (j_1+1)s^{-1}\ell_2 \rfloor\}$ and some edge e_β intersects the segment $[0,\ell_1] \times \{\lceil j_2 s^{-1}\ell_2 \rceil\}$. Choose α and β such that $|\beta-\alpha|$ is minimal. For the sake of argument let $\alpha \le \beta$. Then the piece $(v_\alpha,e_{\alpha+1},\dots,e_\beta,v_\beta)$ of r is an occupied vertical crossing of $[0,\ell_1] \times [\lfloor (j_1+1)s^{-1}\ell_2 \rfloor, \lceil j_2 s^{-1}\ell_2 \rceil]$ (see Fig. 6.5) Thus for $j = j_2 - j_1 - 1$ the left hand side of (6.42) is (by virtue of the periodicity and the monotonicity property of Comment 3.3(v)) at least

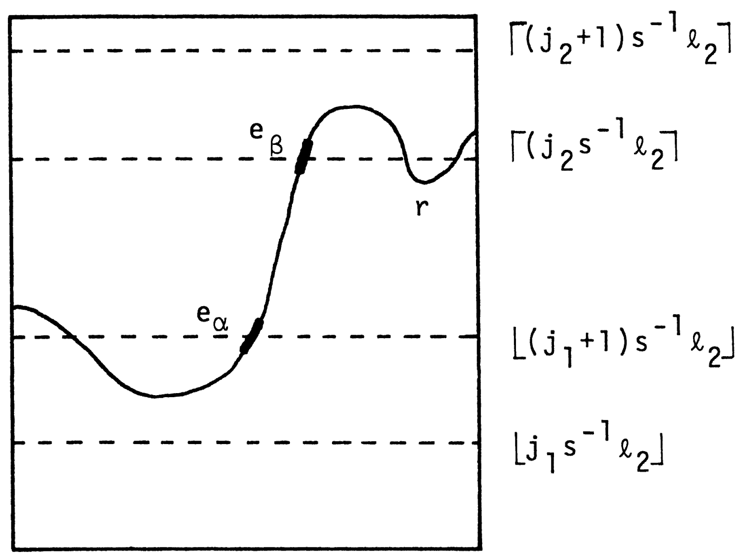

Figure 6.5. The boldly drawn pieces of r represent the edges e_α and e_β.

$P\{\exists$ occupied horizontal crossing of $[0,\ell_1] \times [\lfloor \frac{j_1}{s} \ell_2 \rfloor,$
$\lceil \frac{j_2+1}{s} \ell_2 \rceil]$ and $\exists$ occupied vertical crossing of
$[0,\ell_1] \times [\lfloor \frac{j_1+1}{s} \ell_2 \rfloor , \lceil \frac{j_2}{s} \rceil]\} \geq P\{E(j_1,j_2)\} \geq \delta_6$.

Thus, (6.44) for some $j_1 < j_2-300$ implies (6.42) for a $j \geq 300$. If, on the other hand, (6.44) holds for some $j_1 \geq j_2 - 300$ then the first part of the above argument and periodicity show that (6.41) holds. □

Lemma 6.4. Assume (6.13) and (6.14) hold for some integers $\ell_1,\ell_2,\ell_3 \geq 1$ with

(6.45) $$\ell_3 \leq t\ell_1, \ell_1 \geq 302 + 32\Lambda,\ \ell_2 > \Lambda$$

for some t. Then for each k there exists an $f_3(\delta_3,\delta_4,t,k) > 0$ such that

(6.46) $$\sigma((k\ell_1,\ell_2);1,p,\mathcal{G}_{p\ell}) \geq f_3(\delta_3,\delta_4,t,k) > 0$$

and

(6.47) $$\lim_{\substack{\delta_3 \to 1 \\ \delta_4 \to 1}} f_3(\delta_3,\delta_4,t,k) = 1 .$$

For $t = \frac{3}{2}$ this is Lemma 6.1. Here we relax condition (6.15) considerably.

Proof: For $\ell_3 \leq 3\ell_1/2$ Lemma 6.1 already implies (6.46) and (6.47), so that we may assume $\ell_3 \geq 3\ell_1/2$. We now apply Lemma 6.3 with the horizontal and vertical direction interchanged. Take $s = \lceil 302\ell_3\ell_1^{-1} \rceil \leq 303\ell_3\ell_1^{-1} \leq 303t$. We then have

$$\sigma((\frac{302}{s} \ell_3 + 2,\ \ell_2);\ 2,p,\mathcal{G}_{p\ell}) = P\{\exists \text{ occupied vertical crossing of } [0, \frac{302}{s}\ell_3 + 2] \times [0,\ell_2]\} \geq \delta_7 := (1-\delta_4)^{(s+2)^{-2}}$$

or for some $j \geq 300$

(6.48) $\quad P\{\ \exists$ occupied vertical crossing of

$[0,\lceil \frac{j+2}{s}\ \ell_3 \rceil + 2] \times [0,\ell_2]$ and $\exists$ occupied horizontal crossing of $[\lceil \frac{\ell_3}{s} \rceil + 1,\ \lfloor \frac{j+1}{s}\ \ell_3 \rfloor] \times [0,\ell_2]\} \geq \delta_7$.

In the first case (6.14) and (6.15) hold for ℓ_3 replaced by

$$\lceil \frac{302}{s}\ \ell_3 \rceil + 2 \ \leq\ \frac{302\ell_3}{302\ell_3\ell_1^{-1}} + 3 \ \leq\ \frac{3}{2}\ \ell_1\ ,$$

and δ_4 by δ_7. Thus in this case (6.46) and (6.47) follow from (6.16), (6.17) and the fact that $\delta_7 \to 1$ as $\delta_4 \to 1$ (uniformly under the condition $s \leq 303t$ implied by (6.45)).

In the second case (6.48) implies the following replacements of (6.13) and (6.14):

$$\sigma((\lfloor \frac{j}{s}\ \ell_3 \rfloor -3,\ \ell_2);\ 1,p,\mathcal{G}_{p\ell}) \ \geq\ \delta_7$$

(use periodicity again) and

$$\sigma((\lceil \frac{j+2}{s}\ \ell_3 \rceil + 2,\ \ell_2);\ 2,p,\mathcal{G}_{p\ell}) \ \geq\ \delta_7 \quad .$$

Thus ℓ_1 is replaced by

$$\lfloor \frac{j}{s}\ \ell_3 \rfloor -3 \ \geq\ \frac{j\ell_3}{303\ell_3\ell_1^{-1}} - 4 \ \geq\ \frac{300}{303}\ \ell_1 - 4 \geq 32 + 16\Lambda$$

and ℓ_3 by

$$\lceil \frac{j+2}{s}\ \ell_3 \rceil + 2 \leq \frac{j+2}{s}\ \ell_3 + 3 \leq \frac{3}{2}(\frac{j}{s}\ \ell_3 - 4) \leq \frac{3}{2}(\lfloor \frac{j}{s}\ \ell_3 \rfloor - 3)$$

(recall $\ell_1 \geq 302 + 32\Lambda, j \geq 300$). With these replacements, and δ_7 instead of δ_3, δ_4, (6.16) and (6.17) give us (6.46) and (6.47). $\square$

Now assume (6.5) and (6.6) hold. Assume also that m_i, n_i satisfy (6.7) for a given $\pi \geq 1$ and take for the remainder of the proof

(6.49) $$s = 400\pi \quad .$$

We then have (6.13) with $\ell_i = n_i$, $\delta_3 = \delta_1$ and by Lemma 6.3 (6.41) holds or (6.42) holds for some $300 \leq j \leq s$. Also (6.14) holds with

$\ell_2 = m_2$, $\ell_3 = m_1$, $\delta_4 = \delta_2$. In the next lemma we take care of the case where (6.41) holds, and then we deal with the case where (6.42) holds in a sequence of reductions in the succeeding lemmas.

Lemma 6.5. Assume (6.6), (6.7) hold and (6.41) for $\ell_i = n_i$, $\delta_3 = \delta_1$ and $s = 400\pi$. Then the conclusion of Theorem 6.1 holds.

Proof: By (6.7) and (6.49)

$$\ell_2' := \frac{302}{s}\ell_2 + 2 = \frac{302}{400\pi} n_2 + 2 \leq \frac{7}{8}\frac{n_2}{\pi} \leq \frac{7}{8} m_2 \tag{6.50}$$

as soon as n_2 exceeds some $n_0(\pi)$. Then by Comment 3.3(v) and (6.6)

$$\sigma((m_1,\ell_2'); 2,p,\mathcal{G}_{p\ell}) \geq \delta_2 ,$$

while by (6.41)

$$\sigma((\ell_1,\ell_2');1,p,\mathcal{G}_{p\ell}) \geq \delta_6 .$$

Since by (6.7) $m_1 \leq \pi n_1 = \pi\ell_1$ it now follows from Lemma 6.4 that for n_1,n_2 greater than some $n_o(\pi)$ one has

$$\sigma((kn_1,\ell_2'); 1,p,\mathcal{G}_{p\ell}) \geq f_3(\delta_6,\delta_2,\pi,k) . \tag{6.51}$$

Since (6.41) holds for $\delta_3 = \delta_1$, δ_6 here has to be read as

$$\delta_6 = 1 - (1-\delta_1)^{-(s+2)^{-2}} . \tag{6.52}$$

(6.51) together with another application of Comment 3.3(v) gives us (6.9).

For (6.10) we use Lemma 6.2. (6.35) with $\ell_1 = m_1$, $\ell_4 = m_2$, $\delta_5 = \delta_2$ holds by virtue of (6.6). Also, if we apply (6.51) with $k = \pi$, then we find (again using Comment 3.3(v))

$$\sigma((m_1,\ell_2'); 1,p,\mathcal{G}_{p\ell}) \geq \sigma((\pi n_1,\ell_2');1,p,\mathcal{G}_{p\ell}) \geq \delta_8 ,$$

where

$$\delta_8 = f_3(\delta_6,\delta_2,\pi,\pi) . \tag{6.53}$$

This takes the place of (6.13). Since $\ell_2' \leq \frac{98}{100} m_2$ (see (6.50)) (6.37) now gives

$$\sigma((m_1,km_2);\ 2,p,\mathcal{G}_{p\ell}) \ \geq\ f_2(\delta_8,\delta_2,k)$$

and hence (6.10). Finally (6.11) follows from (6.38), (6.47) and the fact that $\delta_6 \uparrow$, $\delta_8 \uparrow 1$ as $\delta_1 \uparrow 1$, $\delta_2 \uparrow 1$. □

In view of the last lemma and the comments immediately before it we may assume from now on that (6.42) holds for some $300 \leq j \leq s$ and $\ell_1 = n_1$, ℓ_2, $= n_2$ and $\delta_3 = \delta_1$. If the first coordinate axis were also an axis of symmetry. Theorem 6.1 would now follow from (6.42) and Lemma 6.4. Without this extra symmetry assumption we must first show that (6.42) can be strengthened to (6.85) below.

For the remainder we take $\ell_1 = n_1$, $\ell_2 = n_2$, $s = 400\pi$, $\delta_3 = \delta_1$ and $300 \leq j \leq s$ such that (6.42) holds for these choices. We shall also use the following abreviations and notations:

$$\ell_5 \ = \ \lceil\ (j+2)s^{-1}\ \ell_2 \rceil + 2 = \lceil (j+2)s^{-1}\ n_2 \rceil + 2;$$

if $r = (v_0,e_1,\dots,e_\nu,v_\nu)$ is a horizontal crossing of $[0,\ell_1] \times [0,\ell_5]$, then ζ_1 denotes the last intersection of e_1 with the segment $\{0\} \times [0,\ell_5]$. For any vertical line $L(a)$: $x(1) = a$ with $0 \leq a \leq \ell_1$, $\zeta(a) = \zeta(a,r)$ is the first intersection of r with $L(a)$ and $Y(a) = Y(a,r)$ is the second coordinate of $\zeta(a)$. Thus $\zeta(a) = (a,Y(a))$, and if $\zeta(a) \in e_\rho$, then the segment $[\zeta_1,v_1]$ of e_1, together with the edges $e_2,\dots,e_{\rho-1}$ and the segment $[v_{\rho-1},\zeta(a)]$ of e_ρ form a continuous curve inside $[0,a] \times [0,\ell_5]$ connecting the left and right edge of this rectangle. For $a = \ell_1/8$ we denote by $z_\ell(r)$ and $z_h(r)$ the minimum and maximum value, respectively, of the second coordinates of the points of this curve, i.e., of the piece of r from ζ_1 to $\zeta(\ell_1/8)$.

<u>Lemma 6.6</u>. <u>Let</u> δ_6 <u>be as in (6.52). Assume</u>

(6.54) $\quad P\{\ \exists$ occupied horizontal crossing r of $[0,\ell_1] \times [0,\ell_5]$ with $z_\ell(r) > (.03)\ell_5$ or $z_h(r) < (.97)\ell_5$ and $\exists$ occupied vertical crossing of $[0,\ell_1] \times [\lceil \frac{\ell_2}{5} \rceil + 1,\ \lfloor \frac{j+1}{s}\, \ell_2 \rfloor]\} \ \geq\ \delta_9 := 1 - \sqrt{1-\delta_6}$.

<u>Then the conclusion of Theorem 6.1 holds.</u>

<u>Proof</u>: A horizontal crossing r of $[0,\ell_1] \times [0,\ell_5]$ with $z_\ell(r) > (.03)\ell$ contains a horizontal crossing of $[0,\ell_1/8] \times [(.03)\ell_5,\ell_5\ ]$. Similarly

a horizontal crossing of $[0,\ell_1] \times [0,\ell_5]$ with $z_h(r) < (.97)\ell_5$ contains a horizontal crossing of $[0,\ell_1/8] \times [0,(.97)\ell_5]$. Therefore (6.54) implies

$$P\{\ \exists \text{ occupied horizontal crossing of } [0, \frac{\ell_1}{8}] \times [(.03)\ell_5,\ell_5] \text{ or of } [0, \frac{\ell_1}{8}] \times [0,(.97)\ell_5]\} \geq \delta_9 \quad .$$

By the FKG inequality, or rather (6.21), this implies

$$P\{\ \exists \text{ occupied horizontal crossing of } [0, \frac{\ell_1}{8}] \times [(.03)\ell_5,\ell_5]\} \geq 1 - \sqrt{1-\delta_9} \tag{6.55}$$

or

$$P\{\ \exists \text{ occupied horizontal crossing of } [0, \frac{\ell_1}{8}] \times [0,(.97)\ell_5]\} \geq 1 - \sqrt{1-\delta_9} \quad . \tag{6.56}$$

For the sake of argument let (6.56) hold. From (6.54), Comment 3.3(v) and periodicity it also follows that

$$P\{\ \exists \text{ occupied vertical crossing of } [0,\ell_1] \times [0, \lfloor \tfrac{j}{s}\ell_2 \rfloor - 3]\} \geq \delta_9 \quad . \tag{6.57}$$

Since $j \geq 300$, $\ell_2 = n_2$ we have for n_2 greater than some $n_0(\pi)$

$$(.97)\ell_5 + 1 \leq (.97)\frac{j+2}{s}\ell_2 + 4 \leq (.98)(\lfloor \tfrac{j}{s}\ell_2 \rfloor - 3).$$

We are therefore in the same situation as in the beginning of Lemma 6.5 and (6.9) - (6.11) for suitable $f(\cdot)$ follow from Lemmas 6.4, 6.2 and Comment 3.3(v). $\square$

By virtue of the last lemma we only have to consider the case where (6.54) fails. Denote by E_1 the event in the left hand side of (6.54) and set

$$E_2 = E_2(\ell_1,\ell_5) = \{\ \exists \text{ occupied horizontal crossing } r \text{ of } [0,\ell_1] \times [0,\ell_5] \text{ with } z_\ell(r) \leq (.03)\ell_5 \text{ and } z_h(r) \geq (.97)\ell_5 \text{ and } \exists \text{ occupied vertical crossing of } [0,\ell_1] \times [\lceil \tfrac{\ell_2}{5} \rceil + 1, \lfloor \tfrac{j+1}{s}\ell_2 \rfloor]\} \quad .$$

Then $E_1 \cup E_2$ is the event in the left hand side of (6.42). Thus if (6.42) holds with $\delta_3 = \delta_1$, but (6.54) fails, then by virtue of (6.21)

$$P\{E_2\} \geq 1 - \frac{(1-\delta_6)}{(1-\delta_6)^{1/2}} = \delta_9 \quad . \tag{6.58}$$

It therefore remains to derive Theorem 6.1 if (6.58) prevails (with $\ell_1 = n_1$, $\ell_2 = n_2$, $\delta_3 = \delta_1$). First we observe that we may assume an even stronger condition than (6.58). Specifically set

$$E_3(k) \geq E_3(\ell_1,\ell_5,k) = \{ \exists \text{ occupied horizontal crossing } r \text{ of } [0,\ell_1] \times [0,\ell_5] \text{ with } z_\ell(r) \leq (.03)\ell_5 ,\ z_h(r) \geq (.97)\ell_5 \text{ and } Y(\lfloor \tfrac{\ell_1}{2} \rfloor, r) \in [\tfrac{k\ell_5}{100}, \tfrac{(k+1)\ell_5}{100}]\} .$$

Since $Y(\lfloor \frac{\ell_1}{2} \rfloor , r) \in [k\ell_5/100, (k+1)\ell_5/100]$ for some $0 \leq k < 100$ it follows from (6.58) that

$$P\{\bigcup_{0 \leq k < 100} E_3(k)\} = P\{E_2\} \geq \delta_9 \quad .$$

As in (6.43), (6.44) this, together with the FKG inequality shows that for some $0 \leq k_1 < 100$

$$P\{E_3(k_1)\} \geq \delta_{10} := 1 - (1-\delta_9)^{1/100} \quad . \tag{6.59}$$

The next lemma will show that we can assume that the intersections of an occupied horizontal crossing of $[0,\ell_1] \times [0,\ell_5]$ with any line $L(a)$, $\frac{\ell_1}{2} \leq a \leq \ell_1$, lie with high probability in

$$\{a\} \times [\frac{k_1 - 11}{100}\, \ell_5, \frac{k_1 + 12}{100}\, \ell_5] \; . \tag{6.60}$$

In order to state the lemmas to follow we need to introduce a further integer $t = t(\mathcal{G}_{p\ell})$. By Lemma A.3 there exists a vertex v_0 of $\mathcal{G}_{p\ell}$, an integer $\alpha \geq 1$ and a path v_0 on $\mathcal{G}_{p\ell}$ from v_0 to $v_0 + (\alpha,0)$ such that for all $n \geq 1$ the path on $\mathcal{G}_{p\ell}$ obtained by successively traversing the paths $r_0 + (k\alpha,0)$, $k = 0,1,\ldots,n$ (these are translates of v_0) is self-avoiding. We take

$$t = 2\lceil \text{diameter of } r_0) \rceil + 1 \; . \tag{6.61}$$

For later use we observe that this definition of t guarantees that if (b_1, b_2) is any point of r_0 then

(6.62) $$r_0 + (k\alpha, 0) \subset [b_1 - t, \infty) \times \mathbb{R}\,,\ k \geq 0 \quad .$$

Lemma 6.7. Assume that (6.59) holds and that there exists an integer $a \in [\frac{\ell_1}{2}, \ell_1]$ for which

(6.63) $$\begin{aligned} &P\{\ \exists \text{ occupied horizontal crossing } r' \text{ of} \\ &[0,\ell_1] \times [0,\ell_5] \text{ with } z_\ell(r') \leq (.03)\ell_5 \\ &z_h(r') \geq (.97)\ell_5 \text{ and which intersects } L(a) \text{ in} \\ &\{a\} \times [0, \frac{k_1 - 11}{100}\ell_5] \cup \{a\} \times [\frac{k_1 + 12}{100}\ell_5, \ell_5]\} \\ &\geq \delta_{11} := 1 - (1-\delta_{10})^{1/12t} \end{aligned}$$

Then the conclusion of Theorem 6.1 holds.

Proof: Assume that

(6.64) $$\begin{aligned} &P\{\ \exists \text{ occupied horizontal crossing } r' \text{ of} \\ &[0,\ell_1] \times [0,\ell_5] \text{ with } z_\ell(r') \leq (.03)\ell_5\,, \\ &z_h(r') \geq (.97)\ell_5 \text{ and which intersects } L(a) \text{ in} \\ &\{a\} \times [0, \frac{k_1 - 11}{100}\ell_5]\} \geq 1 - (1-\delta_{11})^{1/2} \end{aligned}$$

If (6.64) does not hold, then it will become valid after replacing the interval $\{a\} \times [0, (k_1-11)\ell_5/100]$ by $\{a\} \times [(k_1+12)\ell_5/100, \ell_5]$, by virtue of (6.63) and (6.21). In this case one only has to interchange the role of top and bottom in the following argument.

The idea of the proof is now roughly as follows. If $E_3(k)$ occurs then there is an occupied path r with $z_\ell(r) \leq (.03)\ell_5$, $z_h(r) \geq (.97)\ell_5$ and which contains a connection, ρ, between the lower edge of the rectangle .

(6.65) $$T := [0, \lfloor \frac{\ell_1}{2} \rfloor] \times [(.03)\ell_5, \ell_5]$$

and the segment

(6.66) $$I := \{\lfloor \frac{\ell_1}{2} \rfloor\} \times [\frac{k_1}{100}\ell_5, \frac{k_1 + 1}{100}\ell_5]$$

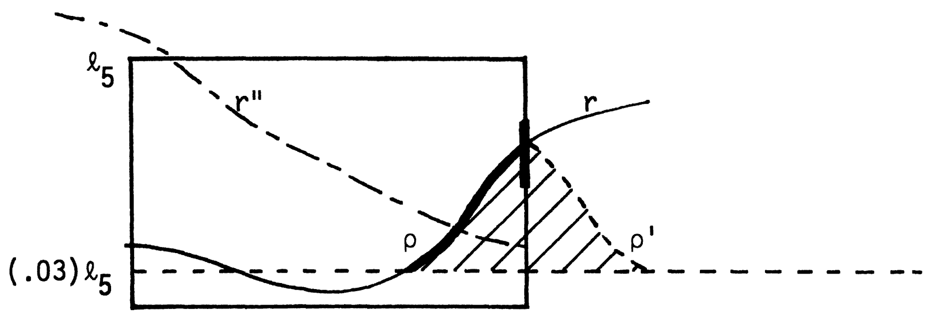

Figure 6.6. The interval I (in the right edge of the rectangle) and the connection ρ are drawn boldly. The reflection ρ' of ρ is dashed (---) —— - —— - denotes r''. The hatched region is Δ

in its right edge (see Fig. 6.6). (Note that (6.64) implies $k_1 \geq 11$ so that I lies entirely in the right edge of the rectangle at (6.65). Also, $z_\ell(r) \leq (.03)\ell_5$ guarantees that r intersects the lower edge of this rectangle. Now if the translate by $(\lfloor \ell_1/2 \rfloor - a, \lfloor (.1)\ell_5 \rfloor)$ of the event in (6.64) occurs, then there exists an occupied horizontal crossing r'' of

$$(6.67). \quad [\lfloor \frac{\ell_1}{2} \rfloor - a, \lfloor \frac{\ell_1}{2} \rfloor - a + \ell_1] \times [\lfloor (.1)\ell_5 \rfloor , \lfloor (.1)\ell_5 \rfloor + \ell_5]$$

which gets above the upper edge of T (in fact its highest point will be on or above the line $x(2) = (.97)\ell_5 + \lfloor (.1)\ell_5 \rfloor$. Also r'' intersects $L(a + \lfloor \frac{\ell_1}{2} \rfloor - a) = L(\lfloor \frac{\ell_1}{2} \rfloor)$ in $\{a\} \times [\lfloor (.1)\ell_5 \rfloor,$ $\frac{k_1 - 11}{100} \ell_5 + \lfloor (.1)\ell_5 \rfloor]$. Thus the intersection of r'' with $L(\lfloor \frac{\ell_1}{2} \rfloor)$ lies in the right edge of T below I. Denote by Δ the "triangle" bounded by ρ, its reflection ρ' in $L(\lfloor \frac{\ell_1}{2} \rfloor)$, and the horizontal line $\mathbb{R} \times (.03)\ell_5$. Then from the above observations we see that r'' contains a point in Δ as well as points outside Δ (to wit points above the upper edge of T). Since r'' is a horizontal crossing of the rectangle (6.67) it lies above the line $x(2) = \lfloor (.1)\ell_5 \rfloor - \Lambda > (.03)\ell_5$ and does not intersect the horizontal bottom edge of Δ. In order to enter Δ r'' must therefore intersect $\rho \cup \rho'$. A symmetry argument will show that we may assume r'' intersects ρ and hence r. But then $r \cup r''$ will contain an occupied vertical crossing of $[-\ell_1, \ell_1 + \Lambda] \times [(.03)\ \ell_5, (.97)\ell_5 + \lfloor (.1)\ell_5 \rfloor]$. By periodicity this gives us a lower bound for

(6.68) $\quad P\{\ \exists$ occupied vertical crossing of

$$[0,\ 3\ell_1] \times [0, \lfloor (1.04)\ell_5 \rfloor - 2]\} \quad .$$

This will take the place of (6.35) and then the lemma will follow directly from Lemmas 6.4, 6.2.

Now for the details. The symmetry argument is really the main part which needs to be filled in. To do this we shall use Prop. 2.3 and this requires a slight change in the definition of ρ and Δ. At various places we tacitly assume n_2, and hence ℓ_5, large. Let B_1 be a continuous path without double points, made up from edges of $\mathcal{G}_{p\ell}$ inside the strip

$$[0, \lfloor \frac{\ell_1}{2} \rfloor] \times ((.03)\ell_5,\ (.04)\ell_5)\ ,$$

and connecting the left and right edge of this strip. It is easy to see from the periodicity and connectedness of $\mathcal{G}_{p\ell}$ that such a B_1 exists as soon as $(.01)\ell_5 \geq 3s^{-1}\ n_2$ is larger than some constant which depends on $\mathcal{G}_{p\ell}$ only (see Lemma A.3 for a more detailed argument). Let the endpoints of B_1 be $(0,c)$ and $(\lfloor \frac{\ell_1}{2} \rfloor, d)$. Next define the straight line segments

$$B_2 = \{\lfloor \frac{\ell_1}{2} \rfloor\} \times [\lfloor (.1)\ell_5 \rfloor,\ \frac{k_1 + 1}{100}\ \ell_5]\ ,$$

$$A = \{\lfloor \frac{\ell_1}{2} \rfloor\} \times [d,\ \lfloor (.1)\ell_5 \rfloor] \quad .$$

Finally, let C be the curve made up of the three segments $\{0\} \times [c, \ell_5]$, $[0, \lfloor \ell_1/2 \rfloor] \times \{\ell_5\}$ and $\lfloor \ell_1/2 \rfloor \times [(k_1+1)\ell_5/100,\ \ell_5]$. Then B_1, A, B_2, C together make up a Jordan curve J which almost equals the perimeter of T, except that the lower edge of T has been replaced by B_1 (see Fig. 6.7). If r is an occupied horizontal

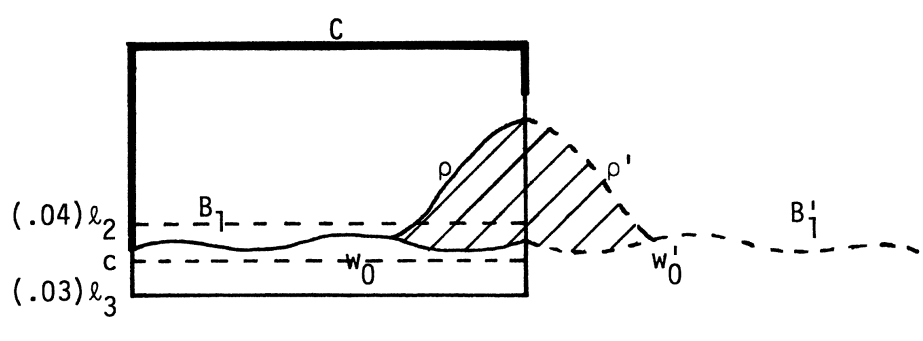

Figure 6.7. C is drawn boldly. The hatched region is Δ.

crossing of $[0,\ell_1] \times [0,\ell_5]$ with $z_\ell(r) \leq (.03)\ell_5$, and $Y(\lfloor \ell_1/2 \rfloor, r) \in [k_1\ell_5/100, (k_1+1)\ell_5/100]$, then since r lies to the left of $L(\lfloor \ell_1/2 \rfloor)$ until it reaches $\zeta(\lfloor \ell_1/2 \rfloor, r)$ and since the piece of r between $L(0)$ and $L(\ell_1/8)$ gets below $\mathbb{R} \times (.03)\ell_5$, r contains an occupied path $\rho = (w_0, f_1, \dots, f_\tau, w_\tau)$ with the following properties:

(6.69) $$w_1, f_2, \dots, f_{\tau-1}, w_{\tau-1} \subset \text{int}(J),$$

(6.70) $$w_0 \in B_1 \text{ and } f_1 \setminus \{w_0\} \subset \text{int}(J),$$

(6.71) f_τ has exactly one point in common with J. This lies in B_2 and is either w_τ or the midpoint of f_τ.

For (6.71) we used Comment 2.4(ii) again. The intersection of f_τ with B_2 is just the point $\zeta(\lfloor \ell_1/2 \rfloor, r) = (\lfloor \ell_1/2 \rfloor, Y(\lfloor \ell_1/2) \rfloor, r))$ in the notation introduced before Lemma 6.6. Also $k_1 \geq 11$. (6.70) holds because B_1 is made up from edges of the planar graph $\mathcal{G}_{p\ell}$; the path r on $\mathcal{G}_{p\ell}$ can intersect B_1 only in a vertex. w_0 is just the first such intersection we reach when going back along r from $\zeta(\lfloor \ell_1/2 \rfloor, r)$ to its initial point. The above shows that

(6.72) $$P\{\exists \text{ occupied path } \rho = (w_0, f_1, \dots, f_\tau, w_\tau) \text{ with the properties (6.69) - (6.71)}\} \geq P\{E_3(k_1)\} \geq \delta_{10}.$$

The properties (6.69) - (6.71) are just the analogues of (2.23) - (2.25) in the present context and we can therefore apply Prop. 2.3 (again with $S = \mathbb{R}^2$). If $J^-(\rho)$ denotes the component of $\text{int}(J) \setminus \rho$ which contains A in its boundary, then we denote by R the path ρ for which $J^-(\rho)$ is minimal among all occupied paths ρ satisfying (6.69) - (6.71). By Prop. 2.3 and (6.72) the probability that R exists is at least δ_{10}. Now for any path ρ_0 satisfying (6.69) - (6.71) denote by ρ_0' its reflection in $L(\lfloor \ell_1/2 \rfloor)$. Also write B_1' for the reflection of B_1 in $L(\lfloor \ell_1/2 \rfloor)$ and $\Delta = \Delta(\rho_0)$ for the triangular domain bounded by $\rho_0 \cup \rho_0'$ and the piece of $B_1 \cup B_1'$ between w_0 and w_0', the reflection of w_0 in $L(\lfloor \ell_1/2 \rfloor)$. Now let ρ_0 be a given path which satisfies (6.69) - (6.71). Assume the translate of the event in (6.64) by $[\lfloor \ell_1/2 \rfloor - a, \lfloor (.1)\ell_5 \rfloor]$ occurs. Then there exists an occupied horizontal crossing r'' of the rectangle in (6.67). Moreover, the piece of r'' between $L(\lfloor \ell_1/2 \rfloor - a)$ and $L(\lfloor \ell_1/2 \rfloor - a + \ell_1/8)$ contains a point on or above the line $\mathbb{R} \times \{(.97)\ell_5 + \lfloor (.1)\ell_5 \rfloor\}$ (by virtue of the condition on $z_h(r')$ in (6.64)). Also r'' intersects $L(\lfloor \ell_1/2 \rfloor)$ in a point with second coordinate at most

$$\frac{k_1-11}{100}\ell_5 + \lfloor (.1)\ell_5 \rfloor < \frac{k_1}{100}\ell_5 .$$

Lastly, r'' lies above the horizontal line $\mathbb{R} \times \{\lfloor (.1)\ell_5 \rfloor - \Lambda\}$ and a fortiori does not intersect $B_1 \cup B_1'$. In particular r'' contains a point outside Δ(since Δ lies below $\mathbb{R} \times \{\ell_5\}$) and a point on $L(\lfloor \ell_1/2 \rfloor)$ inside Δ. Since r'' does not intersect $B_1 \cup B_1'$ it must intersect $\rho_0 \cup \rho'_0$, necessarily in a vertex of $\mathcal{G}_{p\ell}$. Therefore r'' contains a path $\sigma = (u_0, g_1, \ldots, g_\theta, u_\theta)$ with the properties (6.73) - (6.76) below.

(6.73) g_1 intersects the horizontal line $\mathbb{R} \times \{\lfloor (1.07)\ell_5 \rfloor - 1\}$.

(6.74) $(u_0, g_1, \ldots, g_\theta \setminus \{u_\theta\}) = \sigma \setminus \{u_\theta\}$ is contained in the vertical strip $[\lfloor \ell_1/2 \rfloor - a - \Lambda, \lfloor \ell_1/2 \rfloor + a + \Lambda] \times \mathbb{R}$ but outside $\overline{\Delta}(\rho_0)$. (Use the inequality $\lfloor \ell_1/2 \rfloor - a + \ell_1 \le \lfloor \ell_1/2 \rfloor + a$).

(6.75) $$u_\theta \in \rho_0 \cup \rho_0' .$$

(6.76) $$u_0, \ldots, u_{\theta-1} \text{ are occupied} .$$

It follows from these observations and (6.64) that

$$P\{\exists \text{ a path } \sigma = (u_0, g_1, \ldots, g_\theta, u_\theta) \text{ satisfying (6.73) - (6.76)}\} \ge 1 - (1-\delta_{11})^{1/2} .$$

Since $L(\lfloor \ell_1/2 \rfloor)$ is an axis of symmetry we obtain exactly as in the derivation of (6.20) from (6.22) that

(6.77) $$P\{\exists \text{ a path } \sigma = (u_0, g_1, \ldots, g_\theta, u_\theta) \text{ satisfying (6.73), (6.74), (6.76) and } u_\theta \in \rho_0\} \ge 1-(1-\delta_{11})^{1/4}.$$

Now, by Prop. 2.3 the event $\{R = \rho_0\}$ depends only on vertices in $\bar{J}^-(\rho_0) \cup$ the reflection of $\bar{J}^-(\rho_0)$ in $L(\lfloor \ell_1/2 \rfloor)$ i.e., on vertices in $\bar{\Delta}(\rho_0)$. The event in (6.79) depends only on a set of vertices which is disjoint from the above one, and is therefore independent of $\{R = \rho_0\}$. As in the proof of (6.25) we now obtain

(6.78) $P\{R$ exists and $\exists$ path $\sigma = (u_0, g_1, \ldots, g_\theta, u_\theta)$ which satisfies (6.73) and (6.76), is contained in the vertical strip $[\lfloor \ell_1/2 \rfloor - a - \Lambda, \lfloor \ell_1/2 \rfloor + a + \Lambda]$ and has $u_\theta \in R\}$
$$\geq \{1 - (1-\delta_{11})^{1/4}\} P\{R \text{ exists}\}$$
$$\geq \{1 - (1-\delta_{11})^{1/4}\} \quad \delta_{10} \quad .$$

But if R exists, then it is occupied and contains a point on B_1. Thus, if the event in the left hand side of (6.78) occurs, then $\sigma \cup R$ contains an occupied vertical crossing of

$$[\lfloor \frac{\ell_1}{2} \rfloor - a - \Lambda - 1, \lfloor \frac{\ell_1}{2} \rfloor + a + \Lambda + 1] \times [(.04)\ell_5, \lfloor (1.07)\ell_5 \rfloor - 1]$$

Since $a \leq \ell_1$ we obtain from periodicity and the monotonicity property in Comment 3.3(v).

$$\sigma((4\ell_1, \lfloor 1.03\ell_5 \rfloor - 3); 2, p, \mathcal{G}_{p\ell})$$
$$\geq \delta_{10} \{1 - (1-\delta_{11})^{1/4}\} \quad .$$

This is just (6.35) for the values

$$\ell_4 = \lfloor (1.03)\ell_5 \rfloor - 3, \ \delta_5 = \delta_{10}\{1 - (1-\delta_{11})^{1/4}\} \quad ,$$

and ℓ_1 replaced by $4\ell_1$. But we also have

$$\sigma((\ell_1, \ell_5); 1, p, \mathcal{G}_{p\ell}) \geq \delta_9$$

(by virtue of (6.58)) as replacement for (6.13), and

$$\ell_5 \leq \frac{98}{100} (\lfloor (1.03)\ell_5 \rfloor - 3).$$

We can therefore obtain (6.9) - (6.11) again from Lemmas 6.4 and 6.2 in the same way as in Lemma 6.5. $\square$

One more reduction is necessary. Lemma 6.7 discusses intersections of horizontal crossings with $L(a)$ for a single integer a. The next lemma considers the intersections with a vertical strip around such a line.

Lemma 6.8. Assume that (6.59) holds and that for the t of (6.61) there exists an integer $a \in [\frac{5\ell_1}{8}, \frac{7\ell_1}{8}]$ for which

(6.79) $$P\{\exists \text{ occupied horizontal crossing } r' \text{ of } [0,\ell_1] \times [0,\ell_5] \text{ with } z_\ell(r') \le (.03)\ell_5, z_h(r') \ge (.97)\ell_5, \text{ and which contains some vertex } v = (v(1), v(2)) \text{ with } |v(1) - a| \le t \text{ and } v(2) \in [0, \frac{k_1 - 12}{100}\ell_5] \cup [\frac{k_1 + 13}{100}\ell_5, \ell_5]\} \ge \delta_{10}$$

Then the conclusion of Theorem 6.1 holds.

Proof: If the event in (6.79) occurs, then $v(1)$ must lie in one of the intervals $[b,b+1]$, $a - t \le b < a + t$ and $v(2)$ in one of the two intervals $[0, \frac{k_1 - 12}{100}\ell_5]$, $[\frac{k_1 + 13}{100}\ell_5, \ell_5]$. From the by now familiar argument using the FKG inequality it follows that one of these eventualities has a probability at least

$$\delta_{12} := 1 - (1-\delta_{10})^{1/4t} .$$

For the sake of argument let b be an integer with

$$\frac{1}{2}\ell_1 \le a - t \le b < a + t < \ell_1$$

and such that

(6.80) $$P\{\exists \text{ occupied horizontal crossing } r' \text{ of } [0,\ell_1] \times [0,\ell_5] \text{ with } z_\ell(r') \le (.03)\ell_5, z_h(r') \ge (.97)\ell_5 \text{ and which contains a vertex } v = (v(1), v(2)) \text{ with } b \le v(1) < b + 1 \text{ and } v(2) \in [0, \frac{k_1 - 12}{100}\ell_5]\} \ge \delta_{12} .$$

If for $a = b$ or $a = b + 1$

$$(6.81) \qquad P\{\ \exists \text{ occupied horizontal crossing } r' \text{ of } [0,\ell_1] \times [0,\ell_5] \text{ with } z_\ell(r') \le (.03)\ell_5,\ z_h(r') \ge (.97)\ell_5 \text{ and which intersects } L(a) \text{ in } \{a\} \times [0, \frac{k_1 - 11}{100}\ \ell_5]\} \ge \delta_{11}$$

then we are done, by virtue of Lemma 6.7. Thus we may assume that (6.81) fails for $a = b$ and $a = b + 1$. The obvious generalization of (6.21) to three events together with (6.80) then gives

$$(6.82) \qquad P\{\ \exists \text{ occupied horizontal crossing } r' \text{ of } [0,\ell_1] \times [0,\ell_5] \text{ which intersects } L(b) \text{ only in } \{b\} \times (\frac{k_1 - 11}{100}\ \ell_5,\ell_5] \text{ and } L(b+1) \text{ only in } \{b+1\} \times (\frac{k_1 - 11}{100}\ \ell_5,\ell_5], \text{ but contains a vertex } v = (v(1), v(2)) \text{ with } b \le v(1) < b + 1,\ 0 \le v(2) \le \frac{k_1 - 12}{100}\ \ell_5\} \ge 1 - \frac{1-\delta_{12}}{(1-\delta_{11})^2} = \delta_{11}\ .$$

When the event in (6.82) occurs, then the piece of r' from the last edge of r' before v which intersects $L(b) \cup L(b+1)$ through the first edge of r' after v which intersects $L(b) \cup L(b+1)$ contains a vertical crossing of $[b,b+1] \times [\frac{k_1 - 12}{100}\ \ell_5, \frac{k_1 - 11}{100}\ \ell_5]$. Thus, (6.82) and periodicity implies

$$(6.83) \qquad P\{\ \exists \text{ occupied vertical crossing of } [0,1] \times [0, \frac{\ell_5}{100} - 2]\} \ge \delta_{11}\ .$$

As before let μ be the number of vertices of $\mathcal{G}_{p\ell}$ in the unit square $[0,1) \times [0,1)$, and let Λ as in (6.4). Any vertical crossing $r'' = (w_0,f_1,\dots,f_\rho,w_\rho)$ of $[0,1] \times [0, \frac{\ell_5}{100} - 2]$ intersects all the segments $[0,1] \times \{\lfloor \frac{j\ell_5}{300\mu} \rfloor\}$, $1 \le j \le \mu + 1$. Let $w_{i(j)} = (w_{i(j)}(1), w_{i(j)}(2))$ be the last vertex on r'' on or below the j^{th} segment of this form. Then

$$0 < w_{i(j)}(1) < 1 \quad , \quad 1 \leq j \leq \mu + 1,$$

while for $j \neq k$, $1 \leq j,k \leq \mu + 1$

$$| w_{i(j)}(2) - w_{i(k)}(2)| \geq \frac{\ell_5}{300\mu} - \Lambda - 1 \geq \frac{\ell_5}{400\mu} \quad ,$$

provided ℓ_5 is large enough, or equivalently, $n_2 \geq n_0(\mathcal{G},\pi)$ for suitable n_0. Any such point $w_{i(j)}$ is the translate by a vector $(0,m)$, $m \in \mathbb{Z}$, of some vertex in $[0,1) \times [0,1)$. Thus, by Dirichlet's pigeon hole principle there must be a pair $w_{i(j)}$ and $w_{i(k)}$ with equal first coordinates, i.e., with

$$w_{i(j)} - w_{i(k)} = (0,m) \quad \text{for some integer} \quad m \geq \frac{\ell_5}{400\mu} \quad .$$

Since $1 \leq j \leq \mu + 1$ and

$$w_{i(j)} \in (0,1) \times [\lfloor \frac{j\ell_5}{300\mu} \rfloor - \Lambda, \lfloor \frac{j\ell_5}{300\mu} \rfloor],$$

there are at most $\lambda:=(\Lambda+1)^2 \mu^2(\mu+1)^2$ possibilities for the pair $w_{i(j)}, w_{i(k)}$. Thus, by periodicity and the FKG inequality, (6.83) implies the existence of a vertex $w \in [0,1) \times [0,1)$ and integer $m \geq (400\mu)^{-1}\ell_5$ such that

(6.84) $$P\{ \exists \text{ occupied path in } [0,1] \times \mathbb{R} \text{ from } w \text{ to } w + (0,m)\} \geq \delta_{13} := 1 - (1-\delta_{11})^{1/\lambda} \quad .$$

By periodicity (6.84) remains valid if w is replaced by $w + (0,jm)$. Moreover, if we combine occupied paths from $w + (0,jm)$ to $w + (0,(j+1)m)$ for $j = 0,\dots,\nu - 1$ we obtain an occupied path with possible double points from w to $w + (0,\nu m)$. We can remove the double points by loop-removal (see Sect. 2.1). Since all the paths which we combined lie in the strip $[0,1] \times \mathbb{R}$ we obtain an occupied vertical crossing of $[-1,2] \times [1,\nu m-1]$. Thus, by virtue of the FKG inequality (6.84) implies

$$P\{ \exists \text{ occupied vertical crossing of } [0,3] \times [0,\nu m-2]\} \geq \delta_{13}^{\nu} \quad .$$

This, together with (6.5), implies (6.9) - (6.11) (this time we need only Lemma 6.1). □

Lemma 6.8 was the last reduction. With t fixed as in (6.61) it follows from the preceding lemmas that it suffices to prove Theorem 6.1 under the additional hypotheses that (6.58) holds, but (6.79) fails for every $5\ell_1/8 \le a \le 7\ell_1/8$. Again by (6.21) we may therefore assume that for such a in this interval.

$$(6.85) \qquad P\{\exists \text{ occupied horizontal crossing } r \text{ of } [0,\ell_1] \times [0,\ell_5] \text{ with } z_\ell(r) \le (.03)\ell_5,\ z_h(r) \ge (.97)\ell_5, \text{ and which intersects the strip } [a-t, a+t] \times \mathbb{R} \text{ only in } [a-t, a+t] \times [\frac{k_1 - 12}{100}\ell_5, \frac{k_1 + 13}{100}\ell_5]\} \ge 1 - \frac{1-\delta_9}{1-\delta_{10}} \ge \delta_{10}.$$

<u>Lemma 6.9</u>. <u>If (6.85) holds for every integer</u> $a \in [\frac{5\ell_1}{8}, \frac{7\ell_1}{8}]$, <u>then the conclusion of Theorem 6.1 holds</u>.

<u>Proof</u>: Assume $0 \le k_1 \le 50$. The case $50 < k_1 < 100$ again only involves an interchange of the role of top and bottom. If the event in (6.85) occurs, then the segment of r between the points where $z_\ell(r)$ and $z_h(r)$ are achieved lies (by definition of z_ℓ and z_h) in the vertical strip $[0,\ell_1/8] \times \mathbb{R}$. Consequently, by periodicity and (6.85).

$$(6.86) \qquad P\{\exists \text{ occupied vertical crossing } r' \text{ of } [\lfloor \tfrac{11}{16}\ell_1 \rfloor - 1, \lceil \tfrac{13}{16}\ell_1 \rceil + 1] \times [-(.01)\ell_5, (.93)\ell_5 - 1] \ge \delta_{10}.$$

We shall again use Prop. 2.3 to find the "right most" of the vertical crossings in (6.86). More precisely, let $v_0 \in [0,1) \times [0,1)$, α and r_0 have the properties discussed before the definition (6.61) of t; (see also Lemma A.3). For a suitable choice of the integers ν_1, ν_2 and m the path obtained by traversing successively $r_0 + (\nu_1 + j\alpha, \nu_2)$, $j = 0,1,\dots,m$ will be a self-avoiding path s on $\mathcal{G}_{p\ell}$ in the horizontal strip $\mathbb{R} \times (-(.01)\ell_5, 0)$ (provided $n_2 \ge n_0(\mathcal{G},\pi)$ again) which intersects both the vertical lines

$$L(\lfloor \tfrac{11}{16}\ell_1 \rfloor - 1) \quad \text{and} \quad L(\lceil \tfrac{13}{16}\ell_1 \rceil + 1).$$

Denote by B_1 the segment of the path s from its last intersection with $L(\lfloor 11\ell_1/16 \rfloor - 1)$ to its first intersection with $L(\lceil 13\ell_1/16 \rceil + 1)$. Similarly s' will be a path in the horizontal strip $\mathbb{R} \times ((.92)\ell_5 - 1, (.93)\ell_5 - 1)$ obtained by traversing successively $r_0 + (\nu_3 + j\alpha, \nu_4)$, $j = 0,1,\ldots,m$, and B_2 will be the segment of s' from its last intersection with $L(\lfloor \frac{11}{16}\ell_1/16 \rfloor - 1)$ to its first intersection with $L(\lceil 13\ell_1/16 \rceil + 1)$ (see Fig. 6.8).

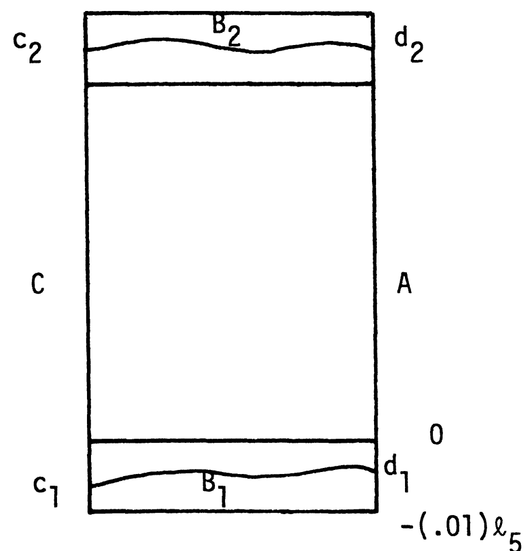

Figure 6.8

By property (6.62), if a vertical line $L(b)$ intersects B_2 in a point of $r_0 + (\nu_3 + j_0\alpha, \nu_4)$ then

(6.87) the paths $r_0 + (\nu_3 + j\alpha, \nu_4)$, $j_0 \le j \le m$, are contained in the halfplane $[b-t, \infty) \times \mathbb{R}$.

We denote the endpoints of B_i, $i = 1,2$, by

$$(\lfloor \tfrac{11}{16}\ell_1 \rfloor - 1, c_i) \text{ and } (\lceil \tfrac{13}{16}\ell_1 \rceil + 1, d_i).$$

Furthermore A denotes the straightline segment $\{\lceil 13\ell_1/16 \rceil + 1\} \times [d_1, d_2]$ and C the straightline segment $\{\lfloor 11\ell_1/16 \rfloor - 1\} \times [c_1, c_2]$. (see Fig. 6.8). The composition of B_1, A, B_2 and C is a Jordan curve which we denote by J. If the event in (6.86) occurs, then the path r' begins below B_1 and ends above B_2. Since $\mathcal{G}_{p\ell}$ is planar r' intersects B_1 as well as B_2 only in vertices of $\mathcal{G}_{p\ell}$. In particular r' must contain an occupied path $\rho = (w_0, f_1, \dots, f_\tau, w_\tau)$ with the following two properties

$$(6.88) \qquad \rho \setminus \{w_0, w_\tau\} \subset \operatorname{int}(J)$$

$$(6.89) \qquad w_0 \in B_1,\ w_\tau \in B_2 \quad .$$

These are the analogues of (2.23) - (2.25). Again we denote the component of $\operatorname{int}(J) \setminus \rho$ which contains A in its boundary by $J^-(\rho)$ whenever ρ is a path satisfying (6.88) and (6.89). Prop. 2.3 with $S = \mathbb{R}^2$ shows that as soon as such an occupied path ρ exists, there also exists one with minimal $J^-(\rho)$. As in Prop. 2.3 we denote the occupied path ρ for which $J^-(\rho)$ is minimal by R whenever it exists. By Prop. 2.3 and (6.86).

$$(6.90) \qquad P\{\ R \text{ exists } \} \geq P\{\ \exists \text{ occupied path } \rho \text{ which satisfies (6.88) and (6.89)}\} \geq \delta_{10} \quad .$$

Now assume that R exists and equals some fixed path $\rho_0 = (w_0, f_1, \dots, f_\tau, w_\tau)$. Set

$$b = w_\tau(1)\ ,\ a = \lfloor b \rfloor = \lfloor w_\tau(1) \rfloor\ ,$$

and denote the highest intersection of ρ_0 with $L(b)$ by (b, b_2). Since the endpoint of ρ_0, $w_\tau = (w_\tau(1), w_\tau(2))$ lies on $L(b)$ we have $b_2 \geq w_\tau(2)$. We write I for the segment $\{b\} \times [b_2, \lfloor 5\ell_5/4 \rfloor]$ of $L(b)$, and ρ_1 for the segment of ρ_0 from its initial point w_0 to the intersection (b, b_2) of ρ_0 and $L(b)$. Then $\rho_1 \cup I$ contains a crosscut of the rectangle

$$T := (\lfloor \tfrac{11}{16} \ell_1 \rfloor - 1, \lceil \tfrac{13}{16} \ell_1 \rceil + 1) \times (0, \lfloor 5\ell_5/4 \rfloor),$$

because ρ_0 begins on B_1 which lies below the lower edge of this rectangle (see Fig. 6.9). This crosscut divides T in a left and a right component, w_τ lies on B_2, hence belongs to $(r_0 + (\nu_3 + j_0\alpha, \nu_4)$ for some j_0. The piece of B_2 which belongs to $\operatorname{Fr}(J^-(\rho_0))$ then consists of pieces of $r_0 + (\nu_3 + j\alpha, \nu_4)$ with $j_0 \leq j \leq m$. By (6.87) and the construction of B_2 $B_2 \cap \operatorname{Fr}(J^-(\rho_0))$ contained in the rectangle

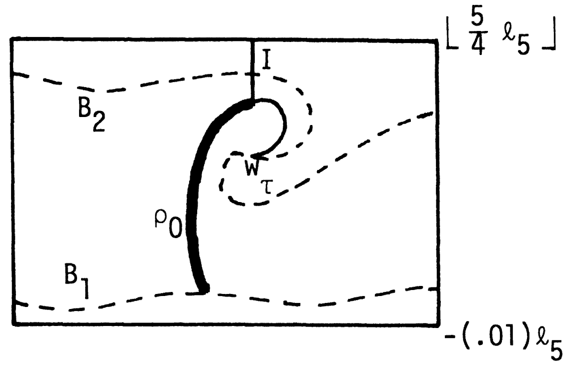

Figure 6.9 B_1 and B_2 are dashed. ρ_0 is drawn solidly; the boldly drawn part of ρ_0 is ρ_1.

$$[b-t, \lceil \frac{13}{16} \ell_1 \rceil + 1] \times [(.92)\ell_5 - 1, (.93)\ell_5] \quad .$$

We show first that this implies

$$(6.91) \qquad B_2 \cap Fr(J^-(\rho_0)) \cap \text{ left component of } T$$
$$\subset [b - t, b + t - 1] \times [(.92)\ell_5 - 1, (.93)\ell_5]$$

From the preceding it follows that it suffices to show that the left hand side of (6.91) is contained in $(-\infty, b + t - 1] \times \mathbb{R}$. Now assume x is a point of $r_0 + (\nu_3 + j\alpha,\nu_4) \cap Fr(J^-(\rho_0))$ for some $j_0 \le j \le m$. If $r_0 + (\nu_3 + j\alpha,\nu_4)$ lies entirely strictly to the right of $L(b)$, then so do $r_0 + (\nu_3 + j'\alpha,\nu_4)$ for $j' > j$, because $\alpha \ge 1$. In this case there is a path from x to the right edge of T which consists of pieces of $r_0 + (\nu_3 + j'\alpha,\nu_4)$, $j' = j, j+1,\dots,m$. This path neither intersects $I \subset L(b)$, nor does it intersect ρ_0, since $\rho_0 \cap B_2 = \{w_\tau\}$. Consequently, x can be connected in $T \setminus \rho_0 \cup I$ to the right edge of T, and x cannot lie in the left component of T. If on the other hand $r_0 + (\nu_3 + j\alpha,\nu_4)$ is not entirely strictly to the right of $L(b)$, then $r_0 + (\nu_3 + j\alpha,\nu_4) \subset (-\infty, b + t - 1] \times \mathbb{R}$ by the choice of t in (6.61). Thus (6.91) holds.

Assume now that the translate of the event in (6.85) by $(0,\lfloor \frac{\ell_5}{4} \rfloor)$ occurs. Then there exists an occupied horizontal crossing r of

$$[0,\ell_1] \times [\lfloor \frac{\ell_5}{4} \rfloor, \lfloor \frac{\ell_5}{4} \rfloor + \ell_5] = [0,\ell_1] \times [\lfloor \frac{\ell_5}{4} \rfloor, \lfloor \frac{5\ell_5}{4} \rfloor]$$

which intersects the strip $[a - t, a + t] \times \mathbb{R}$ only in

$$[a - t, a + t] \times [\frac{k_1 + 12}{100} \ell_5, \frac{k_1 + 38}{100} \ell_5] .$$

Moreover, r passes through a point $z = (z(1), z(2))$ with

$$0 \le z(1) \le \frac{\ell_1}{8} , \; z(2) \ge (1.22)\ell_5 - 1$$

before it reaches $L(\frac{\ell_1}{8})$. Since this horizontal crossing r begins to the left of $L(\lfloor \frac{11\ell_1}{16} \rfloor - 1)$ and ends to the right of $L(\lceil \frac{13\ell_1}{16} \rceil + 1)$ it must intersect the crosscut of T contained in $\rho_1 \cup I$. We claim that r intersects ρ_0, but not I, and does not hit $Fr(J^-(\rho_0))$ before it hits ρ_0). To prove this claim we first note that r cannot intersect

$$I \cup \{B_2 \cap Fr(J^-(\rho_0)) \cap \text{left component of } T\} ,$$

since this set is contained in

$$[a - t, a + t] \times [(.92)\ell_5 - 1, \lfloor 5\ell_5 /4 \rfloor] ,$$

which is disjoint from

$$[a - t, a + t] \times [\frac{k_1 + 12}{100} \ell_5 , \frac{k_1 + 38}{100} \ell_5] .$$

To see this we use (6.91) and the facts $b_2 \ge w(2) \ge (.92)\ell_5 - 1$ (recall that the lower endpoint of I, (b,b_2) lies no lower than $w_\tau \in B_2$) and $k_1 \le 50$. In particular r does not intersect I and must intersect ρ_0. Moreover, r does not get below the horizontal line $\mathbb{R} \times \{\lfloor \ell_5 /4 \rfloor\}$ and therefore cannot hit B_1 or the lower edge of T. Neither does r get above the top edge of T and therefore cannot enter the right component of T through the upper edge of T without hitting $\rho_0 \cup I$ first. Lastly, since r stays between the upper and lower edge of T and begins to the left of T, it cannot reach the right edge of T without hitting $\rho_0 \cup I$. All in all we see that r cannot enter the right component of T without hitting $\rho_0 \cup I$. A fortiori r cannot hit

$$B_2 \cap Fr(J^-(\rho_0)) \cap \text{right component of } T$$

without hitting $\rho_0 \cup I$ first. Combining the above observations we see that r must hit $\rho_0 \cup I$ (and hence ρ_0) before hitting the other parts of $Fr(J^-(\rho_0))$ (since these other parts lie in $\mathbb{R} \times (-\infty, 0] \cup B_2 \cup$ right edge of T. This substantiates our claim.

An immediate consequence of the claim is that the piece of r from its initial point to its first intersection with ρ_0 is a path $s = (u_0, g_1, \dots, g_\sigma, u_\sigma)$ with the following properties:

(6.92) $$s \setminus \{u_\sigma\} = (u_0, g_1, \dots, u_{\sigma-1}, g_\sigma \setminus \{u_\sigma\}) \subset (\bar{J}^-(\rho_0))^c,$$
$$\text{and} \quad s \cap \rho_0 = \{u_\sigma\}$$

(6.93) s is contained in the horizontal strip
$$[-\Lambda, \ell_1] \times \mathbb{R},$$

(6.94) s contains a point $z = (z(1), z(2))$ with
$$z(2) \geq (1.22)\ell_5 - 1 .$$

and

(6.95) $u_0, \dots, u_{\sigma-1}$ are occupied

Clearly the existence of such a path s depends only on the occupancies of vertices outside $\bar{J}^-(\rho_0)$, and by Prop. 2.3, these are independent of the event $\{R = \rho_0\}$. Just as in the proof (6.25) - in particular the estimates following (6.28) - it follows from this and (6.85) that

(6.96) $$P\{R = \rho_0 \text{ and there exists a path } s \text{ with the properties } (6.92) - (6.95)\} \geq \delta_{10} P\{R = \rho_0\} .$$

Finally observe that if $R = \rho_0$ and there exists a path s with the properties (6.92) - (6.95) then s and ρ_0 together contain an occupied path from the initial point of ρ_0 on B_1, (and hence below $\mathbb{R} \times \{0\}$) via u_σ(the intersection of ρ_0 and s) to z above the horizontal line $\mathbb{R} \times \{(1.22)\ell_5 - 1\}$. This path also lies in the strip $[-\Lambda, \ell_1] \times \mathbb{R}$ and consequently $\rho_0 \cup s$ contains an occupied vertical crossing of $[-\Lambda - 1, \ell_1 + 1] \times [0, (1.22)\ell_5 - 1]$. Thus,

$$P\{\exists \text{ occupied vertical crossing of } [-\Lambda - 1, \ell_1 + 1] \times [0, (1.22)\ell_5 - 1]\} \geq \sum_{\substack{\rho_0 \text{ satisfying} \\ (6.88),\ (6.89).}} P\{R = \rho_0 \text{ and}$$

there exists a path s with properties (6.92) - (6.95)}

$$\geq \delta_{10} \sum_{\substack{\rho_0 \text{ satisfying} \\ (6.88),\ (6.89)}} P\{R = \rho_0\} \quad \text{(by (6.96))}$$

$$\geq \delta_{10}^2 \text{ (by (6.90)}.$$

By periodicity this implies for $\ell_1 = n_1 \geq 2\Lambda + 3$,

$$\sigma((2\ell_1, (1.22)\ell_5 - 1); 2, p, \mathcal{G}_{p\ell}) \geq \delta_{10}^2 \ .$$

Since we also have

$$\sigma((\ell_1, \ell_5); 1, p, \mathcal{G}_{p\ell}) \geq \delta_{10}$$

(by virtue of (6.85)), and $(1.22)\ell_5 - 1 \geq \frac{100}{98}\ell_5$ we can now obtain (6.9) - (6.11) from Lemma 6.4 and 6.2 in the same way as in Lemma 6.5 (provided $n_i \geq n_0(\mathcal{G}, \pi)$ again). □

As pointed out before Lemma 6.9 takes care of the last case and the proof of Theorem 6.1 is therefore complete. □

Proof of Corollary 6.1. It is easy to see that if r_1 and r_2 are occupied horizontal crossings of $[-2(\pi+3)n_1, 2(\pi+3)n_1] \times [-3n_2, -n_2]$ and $[-2(\pi+3)n_1, 2(\pi+3)n_1] \times [n_2, 3n_2]$, respectively, and if r_3

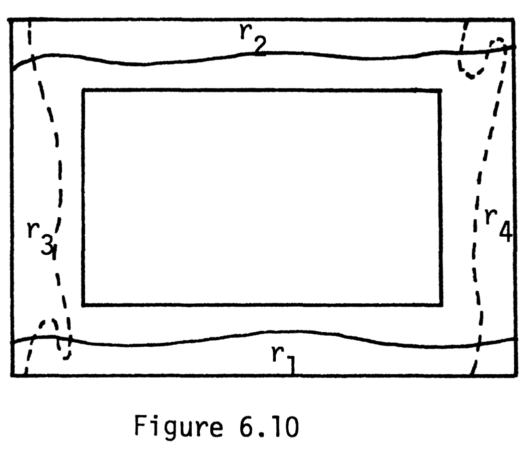

Figure 6.10

and r_4 are occupied vertical crossings of $[-2(\pi+3)n_1, -(\pi+3)n_1] \times [-3n_2, 3n_2]$ and $[(\pi+3)n_1, 2(\pi+3)n_1] \times [-3n_2, 3n_2]$, respectively, then $r_1 \cup r_2 \cup r_3 \cup r_4$ contains an occupied circuit surrounding 0 inside the annulus $[-2(\pi+3)n_1, 2(\pi+3)n_1] \times [-3n_2, 3n_2] \setminus (-(\pi+3)n_1, (\pi+3)n_1) \times (-n_2, n_2)$. (See Fig. 6.10).

Therefore the left hand side of (6.12) is at least equal to the probability of such r_1-r_4 existing. However, by the FKG inequality this is at least

$$\prod_{i=1}^{4} P\{r_i \text{ exists}\} \geq f^4(\delta_1, \delta_2, \pi, 4\pi+12)$$

(by (6.9) and (6.10)).