Chapter 1

Surfaces and Straightness†

In the first six chapters of this book our study of differential geometry focuses on curves and surfaces. In later chapters we will see how to extend the results about surfaces to higher dimensional manifolds (the higher dimensional analogues of surfaces), especially to our physical universe which is a 3-dimensional (or 4-dimensional, if you include time) manifold.

In this chapter we will begin our study by examining a diverse collection of surfaces which will serve as examples throughout the remainder of the book. We will investigate each surface as much as we can without bringing in the differential notions of calculus. For each surface, starting with the plane, we will say what we can about what it means to be straight on the surface.

We begin with a question that encourages you to explore deeply a concept that is fundamental to all that will follow: We ask you to build a notion of straightness for yourself rather than accept a certain number of assumptions about straightness. Although difficult to formalize, straightness is a natural human concept.

PROBLEM 1.1. When Do You Call a Line Straight?

Look to your experiences. It might help to think about how you would explain straightness to a 5-year-old (or how the 5-year-old might explain it to you!). If you use a "ruler," how do you know if the ruler is straight? How can you check it? What properties do straight lines have that distinguish them from non-straight lines?

Think about the question in four related ways:

a. *How can you check in a practical way if something is straight—without assuming that you have a ruler, for then we will ask, "How can you check that the ruler is straight?"*

b. *How do you construct something straight—lay out fence posts in a straight line, or draw a straight line?*

c. *What symmetries does a straight line have? A symmetry of a geometric figure is a transformation (such as reflection, rotation, translation, or composition of them) which preserves the figure. For example, the letter* "T" *has reflection symmetry about a vertical line through its middle, and the letter* "Z" *has rotation symmetry if you rotate it half a revolution about its center.*

d. *Can you write a definition of "straight line"?*

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† A small portion of this chapter is taken (somewhat revised) from the author's *Experiencing Geometry on Plane and Sphere* [**Tx**: Henderson]. It is used here with the permission of the publisher, Prentice-Hall, Inc.

Suggestions

Look at your experience. At first, you will look for examples of physical world (or natural) straightness. Go out and actually try walking along a straight line and then along a curved path; try drawing a straight line and checking that a line already drawn is straight.

Look for things that you call "straight." Where do you see straight lines? Why do you say they are straight? Look for both physical lines and non-physical uses of the word "straight." You are likely to bring up many ideas of straightness. It is necessary then to think about what is common among all of these "straight" phenomena.

As you look for properties of straight lines that distinguish them from non-straight lines, you will probably remember the following statement (which is often taken as a definition in high school geometry): *A line is the shortest distance between two points.* But can you ever measure the lengths of all the paths between two points? How do you find the shortest path? If the shortest path between two points is in fact a straight line, then is the converse true? Is a straight line between two points always the shortest path? We will return to these questions later in this chapter.

A powerful approach to this problem is to think about lines in terms of symmetry. Two symmetries of lines in the plane are:

- Reflection symmetry in the line, also called bilateral symmetry—reflecting (or mirroring) an object over the line (Figure 1.1).

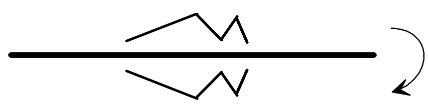


Figure 1.1. Bilateral symmetry.

- Half-turn symmetry—rotating 180° about any point on the line (Figure 1.2).

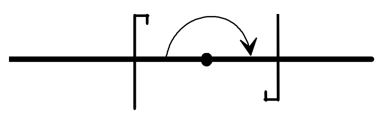


Figure 1.2. Half-turn symmetry.

Although we are focusing on a symmetry of the line in each of these examples, notice that the symmetry is not a property of the line by itself but includes the line and the space around the line. The symmetries preserve the local environment of the line. Notice how in reflection and half-turn symmetry the line and its local environment are both part of the symmetry action. The relationship between them is integral to the action. In fact, reflection in the line does not move the line at all but exhibits a way in which the spaces on the two sides of the line are the same.

Try to think of other symmetries as well (there are quite a few). Some symmetries hold only for straight lines, while some work with other curves too. Try to determine which ones are specific to straight lines and why. Also think of practical applications of these symmetries for constructing a straight line or for determining if a line is straight.

How Do You Construct a Straight Line?

As for how to construct a straight line, one method is simply to fold a piece of paper; the edges of the paper needn't even be straight. This utilizes symmetry (can you see which one?) to produce the straight line. Carpenters also use symmetry to determine straightness—they put two boards face to face and plane the edges until they look and feel straight. They then turn one board over so the planed edges are touching, then hold the boards up to the light. If the edges are not straight, there will be gaps between the boards through which light will shine. (See Figure 1.3.)

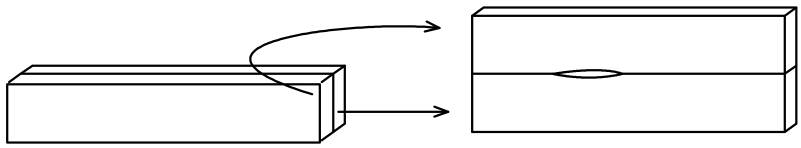


Figure 1.3. Carpenter's method for checking straightness.

To grind an extremely accurate flat mirror, the following technique is sometimes used: Take three approximately flat pieces of glass and put pumice between the first and second pieces and grind them together. Then do the same for the second and the third pieces and then for the third and first pieces. Repeat many times and all three pieces of glass will become very accurately flat. (See Figure 1.4.) Do you see why this process works? What does this have to do with straightness?

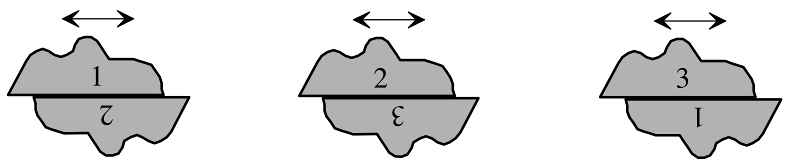
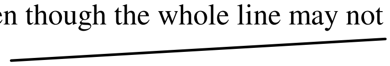


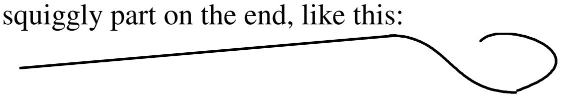
Figure 1.4. Grinding flat mirrors.

Imagine walking (or actually do it!) while pulling a long silk thread with a small stone attached. When will the stone follow along your path? Why? To illustrate this phenomenon, consider how a fallen water skier can be rescued. The boat passes by the skier at a safe distance in a straight path, and the tow rope follows the path of the boat. The boat then turns in an arc in front of the skier. Since the boat is no longer following a straight path, the tow rope will move in toward the fallen skier. In these two examples there is a stretched thread or rope which follow the shortest distance, thus these illustrate the property that straight lines are locally the shortest distance. See, also, the section Is "Shortest" Always "Straight"? later in this chapter.

Another idea to keep in mind is that straightness must be thought of as a local property. Part of a line can be straight even though the whole line may not be. For example, if we agree that this line is straight,



and then we add a squiggly part on the end, like this:



Would we now say that the original part of the line is not straight, even though it has not changed, only been added to? Also note that we are not making any distinction here between "line" and "line segment." The more generic term "line" generally works well for referring to any and all lines and line segments, both straight and non-straight.

Can we use any of the symmetries of a line to define straightness? What symmetries does a straight line have? How do they fit with the examples that you have found and those mentioned above?

Returning to one of the original questions, how would we construct a straight line? One way would be to use a "straight edge" — something that we accept as straight. Notice that this is different from the way that we would draw a circle. When using a compass to draw a circle, we are not starting with a figure that we accept as circular; instead, we are using a fundamental property of circles that the points on a circle are a fixed distance from the center. Can we use the symmetry properties of a straight line to construct a straight line? Is there a tool (serving the role of a compass) which will draw a straight line?

For a historical discussion of straight lines see [**Tx**: Henderson/Taimina, Chapter 1]. For another interesting discussion of this question see *How to Draw a Straight Line: A Lecture on Linkages* (1877) [**Z**: Kempe, p. 12] which shows diagram of the apparatus for drawing a straight line that is pictured in Figure 1.5. See also [**SE**: Hilbert, pp. 272-3] and [**Tx**: Henderson/Taimina, Chapter 16] for other discussions of this linkage. The discovery of this linkage about 1870 is variously attributed to the French army officer, Charles Nicolas Peaucellier, and to Lippman Lipkin, who lived in Lithuania. (See [**Z**: Kempe] and Hilbert and Phillip Davis' delightful little book *The Thread* [**Z**: Davis], Chapter IV.)

Think about and formulate some answers to these questions before you read any further. You are the one laying down the definitions. Do not take anything for granted unless you see why it is true.

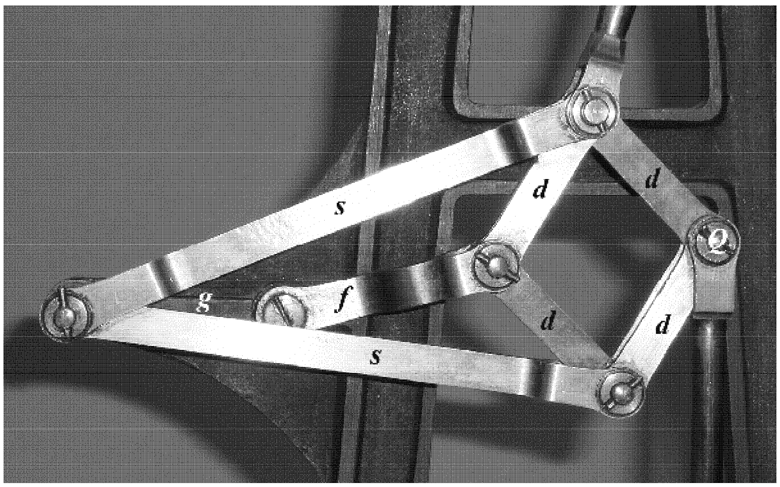


Figure 1.5. If $g = f$ then the point Q always traces a straight line. See [Tx: Henderson/Taimina, Problem 16.3].

Local (and Infinitesimal) Straightness

Previously, you saw how a straight line has reflection-in-the-line symmetry and half-turn symmetry: One side of the line is the same as the other. But, as pointed out above, straightness is a local property in that whether a segment of a line is straight depends only on what is near the segment and does not depend on anything happening away from the line. Thus each of the symmetries must be able to be thought of (and experienced) as applying only locally. This will become particularly important later when we investigate straightness on the cone and cylinder. For now, it can be experienced in the following way:

> *When a piece of paper is folded not in the center (like in Figure 1.6), the crease is still straight even though the two sides (A and B) of the crease on the paper are not the same. So what is the role of the sides when we are checking for straightness using reflection symmetry? Think about what is important near the crease in order to have reflection symmetry.*

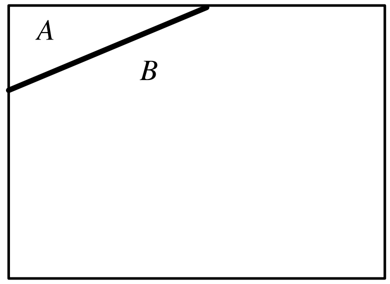


Figure 1.6. Reflection symmetry is local.

When we talk about straightness as a local property, you may bring out some notions of scale. For example, if one sees only a small portion of a very large circle, it will be indistinguishable from a straight line. This can be experienced easily on many of the modern graphing programs for computers. Also, a microscope with a zoom lens will provide an experience of zooming. For some curves, if one "zooms in" on any point of the curve, eventually the curve will be indistinguishable from a straight line segment. (See Figure 1.7.)

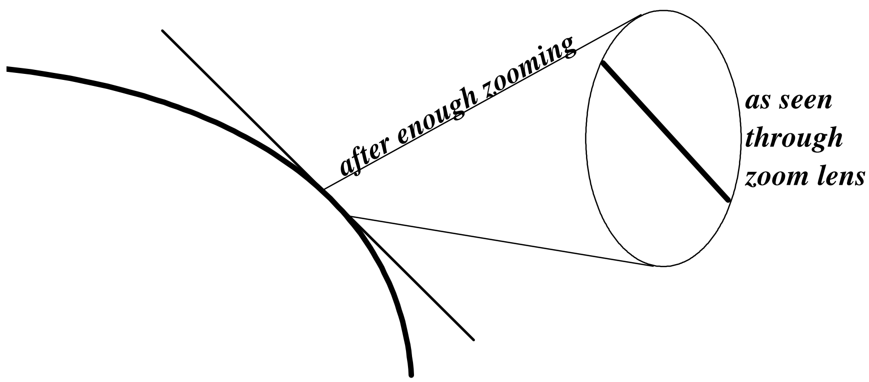


Figure 1.7. Infinitesimally straight.

We call such a curve *infinitesimally straight*. In Chapter 2 we show that this is equivalent to the more standard term, *differentiable*. We also show that a curve is *smooth* (or *continuously differentiable*) if and only if it is *uniformly infinitesimally straight* in the sense made clear in Chapter 2. When the curve is parametrized by arc length this is equivalent to the curve having a continuously defined velocity vector at each point.

In contrast, we can say that a curve is *locally straight at a point* if that point has a neighborhood that is straight. In the physical world the usual use of both *smooth* and *locally straight* are dependent on the scale at which they are viewed. For example, we may look at an arch made out of wood that at a distance appears as a smooth curve (Figure 1.8a); then as we move in closer we see that the curve is made by many short straight pieces of finished (planed) boards (Figure 1.8b), but when we are close enough to touch it, we see that its surface is made up of smooth waves or ripples (Figure 1.8c), and under a microscope we see the non-smoothness of numerous twisting fibers (Figure 1.8d).

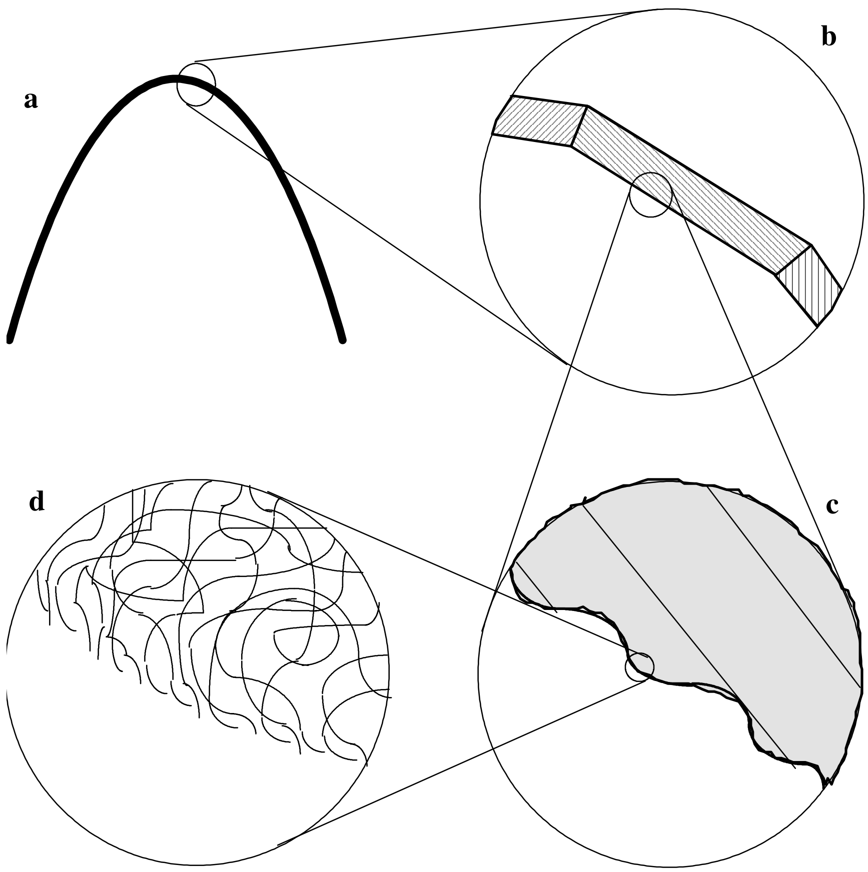


Figure 1.8. Straightness and smoothness depend on the scale.

PROBLEM 1.2. Intrinsic Straight Lines on Cylinders

Take a piece of paper and draw a straight line on the paper. Now bend the paper (without stretching it) so that the line is no longer straight in 3-space. We express this situation by saying that the line is ***extrinsically not straight***; but is the line straight intrinsically on the sheet of paper? That is, if you consider the paper to be the universe then is the line straight in that universe? Or, consider a 2-dimensional bug which crawls on the surface of the paper such that the bug has no awareness of any space off the surface of the paper and is not influenced by gravity. Will the bug experience the line on the paper as straight?

a. *Argue that distances (as measured along the surface of the bent paper) and angles have not changed and thus that the bent paper will intrinsically have the same geometric properties as a flat piece of paper. Argue that the bug would experience the line on the paper as straight. Argue that the bug would experience the line as having the same (local and intrinsic) symmetries as straight lines on the plane.*

The important thing to remember here is to **think in terms of the surface, not in 3-space**. Always try to imagine how things would look from the bug's point of view. A good example of how this type of thinking works is to look at an insect called a water strider. The water strider walks on the surface of a pond and has a very 2-dimensional perception of the world around it — to the water strider, there is no up or down; its whole world consists of the plane of the water. The water strider is very sensitive to motion and vibration on the water's surface, but it can be approached from above or below without its knowledge. If you find a pond with water striders you can actually, by moving slowly (so as not to disturb the surface of the water with air currents), touch the water strider with your finger. Hungry birds and fish can also take advantage of this 2-dimensional perception. This is the type of thinking needed to adequately visualize the intrinsic geometric properties of any surface.

This leads us to consider the concept of ***intrinsic***, or ***geodesic***, ***curvature*** versus ***extrinsic curvature***. An outside observer in 3-space looking at the bent paper will see the line drawn on the paper as curved — that is, the line exhibits *extrinsic* curvature. But relative to the surface (*intrinsically*), the line has no *intrinsic curvature* and thus is straight. **Be sure to understand this difference**. Lines which are intrinsically straight on a surface are often called ***geodesics***.

All symmetries (such as reflections and half-turns) must be carried out intrinsically, or from the bug's point of view. There will in general not be extrinsic symmetries. For example, on a cylinder there is no extrinsic reflection symmetry except along or perpendicular to one of the generators of the cylinder.

It is natural for you at first to have some difficulty experiencing straightness on surfaces other than the plane, and that consequently you will start to look by looking at the curves on surfaces as 3-dimensional objects. Imagining that you are a 2-dimensional bug walking on the surface emphasizes the importance of experiencing straightness and will help you to shed your limiting extrinsic 3-dimensional vision of the curves on a bent surface. Ask yourself: What does the bug have to do, when walking on a cylindrical surface, in order to walk in a straight line? How can the bug check if it is going straight?

b. *What lines are straight with respect to the surface of a cylinder? Why? Why not? Have you listed all of them? How do you know? If you intersect a cylinder by a flat plane and unroll it, what kind of curve do you get? Is it ever straight?*

Rolling a piece of paper into a cylinder does not change the ***local*** intrinsic geometry, and thus the notions of symmetry should still apply locally and intrinsically for a geodesic on the surface. Thus a helix on a cylinder locally and intrinsically has the two types of reflection symmetry, half-turn symmetry, and rigid-motion-along-itself symmetry. Note that reflection symmetry does not hold globally (that is, as symmetries of the whole cylinder) and does not hold extrinsically (that is, an ordinary extrinsic mirror will not produce symmetry on a helix even locally).

Make paper models, but consider the cylinder as continuing indefinitely with no top or bottom. Again, imagine yourself as a bug whose whole universe is the surface of the cylinder. As the bug crawls around on the surface, what will the bug experience as straight?

Lay a stiff ribbon or straight strip of paper on a cylinder. Convince yourself that it will follow a straight line with respect to the surface. Also, convince yourself that straight lines on the cylinder, when looked at locally and intrinsically, have the same symmetries as on the plane.

Rolling a piece of paper into a cylinder does not change the ***local*** intrinsic geometry but it does change the ***global*** intrinsic geometry. For example, on the cylinder there is a ***closed geodesic*** which returns to its starting point (can you find one?) and this is impossible on the plane. Also note that there is more than one geodesic joining every pair of points on a cylinder.

c. *How many geodesics join two points on a cylinder? How can you find these geodesics? On a cylinder, can a geodesic ever intersect itself? Is an intrinsic straight line on the cylinder always the shortest distance? Is the shortest distance always straight? Why?*

As you begin to explore these questions, it is likely that many other related geometric ideas will arise. Do not let seemingly irrelevant excess geometric baggage worry you. Often, you will find yourself getting lost in a tangential idea, and that is understandable. Ultimately, however, the exploration of related ideas will give you a richer understanding of the scope and depth of the problem. There are several important things to keep in mind while you are working on this problem. First, **you must make models**. You may find it helpful to make models using transparencies. Second, you must think about lines on the cylinder in an intrinsic way — always look at things from a bug's point of view. We are not interested in what is happening in 3-space; only what you would see and experience if you were restricted to the surface of a cylinder. Third, remember that if you cut the cylinder and lay it flat on the plane, then paths that were geodesics on the cylinder will become straight lines on the plane.

Here are some activities that you can try, or visualize, to help you experience what are the geodesics on surfaces. However, it is better for you to come up with your own experiences.

- Stretch something elastic on the surface. It will stay in place along a geodesic, but it will not stay on a curved path if the surface is slippery. Here, the elastic follows a path that is approximately the shortest since a stretched elastic always moves so that it will be shorter. Using the shortest distance criterion directly is not a good way to check for straightness because one cannot possibly measure all paths. But, it serves a good purpose here.

- Roll a cylinder (or other "rollable" surface) on a straight chalk line. The chalk will mark the line of contact on the cylinder and it will be a geodesic.

- Take a stiff ribbon or strip of paper that does not stretch, and lay it "flat" on the surface. It will only lie properly along a geodesic. Do you see how this property is related to local symmetry? This is sometimes called the *Ribbon Test*. (See Problem **3.4**.)

- The feeling of turning and "non-turning" comes up. Why is it that on a geodesic path there is no turning and on a non-geodesic path there is turning? Physically, in order to avoid turning, the bug has to move its left feet the same distance as its right feet. On a non-geodesic path the bug has to walk more slowly with the legs that are on the side to which the path is turning. You can test this same idea by taking a small toy car with its wheels fixed so that, on a plane, it rolls along a straight line. Then on the surface the car will roll along a geodesic but it will not roll along other curves.

PROBLEM 1.3. Geodesics on Cones

We now investigate geodesics on a cone which behave in some ways like the cylinder and in some ways differently.

a. *What lines are geodesics on (straight with respect to) the surface of a cone? Why? Have you listed all of them? How do you know?*

b. *How many geodesics join two points on a cone? Is there always at least one?*

***c.**[†] *On a cone, can a geodesic ever intersect itself? How many times?*

If you attempt to visualize lines on a cone without looking at a paper model, you are bound to make claims that you would easily see are mistaken if you investigated them on an actual cone. **You must make models of cones**. And you must look at cones of different shapes, i.e., cones with varying cone angles (see the next page). Try the activities mentioned in the paragraph preceding Problem **1.3**.

Lay a stiff ribbon or straight strip of paper on a cone. Convince yourself that it will follow a straight line with respect to the surface. Also, convince yourself that straight lines on the cone, when looked at locally and intrinsically, have the same symmetries as on the plane. Finally, also consider line symmetries on the cone. Check to see if the symmetries you found on the plane will work on cones, and remember to think intrinsically and locally. A special class of geodesics on a cone are the ***generators***: the straight lines that go through the cone point. These lines have some extrinsic symmetries (can you see which ones?), but in general, geodesics have only local, intrinsic symmetries. For example, can any geodesic that is not a generator have global extrinsic reflection symmetries? Why?

Walking along a generator: When looking at straight paths on a cone, you will be forced to consider straightness at the cone point. See Figure 1.9. You might decide that there is no way the bug can go straight once it reaches the cone point, and thus a straight path leading up to the cone point ends there. Or

[†] Problems or Sections preceded by an asterisk (*) are not essential for later in this book.

you might decide that the bug can find a continuing path that has most of the symmetries of a straight line on the plane. Do you see which path this is?

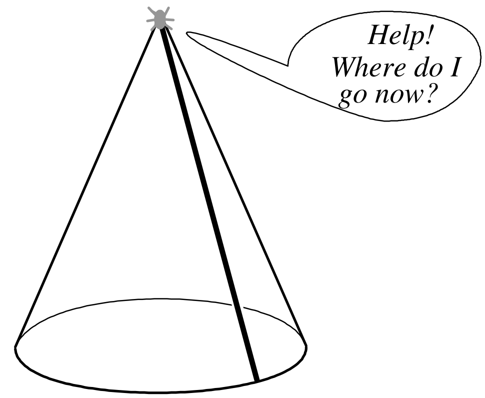


Figure 1.9. What is straight through the cone point?

Geodesics behave differently on differently shaped cones. So an important variable is the cone angle. The ***cone angle*** is generally defined as the angle measured around the point of the cone on the surface. Notice that this is an intrinsic description of angle. The bug could measure a cone angle by first making a model of a one-degree angle and then, determining how many of the one-degree angles it would take to go around the cone point. If we use radian measure, then the cone angle is c/r, where c is the circumference of the circle (on the cone) which is at a distance r from the cone point. We can determine the cone angle extrinsically in the following way: If we cut the cone along a generator and flatten it, then the cone angle is the angle of the planar sector. For example, if we take a piece of paper and bend it so that half of one side meets up with the other half of the same side as in Figure1.10, we will have a 180°-cone:

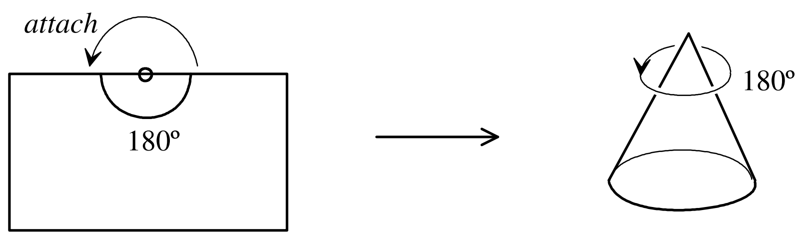


Figure 1.10. 180° cone.

A 90°-cone is also easy to make—just use the corner of a sheet of paper and bring one side around to meet with the adjacent side. Also be sure to look at larger cones. One convenient way to do this is to make a cone with a variable cone angle. Take a sheet of paper and cut (or tear) a slit from one edge to the center. (See Figure 1.11.) A rectangular sheet will work but a circular sheet is easier to picture. Note that it is not necessary that the slit be straight!

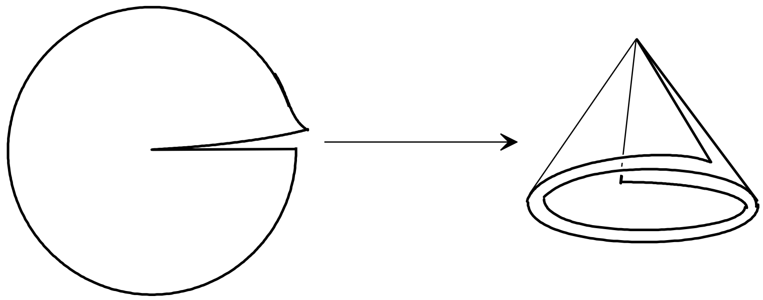


Figure 1.11. A cone with variable cone angle (0 - 360°).

You have already looked at a 360°-cone in some detail—it is just a plane. The cone angle can also be larger than 360°. A common larger cone is the 450°-cone. You probably have a cone like this somewhere on the walls, floor, and ceiling of your room. You can easily make one by cutting a slit in a piece of paper and inserting a 90° slice (360° + 90° = 450°) as pictured in Figure 1.12.

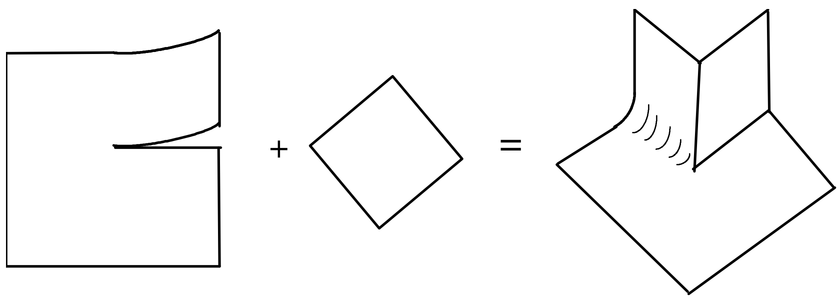


Figure 1.12. How to make a 450°-cone.

You may have trouble believing that this is a cone, but remember that just because it cannot hold ice cream, that does not mean it is not a cone. If the folds and creases bother you, they can be taken out—the cone will look ruffled instead. It is important to realize that when you change the shape of the cone like this (i.e., by ruffling), you are only changing its extrinsic appearance. Intrinsically (from the bug's point of view) there is no difference.

You can also make a cone with variable angle of more than 360° by taking two sheets of paper and slitting them together to their centers as in Figure 1.13. Then tape the left side of the top slit to the right side of the bottom slit as pictured.

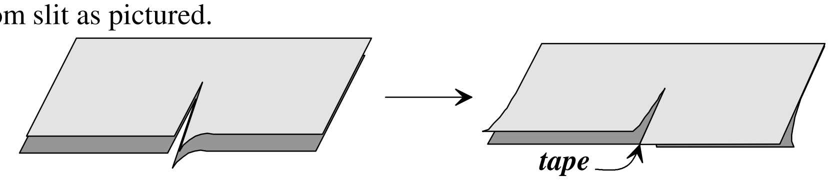


Figure 1.13. Variable cone angle larger than 360°.

It may be helpful for you to discuss some definitions of a cone. The following is one definition: *Take any simple (non-intersecting) closed curve **a** on a sphere and consider a point **P** at the center of the sphere. A **cone** is the union of the rays that start at **P** and go through each point on **a**.* (See Figure 1.14.) The cone angle is then equal to

(length of ***a***)/(radius of sphere),

in radians. Do you see why? Experiment by making out of paper examples of cones like those shown above.

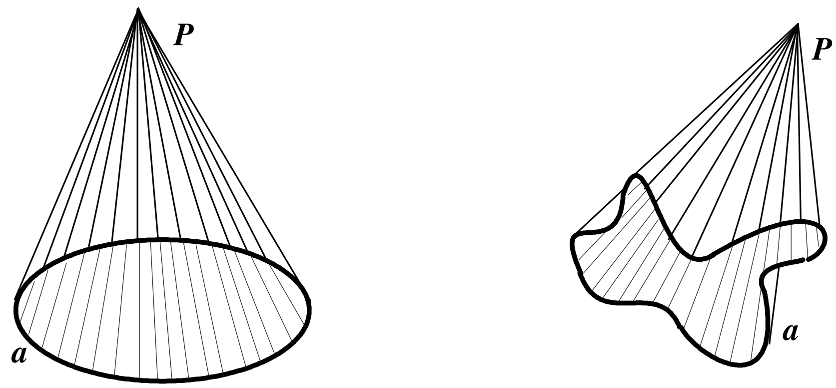


Figure 1.14. General cones.

***d.** *On a 450° cone find a point P (not the cone point) and a geodesic l (not through the cone point) such that there are many geodesics through P that do not intersect l. Compare this situation to the usual parallel postulate for the plane.*

In standard treatments of non-Euclidean geometries, hyperbolic geometry is presented as a geometry in which there are more than one line through a given point parallel to a given line. Euclid's fifth postulate for the plane implies that on the plane there is exactly one line through a given point parallel to a given line. For a discussion about parallel postulates, see Chapter 10 of [**Tx**: Henderson/Taimina].

Is "Shortest" Always "Straight"?

We are often told that "a straight line is the shortest distance between two points," but is this really true? As we have already seen on a cylinder, two points are, in general, connected by at least two straight paths. Only one of these paths is the shortest. The other is also straight, but not the shortest straight path.

Consider a model of a cone with angle 450°. Notice that such cones appear commonly in buildings as so-called "outside corners" (see Figure 1.15). It is best, however, for you to have a paper model that can be flattened. Use your model to investigate which points on the cone can be joined by straight lines. In particular, look at points like those labeled *A* and *B* in Figure 1.15. There is no single straight line on the cone going from *A* to *B*, and thus for these points the shortest path is not straight. Convince yourself that in this case this shortest path is not straight. (This is part of 1.3.b.)

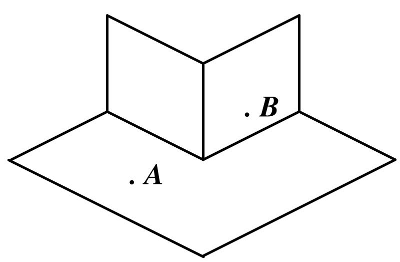


Figure 1.15. There is no geodesic path from A to B.

Here is another example: Think of a bug crawling on a plane with a tall box sitting on that plane (refer to Figure 1.16). This combination surface — the plane with the box sticking out of it — has eight cone points. The four at the top of the box have 270° cone angles, and the four at the bottom of the box have 450° cone angles (180° on the box and 270° on the plane). What is the shortest path joining points *X* and *Y*, which are on opposite sides of the box? Is the straight path the shortest? Is the shortest path straight? To check that the shortest path is not straight, see that at the bottom corners of the box, the two sides of the path have different angular measures.

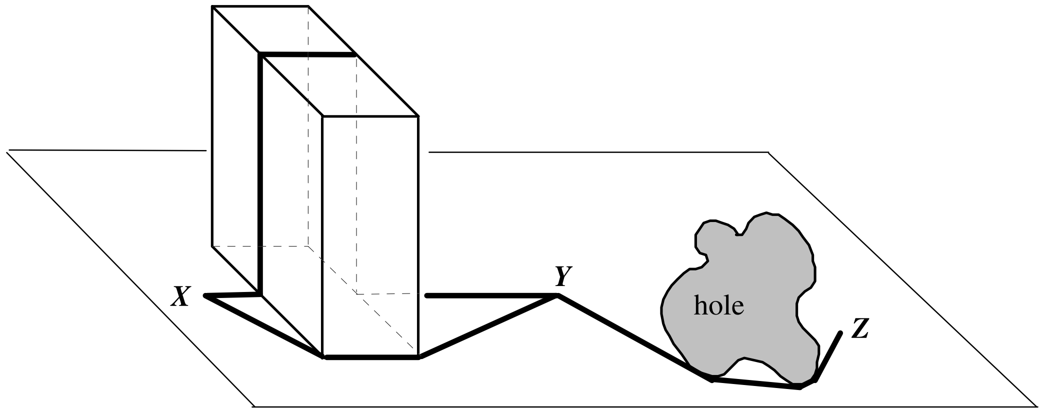


Figure 1.16. Shortest paths that are not straight.

Also consider a planar surface with a hole removed as in Figure 1.16. Check that for points *Y* and *Z*, the shortest path (on the surface) is not straight because the shortest path must go around the hole.

So, we see that sometimes a straight path is not shortest and the shortest path is not straight. However, for surfaces that are "smooth" enough, there is a close relationship between "straight" and "shortest." A *smooth* surface is essentially what it sounds like — a surface is smooth at a point if, when you zoom in on the point, the surface becomes indistinguishable from a flat plane. Note that a cone (with cone angle not equal to 360°) is not smooth at the cone point but is smooth at all other points; also a sphere and a cylinder are both smooth at every point. The surface of a piece of paper with a crease in it is not extrinsically smooth but it is *intrinsically smooth* in the sense that locally and intrinsically its geometry is the same as the plane. The following is a theorem which we will prove in a later chapter:

> **THEOREM:** *If a surface is smooth (in the* C^2 *sense), then a geodesic on the surface is always the shortest path between "nearby" points. If the surface is also geodesically complete (that is, every geodesic on it can be extended indefinitely, for example, there are no holes), then any two points can be joined by a geodesic which is the shortest path between them.* (See Problem **7.4**.)

A surface is C^2 if it can be described by local coordinates (see the next section) whose first and second derivatives exist and are continuous. This is a stronger condition than merely being extrinsically smooth (see Problems **3.1** and **6.1.c**). However, I do not know if there are any surfaces that are intrinsically smooth for which there are is no C^2 embedding into Euclidean space or for which the theorem above is false.

We encourage the reader to discuss how each of the previous examples is in harmony with this theorem. Note that the statement "every geodesic on it can be extended indefinitely" is a reasonable interpretation of Euclid's first postulate, which says "every line can be extended indefinitely." We will begin a detailed discussion of smooth surfaces in Chapter 3.

Locally Isometric Surfaces

We can describe this situation more generally by defining: Two geometric spaces, G and H, are said to be ***locally isometric*** at points *G* in G and *H* in H if the local intrinsic experience at *G* is the same as the experience at *H*. That is, there are neighborhoods of *G* and *H* that are identical in terms of intrinsic geometric properties such as measurement of lengths and angles in the neighborhoods. A cylinder and the plane are locally isometric (at every point), and the plane and a cone are locally isometric except at the cone point. Two cones are locally isometric at their cone points only if the cone angles are the same.

Euclid defines a right angle as follows: "When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**" [**AT**: Euclid's *Elements*]. Note that if you use this definition, then right angles at a cone point are not equal to right angles at points that are locally isometric to the plane. Euclid goes on to state as his fourth postulate: "All right angles are equal to one another." Thus, Euclid's postulate rules out cone points.

A surface that is ***locally isometric*** to the plane is traditionally called ***developable***. The notion of developable is important, for example, in the manufacture of the steel hull of ships. Those portions of the hull that are developable surfaces can be made by bending a sheet of steel; but those portions of hull that are not developable must be covered with more expensive "furnace plates," which are steel sheets that have been softened in a furnace and then molded into the desired shape.

Local Coordinates for Cylinders and Cones†

If we have a surface M in 3-space then ***local coordinates*** (or a ***local coordinate patch***) for M is a continuous function of two real variables, defined in a region R in the plane, which maps R one-to-one onto some region of M by describing the point in the region, which has coordinates, a, b. See Figure 1.17. We will use lower case boldfaced letters to denote the function that defines the local coordinates. We call these ***extrinsic local coordinates*** if the location of the point is described extrinsically (usually this means in terms of its rectangular coordinates in 3-space or $\mathbf{R}^n$) and we call them ***intrinsic local coordinates*** if the location of the point is described intrinsically by referring only to intrinsic geometric properties of the surface.

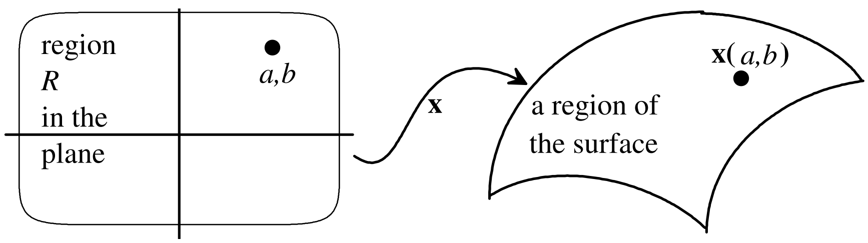


Figure 1.17. Local Coordinates.

For example, if we have a cylinder in 3-space, then to define extrinsic local coordinates, we can:

1. Choose an origin for 3-space on the axis of the cylinder;
2. Choose two straight rays from the origin, both rays perpendicular to each other and to the axis, which we call the x-axis and the y-axis;
3. Choose one of the two directions along the axis of the cylinder as the positive z-axis.

Then one possible extrinsic local coordinates are:

$$\mathbf{x}(\theta,z) = (r\cos\theta,\, r\sin\theta,\, z),$$

where r is the radius of the cylinder.

This definition would not be appropriate for a 2-dimensional bug on the surface because the bug has no awareness of 3-space. Instead the bug would like to define the local coordinates intrinsically. To do this the bug could:

1. Choose any point on the cylinder as its (intrinsic) origin, the point with coordinates, (0,0), written $\mathbf{y}(0,0)$;
2. Choose at $\mathbf{y}(0,0)$ one of two directions along the unique geodesic that comes back to $\mathbf{y}(0,0)$, which he might call the positive direction along the base curve;

† In Chapters 1 though 5 of this text, local coordinates are not necessary for the understanding of any of the main geometric concepts except in Problem **4.7**. Thus readers who find local coordinates distracting at this stage may skip all sections and parts of problems dealing with local coordinates while reading the first five chapters.

3. Choose one of two geodesic rays at $\mathbf{y}(0,0)$ which are perpendicular to the base curve as the positive z-axis.

Then the intrinsic local coordinates can be described as

> $\mathbf{y}(w,z)$ = {The point attained by walking along the base curve a distance w and then turning at right angles in the direction of the positive z-axis walking along that geodesic a distance z.}

(See Figure 1.18.)

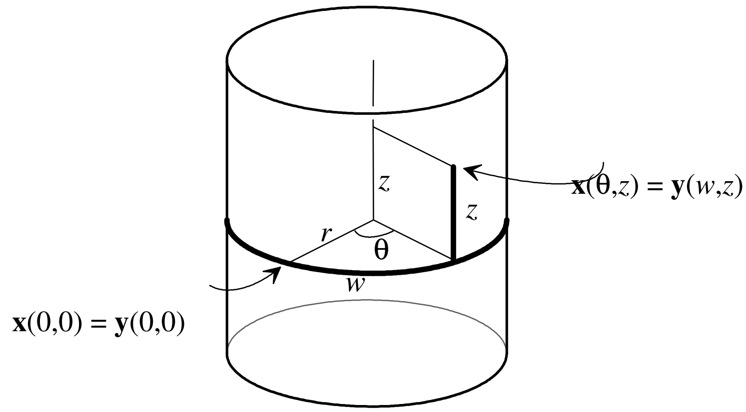


Figure 1.18. Local coordinates on a cylinder.

Such local coordinates are called ***geodesic rectangular coordinates***. In this example the base curve is a geodesic, but that is not necessary. The same intrinsic description will work with any smooth curve chosen as the base curve. However, it is necessary (as we will see later) that the curves defined by the second coordinates be geodesics.

These intrinsic and extrinsic local coordinates are different in the sense that, for most points, $\mathbf{x}(a,b) \neq \mathbf{y}(a,b)$.

For a cone, the natural origin (both intrinsically and extrinsically) is the cone point. The angle that the 2-dimensional bug would measure at the cone point is called the ***cone angle*** α. If we use radian measure, then $\alpha = c/r$, where c is the circumference of the circle on the cone, which is at a distance r from the cone point. The ***geodesic polar coordinates*** on the cone can be described intrinsically by

> $\mathbf{y}(\theta,r)$ = {the point $\mathbf{p}$ on the cone, where r is the length of the line segment from $\mathbf{p}$ to the cone point and θ is the angle along the surface between this segment and a fixed reference ray from the cone point}. (See Figure 1.19.)

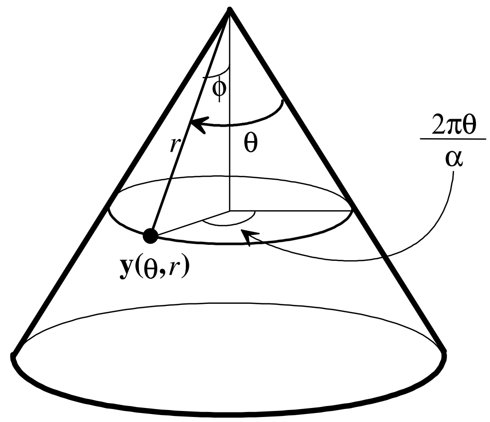


Figure 1.19. Local Coordinates on a Cone.

In the same manner as polar coordinates on the plane, we allow θ to be any angle, but note that two angle coordinates denote the same point on the cone if they differ by an integral multiple of α. You should convince yourself that *these intrinsic polar coordinates work for any cone, even those with cone angle larger than* 2π.

We can define ***geodesic polar coordinates*** extrinsically on a right circular cone with cone angle α by

$$\boldsymbol{x}(\theta, r) = (r \sin\phi \cos\tfrac{2\pi\theta}{\alpha}, r\sin\phi \sin\tfrac{2\pi\theta}{\alpha}, r\cos\phi)\ ,$$

where ϕ is the angle between the axis of the cone and a generator of the cone. By looking at a circle of radius 1 from the cone point the reader can check that its circumference is $\alpha = 2\pi \sin \phi$. Note that *the extrinsic definition only works for cones with cone angle less than* 2π.

Other local rectangular coordinates can be seen by placing graph paper in various orientations on a cone or cylinder.

PROBLEM 1.4. Geodesics in Local Coordinates

***a.** On the cylinder give an intrinsic definition of the coordinate patch* **x**, *which is defined extrinsically above. Also give an extrinsic definition of the coordinate patch* **y** *which is defined intrinsically above.*

b. *Intrinsically define on a cone* ***geodesic rectangular coordinates*** *with the base curve being one of the circles at a fixed distance from the cone point. Compare these coordinates with geodesic polar coordinates on the same cone.*

c. *If γ is a geodesic on the cylinder and α is the angle γ makes with the base curve, then show that in terms of intrinsic rectangular coordinates the parametric equations for γ are*

$$\gamma(s) = \mathbf{y}(s \cos \alpha, s \sin \alpha),$$

where s is the arc-length along γ from the (intrinsic) origin. Write an equation for γ in terms of extrinsic local coordinates. Given two points p and q on the cylinder, determine which geodesics join A to B.

d. *In terms of intrinsic polar coordinates, show that if* λ *is a geodesic on the cone and* $\mathbf{p} = (\beta,d)$ *is the point on the geodesic that is closest to the cone point, then an arbitrary point* $\mathbf{y}(\theta,r)$ *on the geodesic satisfies the equation*

$$r = d\sec(\theta-\beta).$$

Show that the geodesic can be parametrized by arc-length s *as follows:*

$$r = \sqrt{d^2+s^2}\,,\ \theta = \beta + \arctan\tfrac{s}{d}.$$

***e.** *Write an equation for a geodesic on a cone in terms of extrinsic local coordinates.*

***f.** *How many times does a geodesic on the cone intersect itself? How does the number of self-intersections depend on the cone angle?*

Problem 1.5. What Is Straight on a Sphere?

a. *Imagine yourself to be a bug crawling around on a sphere. (This bug can neither fly nor burrow into the sphere.) The bug's universe is just the surface; it never leaves it. What is "straight" for this bug? What will the bug see or experience as straight? How can you convince yourself of this? Use the properties of straightness (such as symmetries), which you talked about in Problem **1.1**, to show that the great circles are straight with respect to the sphere.*

b. *Show* (that is, convince yourself, and give an argument to convince others) *that no other circles* (for example, latitude circles) *on the sphere have the same local symmetries as the great circles.*

In Chapter 3 we will show in another way that the great circles are the only geodesics on the sphere.

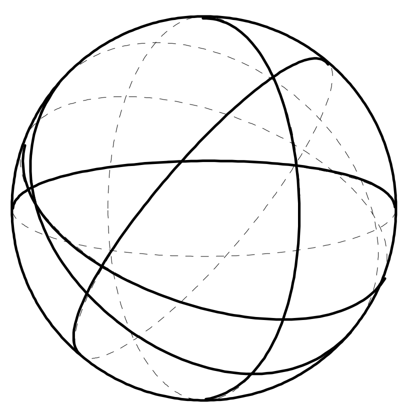


Figure 1.20. Great Circles.

Great circles are those circles that are the intersection of the sphere with a plane through the center of the sphere. Examples: Any longitude line and the equator are great circles on the earth. Consider any pair of opposite points as being the poles, and thus the equator and longitudes with respect to any pair of opposite points will be great circles. (See examples illustrated in Figure 1.20.)

Suggestions

The first step to understanding this problem is to convince yourself that great circles are straight lines on a sphere. Think what it is about the great circles that would make the bug experience them as straight. To better visualize what is happening on a sphere (or any other surface, for that matter), **you must use models**. This is a point we cannot stress enough. You must make lines on a sphere to fully understand what is straight and why. An orange or an old, worn tennis ball works well as a sphere, and rubber bands make good lines. Also, you can use ribbon or strips of paper. Try placing these items on the sphere along different curves to see what happens.

Also look at the symmetries from Problem 1.1 to see if they hold for straight lines on the sphere. The important thing to remember here is to **think in terms of the surface of the sphere, not in 3-space**. Always try to imagine how things would look from the bug's point of view.

Experimentation with models plays an important role here. Working with models *that you create* helps you to experience that great circles are, in fact, the only straight lines on the surface of a sphere. Convincing yourself of this notion will involve recognizing that straightness on the plane and straightness on a sphere have common elements. The activities listed at the end of Problem **1.3** are all useful here also. However, it is better for you to come up with your own experiences.

These activities will provide you with an opportunity to investigate the relationships between a sphere and the geodesics of that sphere. Your experiences should help you to discover how the symmetries of great circles are mostly the same as the symmetries of straight lines on a plane.

Also notice that, on a sphere, straight lines are circles (points on the surface a fixed distance away from a given point) — special circles whose circumferences are straight! Note that the equator is a circle with two centers: the north pole and the south pole. In fact, any circle on a sphere has two centers.

Intrinsic Curvature on a Sphere

You have tried wrapping the sphere with a ribbon and noticed that the ribbon will only lie flat along a great circle. (If you haven't experienced this yet, then do it now before you go on.) Arcs of great circles are the only paths of a sphere's surface that are tangent to a straight line on a piece of paper wrapped around the sphere. If you wrap a piece of paper tangent to the sphere around a latitude circle (see Figure 1.21), then, extrinsically, the paper will form a portion of a cone and the curve on the paper will be an arc of a circle. The *intrinsic* (or *geodesic*) *curvature* of a path on the surface of a sphere can be defined as the curvature that one obtains when one "unwraps" the path onto a plane. For more details on intrinsic curvature see Chapter 3.

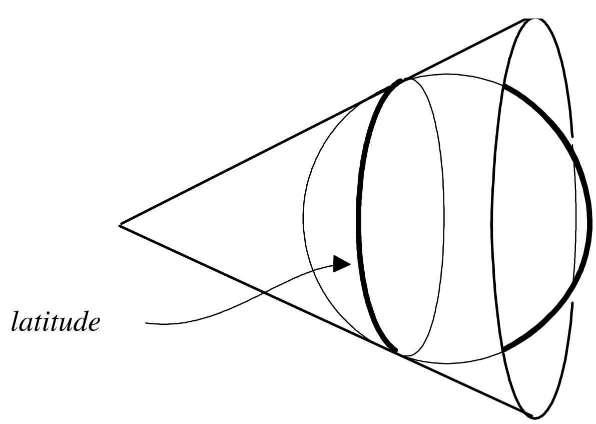


Figure 1.21. Finding the intrinsic curvature.

In Chapter 3 we will talk about geodesics in terms of the velocity vector of the motion as one travels at a constant speed along that path. (The velocity vector is tangent to the curve along which the bug walks.) For example, as you walk along a great circle, the velocity vector to the circle changes direction, extrinsically, in 3-space where the change in direction is toward the center of the sphere. "Toward the center" is not a direction that makes sense to a 2-dimensional bug whose whole universe is the surface of the sphere. Thus, the bug does not experience the velocity vectors at points along the great circle as changing direction. In Chapter 8, the rate of change, from the bug's point of view, is called the *covariant* (or *intrinsic*) *derivative*. We will show that as the bug traverses a geodesic, the covariant derivative of the velocity vector is zero. Geodesics can also be expressed in terms of *parallel transport*, discussed in Chapter 5.

Local Coordinates on a Sphere

Extrinsically we can express the sphere of radius R with center at the origin of 3-space by

$$\mathbf{x}(\theta,\phi) = (R\cos\phi\cos\theta,\ R\cos\phi\sin\theta,\ R\sin\phi). \quad \text{(See Figure 1.22.)}$$

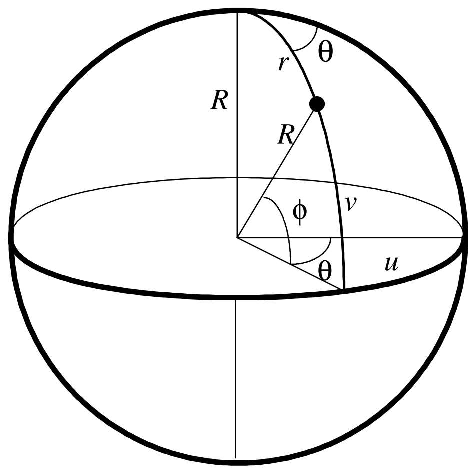


Figure 1.22. Local Coordinates on a Sphere.

This expression provides local coordinates for the sphere except at the North and South Poles ($\phi = \pm\pi/2$). Except when $R = 1$, $\mathbf{x}$ is not strictly geodesic rectangular coordinates because, even though the longitudes (θ equal to a constant) are arcs of great circles, the coordinate ϕ does not express the arc-length as is required by geodesic coordinates. The reader should check that the following is an (extrinsic) expression of geodesic rectangular coordinates on the sphere of radius R:

$$\mathbf{y}(u,v) = (R\cos(v/R)\cos(u/R),\ R\cos(v/R)\sin(u/R),\ R\sin(v/R)).$$

Geodesic Polar Coordinates at the poles can be expressed as

$$\mathbf{z}(\theta,r) = (R\sin(r/R)\cos\theta,\ R\sin(r/R)\sin\theta,\ R\cos(r/R)).$$

Problem 1.6. Strakes, Augers, and Helicoids

To give structural support to large metal cylinders, such as large smoke stacks, engineers sometimes attach a spiraling strip called a *strake*.† (See Figure 1.23.) The strake and related surfaces are common surfaces in the physical world and will serve as illustrative examples throughout this text.

† This example is inspired by an example given in [**DG**: Morgan].

To produce the strake it is convenient to cut annular pieces (the region between two concentric circles) from a flat sheet of steel as illustrated in the Figure 1.23.

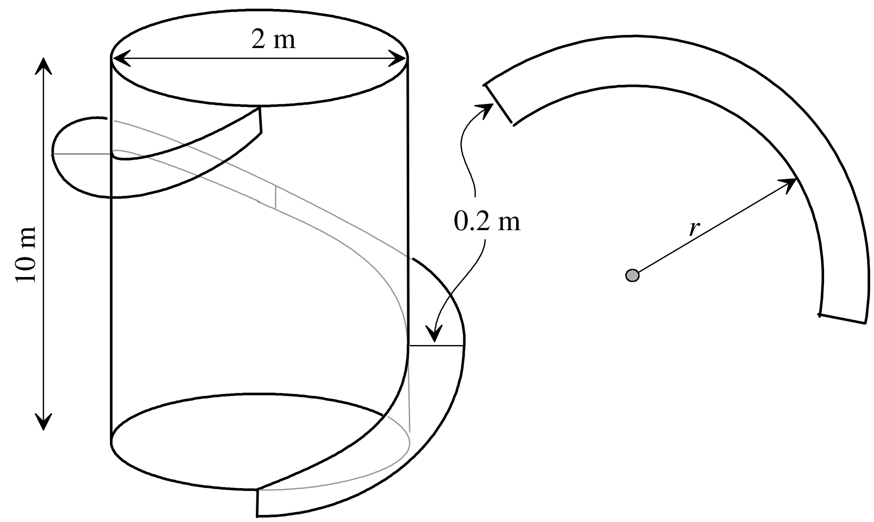
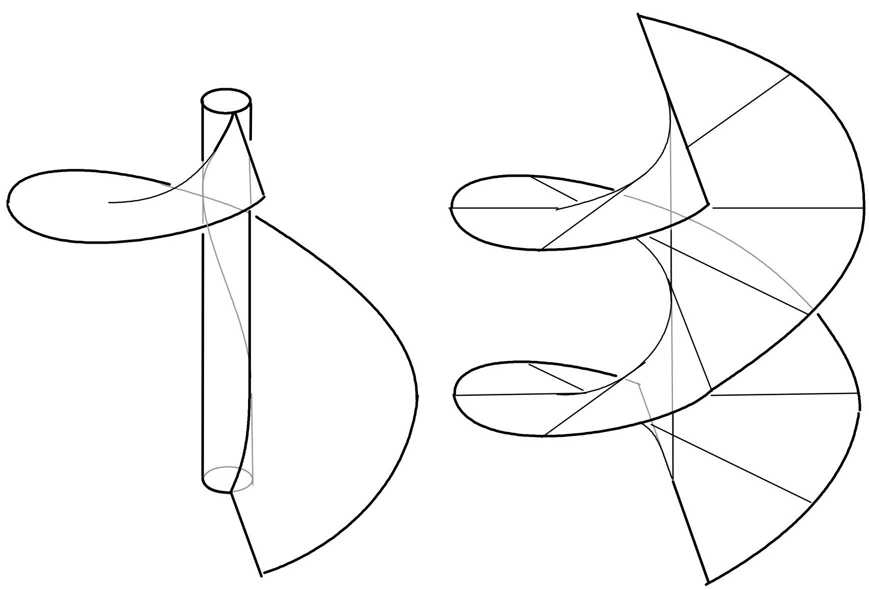


Figure 1.23. A strake.

These annular pieces are then bent along a helix on the cylinder to form the strake. In Chapter 2 we will convince ourselves that the way to compute the ideal value for the radius r is to require that the helix on the cylinder and the inner curve of the annulus to have the same curvature. Also in Chapter 2 we will ask if the flat annulus can be bent (but not stretched) to fit the strake so that the radial line segments of the strake are horizontal straight lines. In other words: *Is the strake developable?*

What happens if we make the strake very wide compared to the diameter of the cylinder — such as happens in an auger? (See Figure 1.24.) If we double the auger (extend it in both directions from the cylinder) and then shrink the cylinder to zero radius, the resulting figure is called a ***helicoid***.



Auger. **Helicoid.**

Figure 1.24

We suggest that at this point you use Computer Exercise 1.6 in Appendix C to display computer images of the strake/auger/helicoid with a cylinder of variable radius.

a. *Show that the horizontal line segments in the strake and helicoid have intrinsic mirror symmetry and thus must be geodesics on the strake or helicoid.*

[Hint: Look at the extrinsic 180° rotation about the segment.]

b. *Show that, if h is the measure of how high the helix goes on the cylinder in one turn of the helix, then arc-length along one turn of the helix is*

$$\sqrt{h^2 + (2\pi R)^2}\,.$$

[Hint: Look at the helix from the bug's 2-dimensional point-of-view.]

c. *Find an extrinsic expression for the helix, parametrized by arc-length. (That is, express the helix as*

$$\gamma(s) = (x(s), y(s), z(s)),$$

where s is arc-length.)

d. *Find an extrinsic expression for geodesic rectangular coordinates on the strake with the helix along the cylinder as the base curve. That is, express the strake as*

$$\mathbf{x}(s,t) = (\, x(s,t), y(s,t), z(s,t)\,),$$

where s is arc-length along the helix and t is arc-length horizontally.

[Hint: Start with Part **c**.]

***e.** *How do you need to modify the coordinates in part* **d** *so that they will be local coordinates for a helicoid with the center line of the helicoid as the base curve?*

[Hint: What happens as $R \to 0$?]

Problem 1.7. Surfaces of Revolution

If f is a positive-valued function of one real variable, then

$$\mathbf{x}(\theta,x) = (x, f(x)\cos\theta,\ f(x)\sin\theta)$$

is extrinsic coordinates defining a surface of revolution (revolved about the x-axis). (See Figure 1.25.)

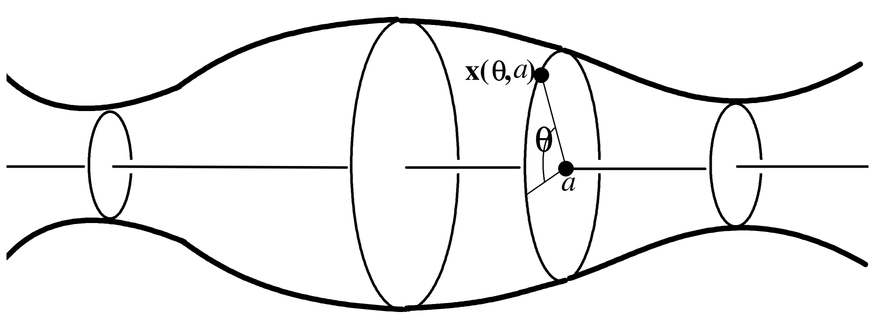


Figure 1.25. Surface of revolution.

Computer Exercise 1.7 will allow you to use the computer to display surfaces of revolution. Can you display a cylinder? a cone? a sphere? a paraboloid? Keep these surfaces in mind as you do Problem 1.7.

a. *Argue that the curves on the surface of revolution that have constant* θ *are geodesics.*

b. *Which generating circles* (x = constant) *appear to be geodesics? Why?* We will return to this question in Chapters 3 and 6.

[Hint: Imagine laying a ribbon on the surface and/or looking for infinitesimal symmetries]

c. *On a surface of revolution describe intrinsic geodesic rectangular coordinates with one of the generating circles* (x = constant) *as base curve.*

[Hint: See pages 14-15 for a description of geodesic rectangular coordinates. Remember that the base curve and the second coordinate curves (geodesic) must be parametrized by arc-length.]

Problem 1.8. Hyperbolic Plane

Hyperbolic geometry, discovered more than 150 years ago by C.F. Gauss (German), J. Bolyai (Hungarian), and N.I. Lobatchevsky (Russian), is special from a formal axiomatic point of view because it satisfies all the axioms of Euclidean geometry except for the parallel postulate. In hyperbolic geometry there are many straight lines through a given point that do not intersect a given line. (Compare with Problem **1.3.d**.)

Hyperbolic geometry and non-Euclidean geometry are often considered as being synonymous, but as we have seen there are many non-Euclidean geometries, particularly spherical geometry. It is also not accurate to say (as many books do) that non-Euclidean geometry was discovered more than 150 years ago. Spherical geometry (which is clearly not Euclidean) was in existence and studied by at least the ancient Babylonians, Indians, and Greeks more than 2,000 years ago. Spherical geometry was of importance for astronomical observations and astrological calculations. Even Euclid in his *Phaenomena* [**AT**: Euclid] (a work on astronomy) discusses propositions of spherical geometry. Menelaus, a Greek of the first century, published a book *Sphaerica*, which contains many theorems about spherical triangles and compares them to triangles on the Euclidean plane. (*Sphaerica* survives only in an Arabic version. For a discussion see [**NE**: Kline, page 120].)

A paper model of the hyperbolic plane may be constructed as follows†: Cut out many identical annular ("annulus" is the region between two concentric circles) strips as in the following Figure 1.26. Attach the strips together by attaching the inner circle of one to the outer circle of the other or the straight ends together. The resulting surface is of course only an approximation of the desired surface. The actual hyperbolic plane is obtained by letting $\delta \to 0$ while holding r fixed. Note that since the surface is constructed the same everywhere (as $\delta \to 0$), it is ***homogeneous*** (i.e. intrinsically and geometrically, every point has a neighborhood that is isometric to a neighborhood of any other point). We will call the results of this construction the ***annular hyperbolic plane.***

I strongly suggest that the reader at this point take the time to cut out carefully several such annuli and to tape them together as indicated.

In Parts **c** and **d**, below, and Problem **3.1f**, there are different descriptions of the hyperbolic plane. Further discussions can be found in the article [**Hy**: Henderson/Taimina] and the book [**Tx**: Henderson-/Taimina, Chapter 5 and Appendix B] -- these two sources also contain descriptions how to crochet a hyperbolic plane and how to make a close polyhedral approximation called the *hyperbolic soccer ball*.

† The idea for this construction was shown to me by William Thurston in 1978.

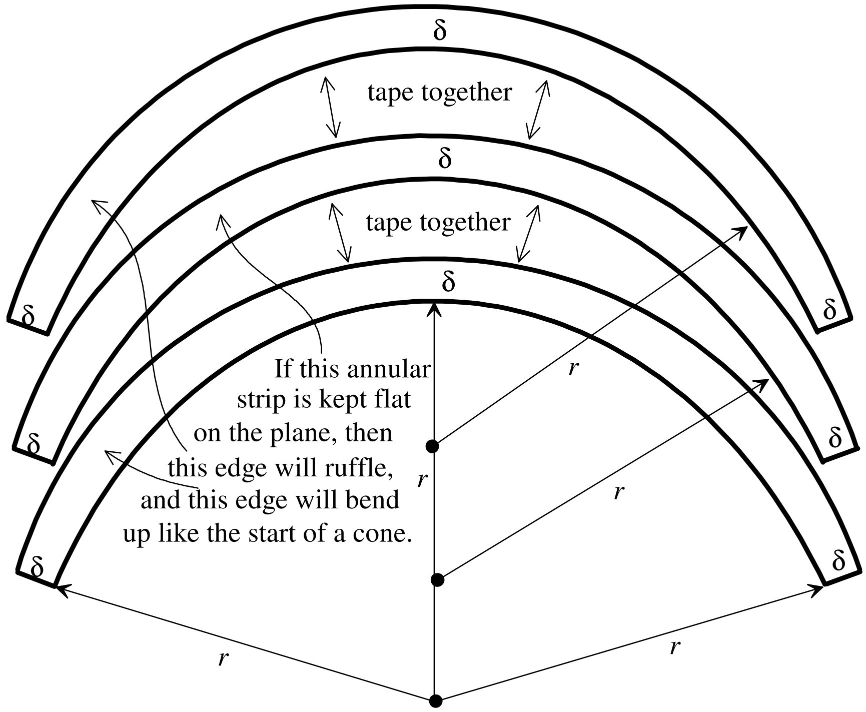


Figure 1.26. Annular strips for making an annular hyperbolic plane.

a. *Argue that the curves on the annular hyperbolic plane that run radially across each annular strip are geodesics.*

[Hint: Look for the local intrinsic symmetries.]

b. *On the annular hyperbolic plane, describe geodesic rectangular coordinates with the base curve being one of the circular edges of one of the annuli and the positive second coordinate direction being along the geodesics that run radially across each annular strip toward the center of that annulus. Draw a picture.*

[Note that it only makes sense now to do this intrinsically.]

c. *Let λ and μ be two of the geodesics described in part* **a**. *If the distance between λ and μ along the base curve is d, then show that the distance between them at a distance $c = n\delta$ from the base curve is on the annular hyperbolic model:*

$$d\left(\tfrac{r}{r+\delta}\right)^n = d\left(\tfrac{r}{r+\delta}\right)^{c/\delta}.$$

Now take the limit as $\delta \to 0$ to show that the distance between λ and μ on the annular hyperbolic plane is

$$d\ \exp(-c/r).$$

[Hint: What happens to the distance between λ and μ as they cross *one* annulus?]

Now we define a new coordinate patch **z** from the upper half plane

$$\mathbf{H}^+ \equiv \{(x,y) \in \mathbf{R}^2 \mid y > 0\}$$

onto the annular hyperbolic plane with $r = 1$ by defining

$$\mathbf{z}(x,y) = \mathbf{x}(x,\ln(y)),$$

where **x** is the geodesic rectangular coordinates defined in Part **b**. This representation of the hyperbolic plane is usually called ***the upper half plane model***.

The particular form of these coordinates is such that **z** is ***conformal***, meaning that as **z** takes the upper half plane onto the hyperbolic plane, it does not change angles (but will distort distances). To see that **z** is conformal, note first that the x and y coordinate curves in the plane and on the annular hyperbolic plane are both orthogonal to each other. Other angles will not be changed if, at each point, the distortion of distances along the two coordinate curves is the same, because then the distortion at each point will be infinitesimally a similarity (which preserves angles). For any curve γ the ***distortion*** at a point $p = \gamma(a)$ is

$$\lim_{x \to a} \frac{\text{the arc length along } \gamma \text{ from } \gamma(a) \text{ to } \gamma(x)}{|x-a|}.$$

If the curve lies in Euclidean space, then this is precisely the speed $|\gamma'(a)|$. Thus we need to:

d. *Show that the distortion of both coordinate curves*

$$x \to \mathbf{z}(x,b) \;\; \textit{and} \;\; y \to \mathbf{z}(a,y)$$

at the point $z(a,b)$ is $1/b$.

[Hint: For the first coordinate direction, use the result of Part **c**. For the second coordinate direction, use the fact that the second coordinate curves in geodesic rectangular coordinates are parametrized by arc-length.]

In Problem **7.5** you will show that there exist in the hyperbolic plane: rotations through any angle about any point, reflections through any geodesic, and translations along any geodesic segment.

It is a theorem (see [Henderson/Taimina, Chapter 5] for references and a historical discussion) that there does not exist a C^2 (second derivatives exist and are continuous) isometric embedding of the whole hyperbolic plane into 3-space. However, the annular model can be extended indefinitely and provides (as $\delta \to 0$) an isometric embedding into 3-space. For a small section of the surface it is possible to have the embedding analytic (see Problem **3.1**), but it is possible to see in the model that as you take larger and larger sections, eventually "ruffles" will form, which cause the surface not to have a tangent plane at some points and thus not to be smooth.

Problem 1.9. Surface as Graph of a Function $z = f(x,y)$

If we have any real-valued function $z = f(x,y)$ defined on a region in the plane, then we can use it to define coordinates extrinsically for a surface

$$\mathbf{x}(x,y) = (\, x, y, f(x,y)\,).$$

On the other hand, if M is a smooth surface in 3-space, which has a tangent plane T_pM at the point p, then we may find a function whose graph is a neighborhood of p in the surface. Choose coordinates (x,y,z) in 3-space so that $p = (0,0,0)$ and the tangent plane T_pM is the (x,y)-plane. The projection

$$g(x,y,z) = (x,y,0)$$

will be one-to-one onto a neighborhood of p. (See Appendix B.) Thus there will be an inverse function $\mathbf{x}(x,y) \in M$ of the form

$$\mathbf{x}(x,y) = (x,y,f(x,y)).$$

Therefore **x** expresses the portion of M in a neighborhood of p as the graph of a function. These coordinates have the extra property that the plane tangent to M at p is the (x,y)-plane and $p = (0,0,0)$. Such special graphs of functions are called a ***Monge patch***. (See Figure 1.27.) In general it is not possible (or is very difficult), given a surface, to explicitly find the function that defines a Monge patch. We will show in Problem **3.1.e** that such Monge patches for a smooth manifold will be continuously differentiable.

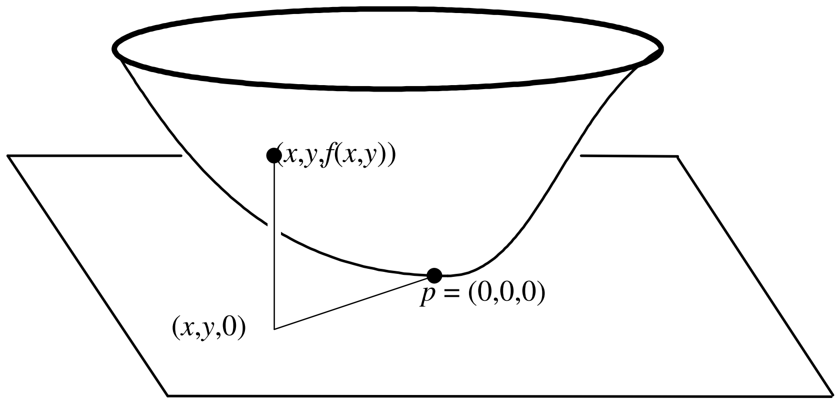


Figure 1.27. Monge patch.

You are encouraged to use Computer Exercise 1.9 to display the graph of any real-valued function of two real variables.

a. *Express locally as a graph of a function a helicoid and a cone with angle less than* 360º.

b. *Find a Monge patch for a cylinder and for a sphere.*

c. *Consider a general surface of revolution*

$$\mathbf{x}(\theta,x) = (x, f(x)\cos\theta,\ f(x)\sin\theta).$$

Express the surface locally as the graph of a function. If the derivative $f'(a) = 0$, *then find a Monge patch for a neighborhood of*

$$(a, f(a)\cos\theta, f(a)\sin\theta).$$