Chapter 1
Straightness and Symmetry

PROBLEM 1.1. When Do You Call a Line Straight?

Try to build for yourself a notion of a straight line. For example, think about how you would build a straight line and how you would check if the line you constructed is straight. Consider ways that you might convince someone that the line is, in fact, straight. Look to your experiences.

At first, you may look for examples of the physical world or natural straightness. You are likely to bring up ideas such as using a straight edge, stretching a string, sighting along a line, or, perhaps, following a laser beam. If so, then think about what is common among all of these "straight phenomena."
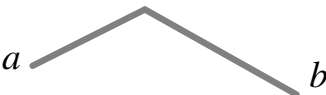
As you look for properties of straight lines that distinguish them from non-straight lines, the following statement (which is often taken as a definition) is likely to arise: *"A line is the shortest distance between two points."*

You may turn to this definition because it is, in fact, a common feature of the straightness expressed in the examples you have examined; perhaps, too, you are compelled to rely on it because this definition is most commonly used in high school mathematics classes. Whatever the case, the following questions may help you take a closer look at this definition:

 ♦ Can you always measure all the paths between two points?

 ♦ How do you find the shortest path?

 ♦ Is the shortest path between two points in fact a straight line?

 ♦ Is a straight line between two points always the shortest path?

You may cite examples of sighting along a line as a way to check for straightness. One example of "sighting" is a technique that is used for laying out fence posts. One can sight along a post on the end to see if any of the others are "out of line." From the view at the end, all the posts should coincide. The notion of sighting along a line comes from a property of light; that is, light always travels along the path that takes the least time. This "least-time" path is straight whenever the light is traveling through a vacuum or a uniform medium such as air at a uniform pressure and temperature. When light crosses into a different medium, such as glass or water, it bends. This is the property that allows a lens to work. Ordinary light beams, even if narrowly constricted or focused, become diffused and fuzzy over a distance because of interference among the various wavelengths of light contained in an ordinary light beam. Laser beams overcome this problem because they consist of only a single wavelength of light which can be narrowly focused into a thin beam that does not degenerate over distances.

Discussions of these and other examples may lead you to a concept of "non-turning." Here are some typical ideas you may present:

 ♦ "It is obvious that: a b is not the shortest path between a and b."

♦ "If one turns one can form a triangle..."

♦ "There is a bend..."

♦ "A straight line is the most direct path..."

There can be at least two expressions of "not turning": (1) going at a constant heading (the physical concept of sighting or light path); or, (2) having a 180° angle at any point on the path. Now ask yourself: *"What is a 180° angle? What is it about a 180° angle that expresses "not turning"?"*
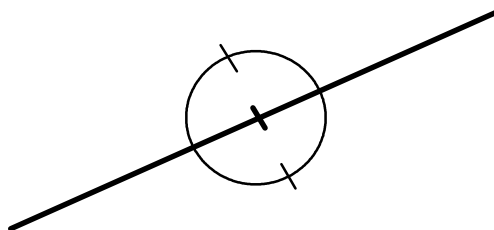
Figure 1.A. What is a 180° angle?

What symmetries does a straight line have? How do they fit with the examples that you have come up with and those mentioned above? Can we use any of the symmetries of a line to define straightness?

Remember that, in this course, nothing will be taken for granted, and that no answers are predetermined. Consequently, it is important that you persist in following your own ideas so that learning can happen for both the instructor and yourself.

Now the time has come to introduce the symmetries of a line. It will be valuable for you to share your ideas with other people. The spirit of this text is about sharing ideas. Mathematicians and other practitioners share and compare ideas all of the time. Most importantly, the act of explaining is a learning experience**.**

The Symmetries of a Line

Reflection-in-the-line symmetry: It is most useful to think of reflection as a "mirror" action with the line as an axis rather than as a "flip-over" action which involves an action in 3-space. In this way one can extend the notion of reflection symmetry to a sphere (the flip-over action is not possible on a sphere). Notice that this symmetry cannot be used as a definition for straightness because we use straightness to define reflection symmetry — the definition would be circular.

♦ *Practical application*: One can produce a straight line by folding a piece of paper because this action forces reflection-in-itself symmetry along the crease. On pages 2-3 we showed a carpenter's example.

Reflection-perpendicular-to-the-line symmetry: A reflection through **any** axis perpendicular to the line will take the line onto itself. Note that circles also have this symmetry about any diameter.

♦ *Practical applications*: You can tell if a straight segment is perpendicular to a mirror by seeing if it looks straight with its reflection. Also, a straight line can be folded onto itself.

Half-turn symmetry: A rotation through a half of a full revolution about any point p on the line takes the part of the line before p onto the part of the line after p and vice versa. Note that some non-straight lines, such as the letter "Z" also have half-turn symmetry.

♦ *Practical applications*: This is the principle behind hinges. If this is applied in 3-space, then you get a universal joint (a hinge that can fold in any direction).

Rigid-motion-along-itself symmetry: For straight lines in the plane, we call this *translation symmetry*. Any portion of a straight line may be moved along the line without leaving the line. This property of being able to move rigidly along itself is not unique to straight lines; circles (rotation symmetry) and circular helixes (screw symmetry) have this property as well.

♦ *Practical applications*: Slide joints such as in trombones, drawers, nuts and bolts, pulleys, etc. all utilize this symmetry.

3-dimensional rotation symmetry: In a 3-dimensional space, rotate the line around itself through *any* angle using itself as an axis.

♦ *Practical applications*: This symmetry can be used to check the straightness of any long thin object such as a stick by twirling the stick with itself as the axis. If the stick does not appear to wobble, then it is straight. This is used for pool cues, axles, etc.

Central symmetry, or point symmetry: Central symmetry through the point *p* sends any point *a* to the point at the same distance from *p* but on the diametrically opposite side. In two dimensions central symmetry does not differ from half-turn symmetry in its end result.

♦ In 3-space, central symmetry produces a result different than any single rotation or reflection (though one can check that it does give the same result as the composition of three reflections through mutually perpendicular planes). To experience central symmetry in 3-space, hold your hands in front of you with the palms facing each other and your left thumb up and your right thumb down. Your two hands now have approximate central symmetry about a point midway between the center of the palms.

Similarity symmetry, or self-similarity: Any segment of a straight line (and its environs) is similar to (that is, can be magnified or shrunk to become the same as) any other segment.

♦ Logarithmic spirals like the chambered nautilus have self-similarity as do many fractals.

PROBLEM *1.2. Intrinsic Straight Lines on Cylinders*

a.

The important thing to remember here is to **think in terms of the surface, not in 3-space**. Always try to imagine how things would look from the bug's point of view. Discuss the distinction between *intrinsic* or *geodesic curvature* versus *extrinsic curvature*. Refer back to your answers to Problem **1.1** and argue that intrinsically the local symmetries have not changed. All symmetries (such as reflections and half-turns) must be carried out intrinsically, or from the bug's point of view. In general, there will not be extrinsic symmetries. For example, on a cylinder there is no extrinsic reflection symmetry except along or perpendicular to one of the generators of the cylinder.

b.

Rolling a piece of paper into a cylinder does not change the ***local*** intrinsic geometry and, thus, the notions of symmetry should still apply locally and intrinsically for a geodesic on the surface. Thus, a helix on a cylinder locally and intrinsically has the two types of reflection symmetry, half-turn symmetry, and rigid-motion-along-itself symmetry. Note that reflection symmetry does not hold globally (that is, as symmetries of the whole cylinder) and does not hold extrinsically (that is, an ordinary extrinsic mirror will not produce symmetry on a helix even locally). Thus, the straight lines (geodesics) on the cylinder are exactly those lines which are straight when the cylinder is unrolled. We have three cases:

Figure 1.B: Straight line path made by a vertical generator. Figure 1.C: Path made by the intersection of a horizontal plane with the cylinder, a *great circle* or a *generator circle*. Figure 1.D: A path which makes a spiral or helix of constant slope around the cylinder.
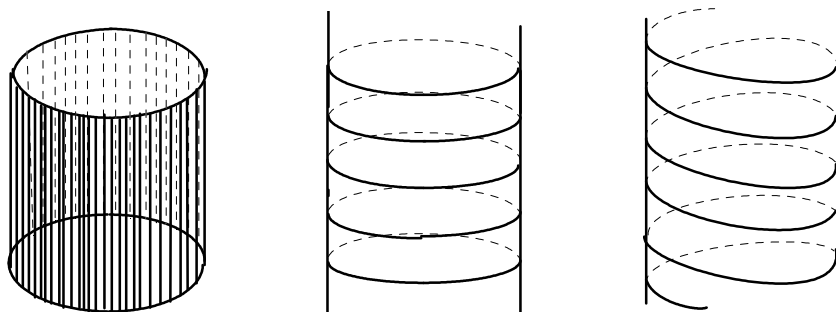
Figure 1.B. Vertical Generators.   Figure 1.C. Generator circles.   Figure 1.D. Helices.

Be sure to explain how you decided these lines were straight: What methods and what experiments did you use? Did you predict that these lines would be straight or were you surprised when you found them? It is important that you explain how you constructed these straight lines.

If you intersect a cylinder by a flat plane you get one or two vertical generators, or you get a generating circle, or you get an ellipse.

c.

We can see that there are an indefinite number of lines joining two points on a cylinder if the two points are not on the same great circle. Suppose we have two points *a* and *b,* and we unroll the cylinder into an *n-sheeted* cover. (See Figure 1.E.) There will be a lift (copy) of *a* and a lift of *b* on each sheet. Pick one of the lifts of *a* and join it to each of the lifts of *b* by lines which are straight on the covering. The images of these *n* lines will be *n* different geodesics on the cylinder.
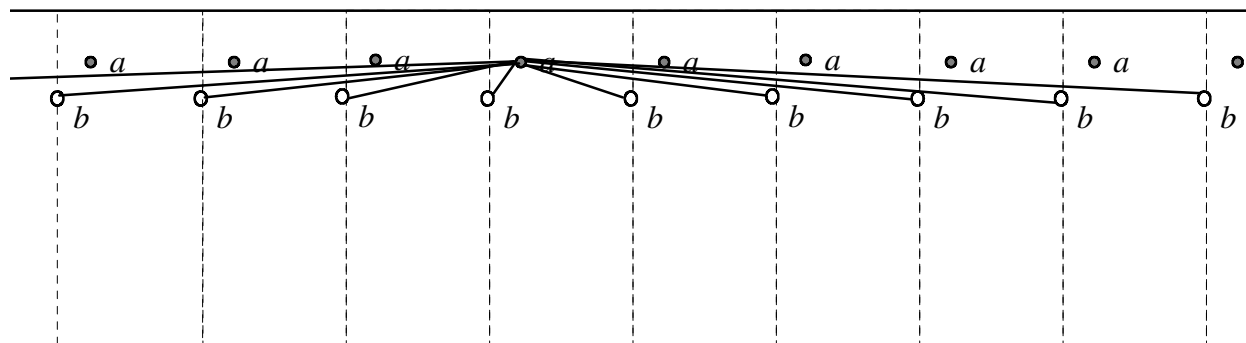
Figure 1.E. An *n*-sheeted cover.

Thus we can construct as many geodesics as we wish joining two points on the cylinder and the covering space gives us a means for constructing these geodesics.

No geodesic intersects itself except for the generating circles.

Any helix with more than half a turn is intrinsically straight but not the shortest path between its endpoints.

Yes. If the distance is shortest on the cylinder, then it will also be shortest on the covering space and, thus, straight.

PROBLEM *1.3. Geodesics on Cones*

a.

Now, let's look at the classes of straight lines on a cone.

Path made by a generator: As you consider the straight path made by a generator on a cone, you will have to make a decision about straightness at the cone point. You might decide that there is no way for the bug to find out how to go straight from the cone point, and, thus, the straight path ends there. Or, you might decide that the bug has some knowledge of its world and knows the cone angle. In this second case, when the bug is at the top, it can go straight down by bisecting the cone angle (see Figure 1.F). Symmetries may be very useful in deciding what it means to walk straight from the cone point. You may say that the line ends at the cone point because some of the straight line symmetries do not hold there, for example, translation symmetry. Or, you may choose to continue the line on the other side of the cone, dividing the cone into two symmetric parts. This path has most of the symmetries of the straight line at the cone point. (Which ones?)
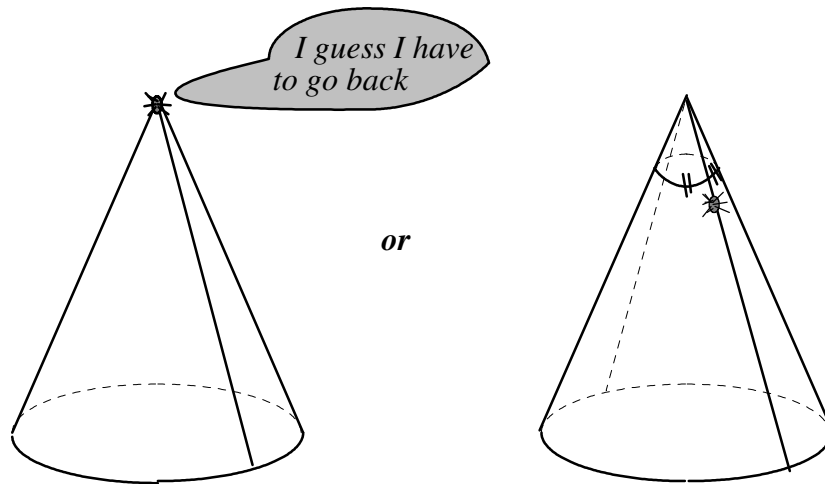

Figure 1.F. Straightness at the cone point

Path straight and around: You may have different perspectives on the kind of geodesic on a cone that goes straight around the cone. You may say that this kind of geodesic wraps around an infinite number of times. Or, you may say that a straight-around geodesic seems to wrap around once, intersect once, and then go straight down on both ends without intersecting again. Or, you may say that these geodesics never intersect even though they are not generators. You should work with cones of varying angle sizes. Soon enough, you will realize that your conjectures depend upon the size of the cone angle, and you can state your findings. The geodesics missing the cone point are straight lines when lifted to the branched covering space and each such geodesic has a closest point to the cone point and at this point the geodesic must be perpendicular to a generator.

b.

It is more complicated to compute the number of lines that connect two points on a cone. The number will depend upon the cone angle and the location of the points that one wants to connect. Given a ϕ angle cone, let us construct covering sheets for that cone until the sheets cover the plane (this is once around). Given two points on the cone, mark the lifts of those points on the sheets of the covering space considered. Now join one of the lifts of *A* (*A′* in Figure 1.G) to all the lifts of *B*. These lines are all different ways of connecting these two points on the cone.
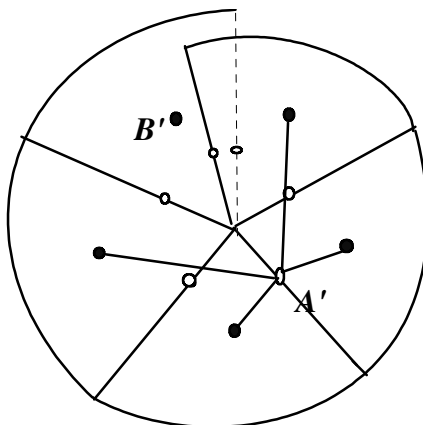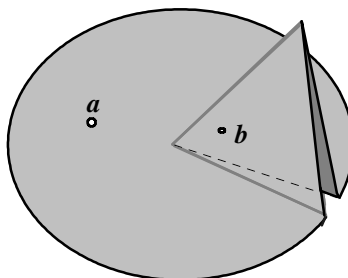
Figure 1.G. Covering of a cone

Contrary to what you might predict, the number of lines connecting two points on a cone with cone angle ϕ is finite; there are approximately 360°/ϕ lines, which is the number of sheets of the covering that produce 360° at the lift of the cone point. Note that in Figure 1.G, the lift of *B*, represented by *B′*, cannot be connected to *A′* by a straight line, thus the relative positioning of *A* and *B*, together with the cone angle, will influence how many geodesics join the two points. Adding more sheets to the covering will not result in more geodesics joining *A* to *B*. Continually make models as you make and confirm conjectures.

Let us see an example where there are no straight lines connecting two particular points. Consider a cone with angle 450°, and two points *a* and *b* such as those in Figure 1.H. In this case there is no geodesic that connects the two points.

Figure 1.H. No geodesic can connect *a* and *b*. Try it!

*c

The number of times that a geodesic can intersect itself depends on the cone angle. You should become comfortable using covering spaces, specifically here, to estimate the maximum number of self-intersections on a cone with cone angle ϕ.
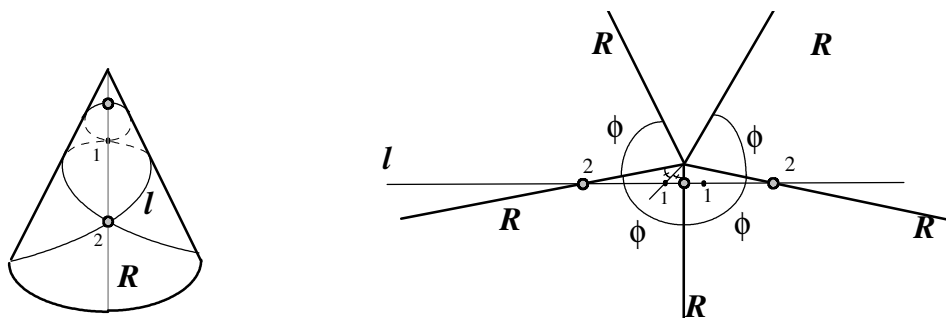

Figure 1.I. A geodesic intersecting itself.

Draw a straight line *l* on the cone. Consider the ray *R* such that the line *l* is perpendicular to it. The lift of the geodesic *l* (this is the corresponding geodesic on the cone) will cross the ray *R* exactly once for every time its lift crosses the seam of the covering. (See Figure 1.I.) Note that the seams between individual wedges are lifts of the ray *R*. The number of times that the geodesic can cross the ray depends upon the number of times the cone angle ϕ fits into 180°. The geodesic goes around the cone once for every time it intersects the seam. And every time it goes around the cone, it intersects itself. A geodesic will intersect itself every time it crosses the ray *R*, and will also intersect again in the back of the cone opposite to *R*. Thus, the number of intersections is roughly the number of times that the cone angle fits into 180°. Consequently, on a cone with an angle greater than 180°, geodesics never intersect themselves.
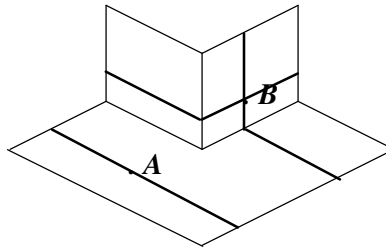
*d.

Figure 1.J. There are many geodesic paths through *B* which do not intersect the line through A.

PROBLEM *1.4. Geodesics in Local Coordinates*

*a.

$$\mathbf{x}(\theta,z) = \mathbf{y}(r\theta,z)$$
$$\mathbf{y}(w,z) = \mathbf{x}(w/r,z)$$
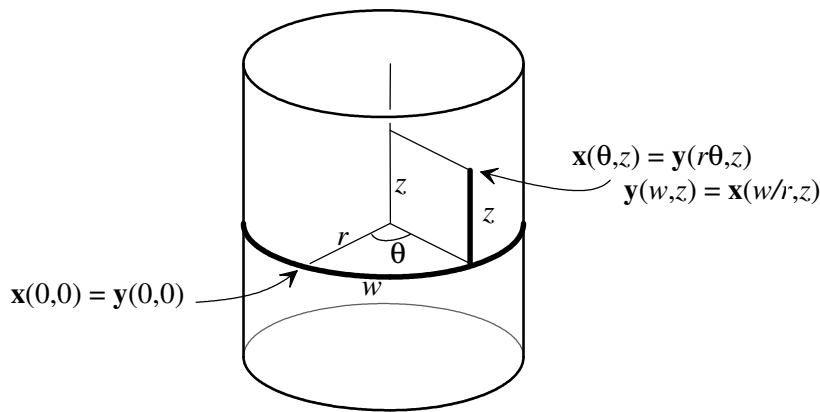
$$\mathbf{x}(0,0) = \mathbf{y}(0,0)$$

Figure 1.K. Coordinates on a cylinder.

b.

Let the base curve be the circle with center at the cone point and radius *s*. Then we can describe geodesic rectangular coordinates by:

$\mathbf{z}(x,y)$ = {the point on the cone attained by walking along the base curve a distance *x* and then walking away from the cone point a distance *y* along a generator. See Figure 1.L.

The *geodesic polar coordinates* on the cone can be described intrinsically by

$\mathbf{y}(\theta,r)$ = {the point **p** on the cone, where *r* is the length of the line segment from **p** to the cone point and θ is the angle along the surface between this segment and a fixed reference ray from the cone point}. (See Figure 1.L.)

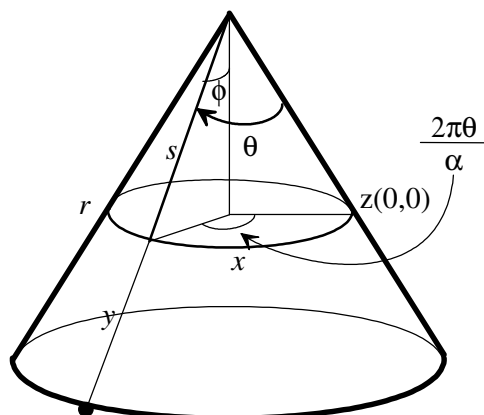**Figure 1.L. z(x,y) = y(θ,r), where x = sθ and y = r − s.**

In the same manner as polar coordinates on the plane we allow θ to be any angle but note that two angle coordinates denote the same point on the cone if they differ by an integral multiple of α. You should convince yourself that *these intrinsic polar coordinates work for any cone even those with cone angle larger than* 2π.

c.
Since we are dealing with intrinsic coordinates we may look at the covering space:
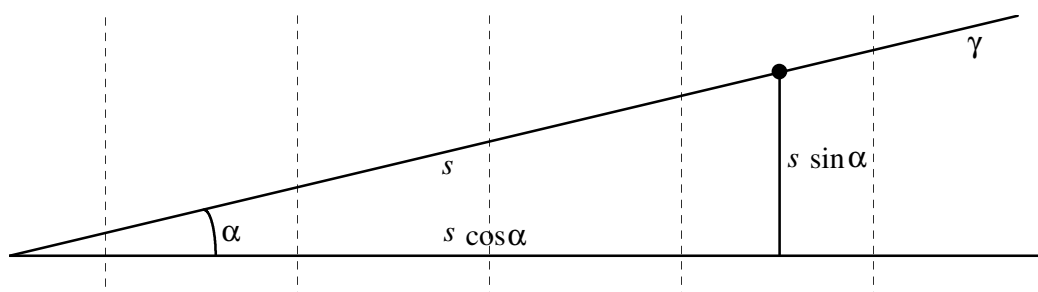

Figure 1.M. Finding parametric equation of geodesic on a cylinder.

An equation for γ in terms of extrinsic local coordinates:

$$\gamma(s) = x(\ r\cos(\tfrac{s\cos a}{r}),\ r\sin(\tfrac{s\cos a}{r}),\ s\sin a\ )$$

Given two points *A* and *B* on the cylinder, determine which geodesics join *A* to *B*. This is easiest done in intrinsic coordinates and using *n*-sheeted coverings (*C* is the circumference of the cylinder):
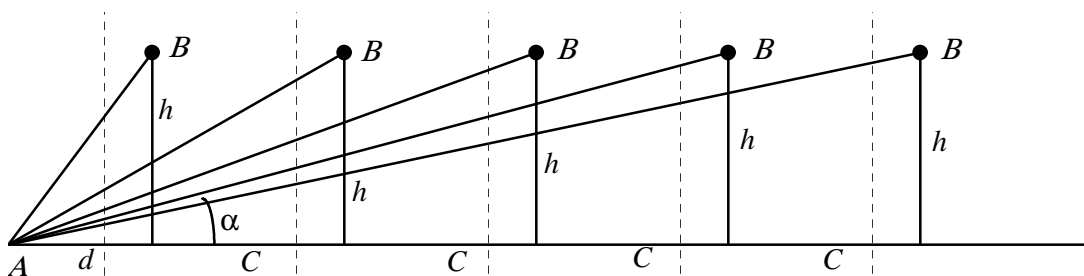

Figure 1.N. Different geodesics joining *A* to *B*.

The angles at which geodesics leave *A* and join to *B* are:

$$a = \arctan(\tfrac{h}{d+nC}), \text{ where } n = 0, \pm1, \pm2, \pm3, \ldots$$

d.

Look in the covering space at the picture in Figure 1.O.
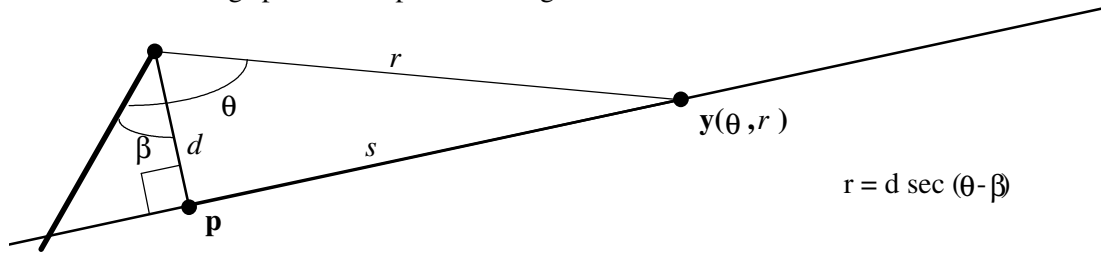

$$r = d \sec(\theta - \beta)$$

Figure 1.O. Geodesic on a cone in intrinsic coordinates.

*e.

$$x(\theta, r) = (r \sin\phi \cos\tfrac{2\pi\theta}{a}, r \sin\phi \sin\tfrac{2\pi\theta}{a}, r \cos\phi), \text{ where } r = \sqrt{d^2 + s^2}, \ \theta = \beta + \arctan\tfrac{s}{d}.$$

*f.

The geodesic will self intersect at the points with $\arctan\tfrac{s}{d} = n(\tfrac{a}{2})$, $n = \pm1, \pm2, \ldots$. Since the arctangent must be less than $\pi/2$, we must have $n < \pi/\alpha$. Thus, the number of self intersections is the greatest integer less than π/α. See also Problem **1.3.c**.

PROBLEM *1.5. What Is Straight on a Sphere?*

a.

It is easy to check that the great circles have intrinsically all the symmetries of a straight line in the plane (except for 3-d rotation symmetry and self-similarity which do not apply on the sphere). All the symmetries of the great circle are actually also extrinsic symmetries except for central symmetry. Central symmetry (about a point on the sphere) only makes sense on the sphere intrinsically.

b.

It is easy to see that non-great circles do not have half-turn symmetry nor reflection-in-the-line symmetry.

PROBLEM *1.6. Strakes, Augers, and Helicoids*

a.

Notice that an extrinsic 180° rotation about the segment will intrinsically be experienced as a reflection through the segment.

b.

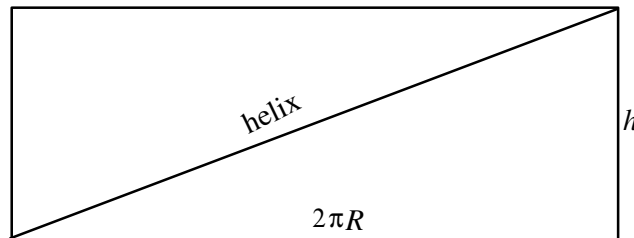This is clear from a 1-sheeted covering of the cylinder. (See Figure 1.P.)

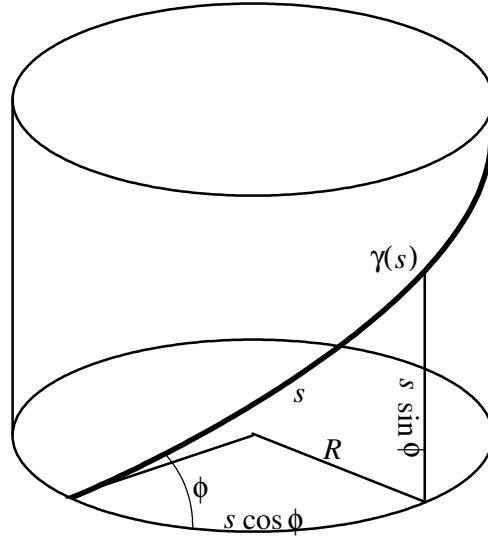**Figure 1.P. Lift of helix to 1-sheeted covering of the cylinder.**

c.

Figure 1.Q. A helix extrinsically.

$$\gamma(s) = \left(R\cos\frac{s\cos\phi}{R}, R\sin\frac{s\cos\phi}{R}, s\sin\phi\right)$$

But

$$\cos\phi = \frac{2\pi R}{\sqrt{h^2+(2\pi R)^2}} \text{ and } \sin\phi = \frac{h}{\sqrt{h^2+(2\pi R)^2}},$$

thus,

$$\gamma(s) = \left(R\cos\frac{2\pi s}{\sqrt{h^2+(2\pi R)^2}}, R\sin\frac{2\pi s}{\sqrt{h^2+(2\pi R)^2}}, \frac{sh}{\sqrt{h^2+(2\pi R)^2}}\right).$$

d.

$$x(s,t) = \left((R+t)\cos\frac{2\pi s}{\sqrt{h^2+(2\pi R)^2}}, (R+t)\sin\frac{2\pi s}{\sqrt{h^2+(2\pi R)^2}}, \frac{sh}{\sqrt{h^2+(2\pi R)^2}}\right)$$

*e.

Take the coordinates from part **d** and let R go to zero:

$$x(s,t) = \left((R+t)\cos\frac{2\pi s}{\sqrt{h^2+(2\pi R)^2}}, (R+t)\sin\frac{2\pi s}{\sqrt{h^2+(2\pi R)^2}}, \frac{sh}{\sqrt{h^2+(2\pi R)^2}}\right) \to \left(t\cos\frac{2\pi s}{h}, t\sin\frac{2\pi s}{h}, s\right).$$

PROBLEM 1.7. Surfaces of Revolution

a.

These curves have reflection-in-the-line (bilateral) symmetry. Also, each of these curves lies in a plane which is perpendicular to the surface and, thus, the curvature (turning) of the curve is perpendicular to the surface. Therefore, intrinsically there is no turning. We will return to this with more precision in Problem **3.4**.

b.

For any a that is a local minima or maxima of the function we can imagine laying a ribbon tangent to the surface along the curve $\gamma(\theta) = x(\theta, a)$ — on the outside of the surface if it is a maximum, on the inside if it is a minimum. For any a where the derivative of f is zero it is also possible to lay a ribbon tangent to the surface but you have to imagine the ribbon cutting through the surface. See Problem **3.4**. There is also infinitesimal symmetry.

c.

Let $x = c$ be the base curve. Then we can describe geodesic rectangular coordinates $\mathbf{z}(w,s)$, thus:

$$z(w, s) = \mathbf{x}\left(\tfrac{w}{f(c)}, x(s)\right), \text{ where } x(s) \text{ is defined by } s = \int_c^{x(s)} \sqrt{1 + (f'(x))^2} \ dx.$$

PROBLEM 1.8. Hyperbolic Plane

a.

For each δ the curves running radially across the annular strips have bilateral symmetry and so they will have symmetry in the limit.
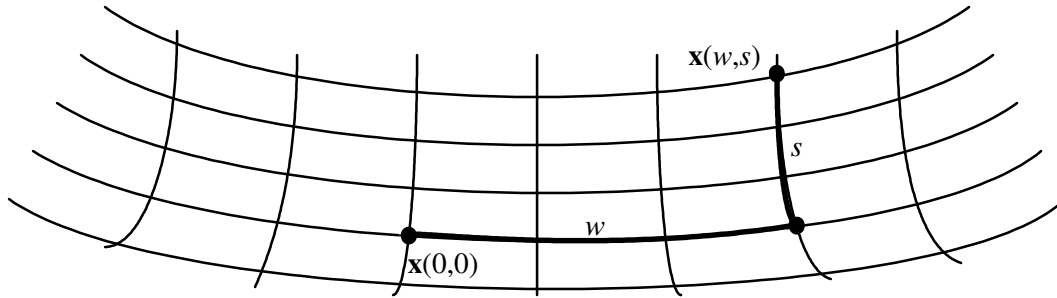
b. .

Figure 1.R. Geodesic rectangular coordinates on annular hyperbolic plane.
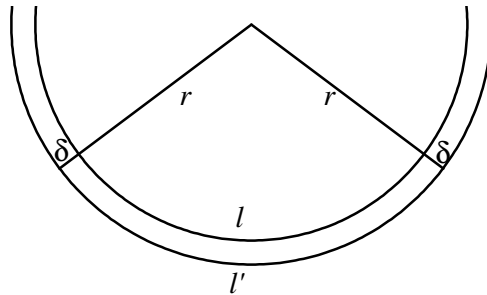
c.

Look at an annulus:

Figure 1.S. Changing arclength on annulus.

Measuring the angle in two different ways we obtain: $\dfrac{l}{r} = \dfrac{l'}{r+\delta}$. Note that the ratio $\dfrac{l}{l'} = \dfrac{r}{r+\delta}$ is independent of the length l and, thus, we can conclude that, as we cross each annular strip, the distance between the geodesics λ and μ is decreased in the ratio $\frac{r}{r+\delta}$. Since we are crossing n strips, the desired equation follows: $\lim_{\delta \to 0} d(\frac{r}{r+\delta})^{c/\delta} = \lim_{\delta \to 0} d(1 + \frac{\delta}{r})^{-c/\delta} = \lim_{t \to 0} d((1+t)^{1/t})^{-c/r} = d \exp(-c/r). \ (t = \delta/r)$

d.

So we now look at the point $\mathbf{z}(a,b) = \mathbf{x}(a, \ln(b))$. As we move along the base curve the change in distance is equal to the change in x and, thus, the speed is 1 on the base curve. By Part **c** as we move along the curve $x \to \mathbf{x}(x, \ln(b))$, then distance traveled is $\exp(-\ln(b)/1)$ times the change in x. Thus, the speed along the first variable curves is $\frac{\Delta x \exp(-\ln(b))}{\Delta x} = 1/b$. Along the second coordinate geodesics the speed is not constant and, thus, we look at $\lim_{\Delta y \to 0} \frac{\ln(b+\Delta y) - \ln(b)}{\Delta y} = 1/b$, since we just have the definition of the derivative of $\ln(y)$.

PROBLEM *1.9. Surface as Graph of a Function z = f (x,y)*

a.

One turn of the helicoid is the graph of (using rectangular coordinates in the *y,z* plane):

$$x = f(y, z) = y \cot \tfrac{2\pi z}{h\pi},$$

or except for the axis it is the graph of (using polar coordinates in the *x,y* plane)

$$z = g(r, \theta) = \tfrac{\theta}{2\pi} h.$$

Refer back to Problem **1.6.e**.

A cone with cone angle α less than 360° is the graph of (using polar coordinates in the *x,y* plane)

$$z = h(r, \theta) = r \cot \phi = r\sqrt{(\tfrac{2\pi}{a})^2 - 1}\,,$$

where φ is the extrinsic cone angle. See Figure 1.19 of the text.

b.

For a cylinder lying on *xy*-plane along the *y*-axis, we have that $x^2 + (R\text{-}z)^2 = R^2$ and, thus,

$$\boldsymbol{x}(x, y) = \left(x, y, R - \sqrt{R^2 - x^2}\,\right).$$

For a sphere (using polar coordinates in the *x,y* plane), we know that $r^2 + (R{-}z)^2 = R^2$, thus,

$$\boldsymbol{y}(r, \theta) = \left(r, \theta, R - \sqrt{R^2 - r^2}\,\right).$$

c.

Because $y^2 + z^2 = [f(x)]^2$, $z = g(x, y) = \sqrt{[f(x)]^2 - y^2}$.

If the derivative f ′(a) = 0, then we can find a Monge patch for a neighborhood of

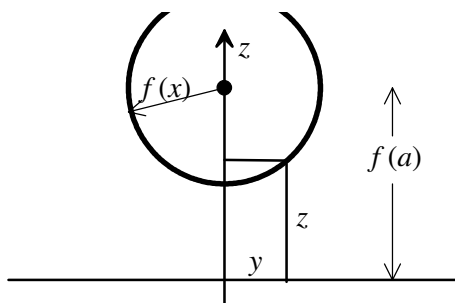$$(a, f(a) \cos \theta, f(a) \sin \theta).$$

Figure 1.T. Monge patch for surface of revolution.

Look at Figure 1.T. We see that $y^2 + [f(a) - z]^2 = [f(x)]^2$, thus, if we set $\hat{x} = x - a$, then

$$z = g(\hat{x}, y) = f(a) - \sqrt{[f(\hat{x} + a)]^2 - y^2}$$

is a Monge patch because $z = g(0, 0) = f(a) - \sqrt{[f(a)]^2 - 0^2} = 0$ and

$$g_1(0, 0) = \tfrac{d}{d\hat{x}} f(\hat{x} + a)|_{\hat{x}=0} = f'(a) = 0, \; g_2(0, 0) = \left[\tfrac{y}{\sqrt{[f(a)]^2 - y^2}}\right]_{y=0} = 0$$