Groups whose set of vanishing elements is the union of at most three conjugacy classes

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Abstract

Let G be a finite group. We say that an element g in G is a vanishing element if there exists some irreducible character χ of G such that $\chi(g) = 0$. In this paper, we prove that if the set of vanishing elements of G is the union of at most three conjugacy classes, then G is solvable.

1 Introduction

Let G be a finite group. We say that an element g in G is a vanishing element if there exists some irreducible character χ of G such that $\chi(g) = 0$. We denote by $\mathrm{Van}(G)$ the set of vanishing elements of G, in other words,

$$\mathrm{Van}(G) = \{g \in G | \chi(g) = 0 \text{ for some } \chi \in \mathrm{Irr}(G)\}$$

in which $\mathrm{Irr}(G)$ is the set of irreducible characters of G. It is clear that $\mathrm{Van}(G)$ is the union of some conjugacy classes. A result of Burnside (see [6, Theorem 3.15]) assert that $\mathrm{Van}(G) = \emptyset$ if and only if G is an abelian group.

Many results show that the structure of $\mathrm{Van}(G)$ has an strong influence on the algebraic structure of G. Let p be a prime number. In [4] Dolfi, Pacifici, and Sanus proved that if the size of every conjugacy class of G contained in $\mathrm{Van}(G)$ is not divisible by p, then G has a normal p-complement and abelian Sylow p-subgroups. Moreover, Brough in [2] show that if the size of every conjugacy class of G contained in $\mathrm{Van}(G)$ is square free, then G is a supersolvable group.

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In this paper, we provide a relatively short proof for the solvability of finite groups whose set of vanishing elements is the union of at most three conjugacy classes, using the Classification of the Finite Simple Groups.

2 Main Theorem

Let p and q be distinct prime numbers. An irreducible character χ of G is said to be of q-defect zero if q does not divide $|G|/\chi(1)$. By Theorem 8.17 of [6], if χ is an irreducible character of q-defect zero of G, then $\chi(g) = 0$ whenever q divides the order of g in G.

Lemma 2.1 ([2], Lemma 2.2)**.** *Let G be a group, and N a normal subgroup of G. If N has an irreducible character of q-defect zero, then every element of N of order divisible by q is a vanishing element in G.*

The following result finds non-abelian simple groups which do not have an irreducible character of q-defect zero for some prime number q.

Corollary 2.2 ([5], Corollary 2)**.** *Every finite simple group G has a p-block of defect 0, for every prime p, except in the following special cases:*

- *G has no 2-block of defect 0 if it is isomorphic to $M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_3, BM$, or $\mathrm{Alt}(n)$ where $n \neq 2m^2 + m$ nor $2m^2 + m + 2$ for any integer m.*
- *G has no 3-block of defect 0 if it is isomorphic to Suz, Co_3, or $\mathrm{Alt}(n)$ with $3n + 1 = m^2 r$ where r is squarefree and divisible by some prime $q \equiv 2 \bmod 3$.*

It follows from Corollary 2.2 that $\mathrm{Alt}(5)$ and $\mathrm{Alt}(6)$ both have p-blocks of defect 0 for all primes p.

Lemma 2.3. *Let S be a non-abelian simple group and assume there exists a prime q such that S does not have an irreducible character of q-defect zero. Then there exist irreducible characters $\theta_1, ..., \theta_4$ of S which extends to $\mathrm{Aut}(S)$ and elements $x_1, ..., x_4$ of distinct orders such that θ_i vanishes on $cl(x_i)$ for $1 \leq i \leq 4$.*

Proof. By Corollary 2.2, the group S is either a sporadic group, or $\mathrm{Alt}(n)$ for some $n \geq 7$. In the former case, using the Atlas [3], we obtain the following table containing pairs $\{\theta_i, x_i\}$ for $1 \leq i \leq 4$, in which characters $\theta_1, ..., \theta_4$ and conjugacy classes $cl(x_1), ..., cl(x_4)$ satisfying the required condition.

Group	θ_1	x_1	θ_2	x_2	θ_3	x_3	θ_4	x_4
M_{12}	χ_7	$6A$	χ_7	$8A$	χ_7	$3B$	χ_6	$5A$
M_{22}	χ_7	$8A$	χ_7	$11A$	χ_2	$7A$	χ_3	$6A$
M_{24}	χ_3	$6A$	χ_7	$3B$	χ_7	$4C$	χ_5	$7A$
J_2	χ_6	$2B$	χ_6	$3B$	χ_6	$6B$	χ_{10}	$5C$
HS	χ_{16}	$4C$	χ_{16}	$2B$	χ_7	$5C$	χ_7	$7A$
Suz	χ_3	$8B$	χ_3	$2B$	χ_9	$3C$	χ_9	$5A$
Ru	χ_2	$6A$	χ_{11}	$3A$	χ_{11}	$4D$	χ_9	$5B$
Co_1	χ_2	$4F$	χ_2	$3D$	χ_2	$9B$	χ_2	$6H$
Co_3	χ_9	$6E$	χ_6	$7A$	χ_6	$4B$	χ_{10}	$5B$
BM	χ_2	$10D$	χ_{20}	$5B$	χ_{20}	$4J$	χ_{27}	$9B$

Now, consider the case where S is an Alternating group Alt(n) for $n \geq 7$. We know that

$$\chi(g) = |\mathrm{Fix}(g)| - 1, \tag{2.1}$$

where $|\mathrm{Fix}(g)|$ is the number of fixed points of g, is an irreducible character of $\mathrm{Alt}(n)$ and $\mathrm{Sym}(n)$. If n is an even number, we set

$$\begin{aligned}
x_1 &= (1, ..., n-1)(n), \\
x_2 &= (1, ..., n-5)(n-4, n-3)(n-2, n-1)(n), \\
x_3 &= (1, ..., n-6)(n-5, n-4, n-3)(n-2, n-1)(n), \\
x_4 &= (1, ..., n-7)(n-6, n-5, n-4)(n-3, n-2, n-1)(n),
\end{aligned}$$

and we set

$$\begin{aligned}
x_1 &= (1, ..., n-4)(n-3, n-2, n-1)(n), \\
x_2 &= (1, ..., n-7)(n-6, ..., n-1)(n), \\
x_3 &= (1, ..., n-5)(n-4, ..., n-1)(n), \\
x_4 &= (1, ..., n-3)(n-2, n-1)(n),
\end{aligned}$$

if n is an odd number. We can check that $\chi(x_i) = 0$ and in each case the order of x_i's are distinct for $n \geq 10$. Moreover, since Aut(Alt(n)) $\cong$ Sym(n) for $n \geq 7$, then the character χ and conjugacy classes of x_i's satisfying the required condition for $n \geq 10$ and $i = 1, ..., 4$. Using [3], for $7 \leq n \leq 9$ we can easily find irreducible characters $\theta_1, ..., \theta_4$ of $\mathrm{Alt}(n)$ which extends to $\mathrm{Sym}(n)$ and elements $x_1, ..., x_4$ of distinct orders such that θ_i vanishes on $cl(x_i)$ for $1 \leq i \leq 4$. ■

Proposition 2.4 ([1], Lemma 5). *Let G be a group, and $M = S_1 \times ... \times S_k$ a minimal normal subgroup of G, where every S_i is isomorphic to a non-abelian simple group S. If $\theta \in \mathrm{Irr}(S)$ extends to $\mathrm{Aut}(S)$, then $\theta \times ... \times \theta \in \mathrm{Irr}(M)$ extends to G.*

In the following results, normal subgroups which are the union of at most four conjugacy classes are characterized.

Theorem 2.5 ([8], Theorem 8 and Proposition 1, 2). *Let G be a finite group and H be a normal subgroup of G which is the union of three conjugacy classes in G. Then one of the following holds:*

(1) H is an elementary abelian p-group of odd order.

(2) H is a metabelian p-group.

(3) H is a Frobenius group with complement $\mathbb{Z}_p$.

Theorem 2.6 ([7], Theorem 1). *Let G be a finite group and let H be the union of four conjugacy classes in G. Then the number of characteristic subgroups of H is at most 4, and one of the following holds:*

(1) H is a p-group and $H'' = 1$.

(2) $H \cong \mathrm{Alt}(5)$, the alternating group of degree 5, and $G/C_G(H) \cong \mathrm{Sym}(5)$.

(3) H is a (solvable) group of order $|H| = p^a q^b$, where a, b are positive integers.

Lemma 2.7. *Let G be a finite group and H be a non-trivial normal subgroup of G which is the union of at most four conjugacy classes in G. Then either H is solvable or the set of vanishing elements of G are the union of at least 6 conjugacy classes.*

Proof. If H is the union two conjugacy classes, then H is an elementary abelian p-group and so solvable. Otherwise, By Theorem 2.5 and 2.6, G is solvable except case (2) of Theorem 2.6. In this case, since each non-trivial elements of $\mathrm{Sym}(5)$ is a vanishing element and $G/C_G(H) \cong \mathrm{Sym}(5)$, then the set of vanishing elements of G are the union of at least 6 conjugacy classes. ■

Now, we are ready to prove Main Theorem.

Theorem 2.8. *Let G be a finite group. If the set of vanishing elements of G are the union of at most three conjugacy classes of G, then G is solvable.*

Proof. We shall prove by induction on the order of the group. Let M be a minimal normal subgroup of G. If M is non-abelian, then $M = S_1 \times ... \times S_n$ in which S_i is isomorphic to a non-abelian simple group S. If S has an irreducible character of q-defect zero θ_q for each prime number q, then $\theta_q \times ... \times \theta_q$ is an irreducible character of q-defect zero of M for each prime number q. By Lemma 2.1, we deduce that every non-trivial element of M is a vanishing element of G and M is the union of at most four conjugacy classes of G. Therefore, by Lemma 2.7 M is solvable which is a contradiction.

Now, we can assume that S does not have any irreducible character of q-defect zero for some prime number q, thus by Corollary 2.2 and Lemma 2.3, there exist elements $x_1, ..., x_4 \in S$ of distinct orders and $\theta_1, ..., \theta_4 \in \mathrm{Irr}(S)$ which extends to $\mathrm{Aut}(S)$, such that $\theta_i(x_i) = 0$ for $1 \leq i \leq 4$. Therefore, by Proposition 2.4, irreducible characters $\theta_i \times ... \times \theta_i$ of M extends to G and vanishes on x_i for $1 \leq i \leq 4$. Since x_i's are of distinct orders, then x_i's lie in distinct conjugacy classes of G for $1 \leq i \leq 4$ and so the conjugacy class of each x_i is vanishing in G which is a contradiction.

Thus M must be abelian and since G/M is solvable by the inductive hypothesis, then G is solvable. ■

Example 1. Let $\mathrm{Alt}(5)$ be a Alternating group of order 60. We can easily check that the set of vanishing elements of $\mathrm{Alt}(5)$ are the union of four conjugacy classes. Thus, Theorem 2.8 may not remain true if the set of vanishing elements of G are the union of at least four conjugacy classes.

Example 2. Let G be a Dihedral group D_{2n} of order $2n$, where n is odd. We can check that the set of vanishing elements of G is a conjugacy class and G satisfies Theorem 2.8.

On the other hand, let k be a finite field of order q. The affine group $G = k \rtimes k^*$ is metabelian, and has at least $q - 1$ conjugacy classes of vanishing elements. Thus, numerous finite soluble groups fail to satisfy the hypothesis of Theorem 2.8.

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