On the multiplication by a polynomial of bounded continued fraction over a finite field

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Abstract

In this paper, we will discuss the period length of the continued fraction of the product of a polynomial with a quadratic power series over a finite field. Furthermore, we will give the first example of bounded continued fraction in characteristic 3 with not flat partial quotients.

1 Introduction

Let $\mathbb{F}$ be a finite field and let $\mathbb{F}((T^{-1}))$ denote the field of formal power series over $\mathbb{F}$. For a nonzero power series:

$$\alpha = \sum_{i \leq n_0} c_i T^i \in \mathbb{F}((T^{-1})), \quad n_0 \in \mathbb{Z}, \quad c_{n_0} \neq 0,$$

we define:

$$\deg(\alpha) = n_0, \quad |\alpha| = |T|^{n_0}, \quad [\alpha] = \sum_{0 \leq i} c_i T^i$$

where $|T|$ is a fixed real number greater than 1. Let $\deg(0) = -\infty$ and $|0| = 0$. Recall that $|\alpha|$ for power series α defines a non- Archimedian absolute value on $\mathbb{F}((T^{-1}))$ and $[\alpha]$ is called the polynomial part of α. Note that $[\alpha]$ is characterized as the unique polynomial E such that $|\alpha - E| < 1$.

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The general theory of continued fractions for power series are expounded by Schmidt in [20]. Here we briefly review the basic facts and establish some notations. The continued fraction expansion for power series α is defined as the unique expression:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [a_0, a_1, a_2, \dots]$$

where $a_n \in \mathbb{F}[T]$ for $n \geq 0$ and $\deg a_n > 0$ for $n > 0$. As usual the tail of the expansion, $[a_n, a_{n+1}, ...]$, called the complete quotient, is denoted by $\alpha_n(\alpha_0 = \alpha)$. The numerator and the denominator of the convergent $[a_0, ..., a_n]$ are denoted by P_n and Q_n. These polynomials, are both defined by the same recursive relation: $K_n = a_n K_{n-1} + K_{n-2}$ for $n \geq 1$, with the initial conditions $P_{-1} = 1$ and $P_0 = a_0$ for the numerator, while the initial conditions are $Q_{-1} = 0$ and $Q_0 = 1$ for the denominator. We can view P_n and Q_n as a function in the $n+1$ variables $a_0, a_1, ..., a_n$. The recursion shows that this function is again a polynomial. We call this polynomial the continuant. These polynomials will simply be denoted by $P_n = \langle a_0, a_1, ..., a_n \rangle$ and $Q_n = \langle a_1, a_2, ..., a_n \rangle$. For more information on continuants, the reader may consult the introduction of [9]. The quotient P_n/Q_n is a rational approximation to α satisfying:

$$|Q_n\alpha - P_n| = |a_{n+1}|^{-1}|Q_n|^{-1}.$$

Thus, if $\deg a_{n+1} = s$, the quotient P_n/Q_n is said to be a convergent of accuracy s. For any irrational $\alpha = [a_0, a_1, a_2, \dots] \in \mathbb{F}((T^{-1}))$, we set

$$\overline{K}(\alpha) = \limsup_n \deg a_n \in \mathbb{N} \cup \{\infty\}. \tag{1.1}$$

We will say that α has bounded partial quotients if $\overline{K}(\alpha) < \infty$.
We will use a basic and technical Lemma concerning continued fractions. The idea involved in this lemma seems to appear for the first time in the works of Mendès France [14] on finite continued fraction in the context of real numbers. First, we recall the following notation. Let $P_n/Q_n := [a_1, a_2, \dots, a_n]$. For all $x \in \mathbb{F}_p(T)$, we will note:

$$\big[[a_1, a_2, \dots, a_n], x\big] := \frac{P_n}{Q_n} + \frac{1}{x}.$$

Lemma 1.1. *Let $a_1, \dots, a_n, x \in \mathbb{F}_p(T)$. We have the following equality:*

$$\big[[a_1, a_2, \dots, a_n], x\big] = [a_1, a_2, \dots, a_n, y], \text{ where } y = \frac{(-1)^{n-1}}{Q_n^2}x - \frac{Q_{n-1}}{Q_n}.$$

The proof of this lemma can be found in Lasjaunias's article [9].
Throughout the paper, we have been dealing with finite sequences (or words). Consequently, we recall the following notation on sequences in $\mathbb{F}[T]$: let

$W = a_0, a_1, \ldots, a_n$ be such a finite sequence, then we set $|W| = n + 1$ for the length of the word W. If we have two words W_1 and W_2, then W_1, W_2 denotes the word obtained by concatenation. Moreover, if $\lambda \in \mathbb{F}^*$, then we define $\lambda \cdot W$ as the following sequence:

$$\lambda \cdot W = \lambda a_0, \lambda^{-1} a_1, \ldots, \lambda^{(-1)^n} a_n.$$

We will also use the same notation of continued fraction where the a_i are constant and the resulting quantity is in $\mathbb{F}$. However, in the last case, by writing $[a_0, a_1, \ldots, a_n]$ we will assume that this quantity is well defined in $\mathbb{F}$, i.e. $a_n \neq 0, [a_{n-1}, a_n] \neq 0, \ldots, [a_1, \ldots, a_n] \neq 0$.

We say that the formal power series α has a n-periodic continued fraction expansion or the continued fraction expansion of α is ultimately periodic of period n if the sequence $(a_i)_{i \geq 0}$ is ultimately periodic of period n. We denote by $Per(\alpha) = n$ and write $\alpha = [a_0, a_1, ..., a_s, \overline{a_{s+1}, ..., a_{s+n}}]$ for the continued fraction expansion of α. We say that the formal power series α has a pure periodic continued fraction expansion of period n if the sequence $(a_i)_{i \geq 0}$ is purely periodic of period n and write $\alpha = [\overline{a_0, ..., a_{n-1}}]$. Let $\alpha \in \mathbb{F}((T^{-1}))$, then α is quadratic if and only if the continued fraction expansion of α is periodic.

In 1974, Cohen [4] studied the function $S(N, n) = \sup_{Per(x)=n} Per(Nx)$ where N is a positive integer, x is a quadratic irrational and $Per(Nx)$ is the length of the period of the continued fraction expansion of Nx. He used an algorithm for computing the continued fraction expansion of Nx and he defined a projective space permitting to evaluate $S(N, n)$ and to study the function $R(N) = \sup_{n \geq 1} \dfrac{S(N, n)}{n}$.

Later, Cusick [6] studied the length of the period of the product of a positive integer with a quadratic irrational by using Raney's algorithm (see [18]). Note that by the Cohen's work [4], we know that the $R(N)$ is always fini and its value is already known for many N. Moreover, Cohen gave a conjecture for the value of $R(N)$ in all the remaining cases. For more details on this topic, the reader is advised to consult the work of Mendès France [15], which summarizes the length of periodic of quadratic irrationals problem.

The case of a finite base field is particularly important and the analogy between these power series and the real numbers is striking. So it is natural to ask how the behavior of $Per(Nx)$ in the function field case becomes. Actually, by adapting Mendès France's result [14] to the polynomial case, Grisel gave in [8], an algorithm for the continued fraction expansion of the product of a formal series by a rational function. In this note, we will study this value for particular polynomial N and for certain quadratic expansion x over a finite field. Our work is based on two nonzero polynomials P_k and Q_k introduced for the first time by Lasjaunias in [9](see also [11]): Let p be an odd prime and $r = p^t$ with $t \geq 1$, we introduce the subset $E(r)$ of integers k such that:

$$k = mp^l + (p^l - 1)/2 \quad for \quad 1 \leq m \leq (p-1)/2 \ and \ 1 \leq l \leq t - 1.$$

Note that $E(r) \subset \{1, \ldots, (r-1)/2\}$ with equality if $r = p$. Also $(r-1)/2 \in E(r)$ in all cases. Let $P_k(T) = (T^2 - 1)^k$ and $Q_k(T) = \int_0^T (y^2 - 1)^{k-1} dy =$

$\sum_{0\leq i\leq k-1}(-1)^{k-1-i}\binom{k-1}{i}(2i+1)^{-1}T^{2i+1}$. Then there exists a $2k$-tuple $(u_1, u_2, \ldots, u_{2k}) \in (\mathbb{F}_p^*)^{2k}$ such that:

$$P_k/Q_k = [u_1T, u_2T, \ldots, u_{2k}T]. \tag{1.2}$$

Note that Q_k is up to a constant factor, the remainder in the Euclidean division of T^r by P_k: There exists $A \in \mathbb{F}_p^*[T]$ such that

$$AP_k - T^r = 2k\theta_k Q_k, \tag{1.3}$$

where $\theta_k = (-Q_k(1))^{-1} = (-1)^k 2^{1-2k}\binom{2k-1}{k} \in \mathbb{F}_p^*$. Furthermore, we have:

$$2k\theta_k[u_1T, u_2T, \ldots, u_{2k}T] = [u_{2k}T, \ldots, u_2T, u_1T]. \tag{1.4}$$

Several works have been interested in studying the quantity $Per(\sqrt{d})$, where d is a positive integer, not a perfect square. In [5], Cohn showed that $Per(\sqrt{d}) \leq \frac{7}{2\pi^2}\sqrt{d}\log(d) + O(\sqrt{d})$. In the case of formal power series, Mkaouar showed in [17] that the period of the square root of any polynomial $Q \in \mathbb{F}_p[T]$ whose degree is even and which is not a perfect square is less than $p^{2\deg Q}$. We will give the exact value of period of the square root of a family of polynomials in $\mathbb{F}_p[T]$.
In this note, we also consider continued fraction expansions for algebraic power series of degree more than 2 over a finite field. Like quadratic real numbers, for which the continued fraction expansion is well known, certain algebraic power series have a continued fraction expansion which can be explicitly described. Most of these power series belong to a particular subset of algebraic elements related to the existence of the Frobenius isomorphism in these power series fields. These power series, now called hyperquadratic, are irrational elements α satisfying an equation $\alpha = f(\alpha^r)$ where r is a power of the characteristic of the base field and f is a linear fractional transformation with integer (polynomials in T) coefficients. The origin of the study of continued fractions for hyperquadratic power series is due to Baum and Sweet [3] who introduced the first example of power series of degree 3 in even characteristic, with bounded partial quotients and others examples with unbounded degree. This studies was been developed in the 1980's by Mills and Robbins [16]. Mills and Robbins pointed out the existence of hyperquadratic continued fractions with all partial quotients of degree one, in odd characteristic with a prime base field. Later, further examples were studied and several methods were introduced. In fact, by the use of computer screen, many continued fractions with regular pattern can be observed which leads to describing them and theoretically proving. The reader can consult [9], [11], [2], [1], [19] and [20] to discover such examples of hyperquadratic continued fractions with bounded and unbounded degree. We cite here a family of hyperquadratic power series with flat continued fraction expansion introduced by Lasjaunias and Ruch in [10]:

Theorem 1.1. *Let p be an odd prime number and let $s, t \geq 1$ be integers. We put $q := p^s$ and $r := q^t$. Let $u \in \mathbb{F}_q^*$ and $k \geq 0$ be an integer. We assume that $u \neq 2$ and* *put $v := 2 - u$. We define $\gamma \in \mathbb{F}_q((T^{-1}))$ by*

$$\gamma = [0, T^{[k]}, \bigoplus_{i\geq 1}(T, (uT, vT)^{[(r^i-1)/2]})^{[k+1]}]$$

Then γ satisfies the algebraic equation:

$$Q_k X^{r+1} - P_k X^r + (uv)^{(r-1)/2} Q_{k+r} X - (uv)^{(r-1)/2} P_{k+r} = 0,$$

where $(P_n/Q_n)_{n\geq 0}$ is the sequence of convergent of γ.

We also recall that, in a recent paper [12], Lasjaunias and Yao could give a description of a large family, including the historical examples due to Mills and Robbins, of hyperquadratic continued fractions with all partial quotients of degree one in the case of an arbitrary base field of odd characteristic. Although many continued fractions in odd characteristic with flat continued fraction(i.,e all partial quotients are of degree one) are described, until now, we haven't known any continued fraction with bounded and not flat partial quotients. In this work, we will construct the first example of continued fraction, in characteristic 3, with bounded and not flat partial quotients.

2 On the length of the period of the product of some periodic continued fractions by P_k

Before giving our main result we note that if $\lambda \in \mathbb{F}_p^*$ and s is an integer such that $[\underbrace{\lambda, \lambda, \ldots, \lambda}_{s}] = 0$, then $[\underbrace{\lambda, \lambda, \ldots, \lambda}_{s'}]$ is not well defined for all $s' > s$. This is due to the following property of continued fraction:

$$[\underbrace{\lambda, \lambda, \ldots, \lambda}_{s'}] = [\lambda, \ldots, \lambda, [\underbrace{\lambda, \lambda, \ldots, \lambda}_{s}]].$$

This proves the uniqueness of s.

Theorem 2.1. *Let $\alpha = [B, \overline{\lambda T^r}]$, where $B \in \mathbb{F}_p[T]$, $r = p^t$ with $t \geq 1$ and $\lambda \in \mathbb{F}_p^*$. Let β be a quadratic power series such that*

$$\beta = P_k \alpha. \tag{2.5}$$

where $P_k(T) = (T^2 - 1)^k$ and $k \in E(r)$. Let $s \geq 2$ be an integer such that $[\underbrace{\lambda, \lambda, \ldots, \lambda}_{s}] = 0$. Then the length of the periodic of continued fraction expansion of β is equal to $(2k+1)(s-1)+2$.

Proof: We have $\alpha_0 = \alpha$ and $\alpha_n = \alpha_1$ for all $n \geq 1$. The equation (2.5) gives that $\beta = BP_k + \dfrac{P_k}{\alpha_1}$. So $b_0 = BP_k$ and

$$\beta_1 = \frac{\alpha_1}{P_k} \tag{2.6}$$

As $\alpha_1 = \lambda T^r + \dfrac{1}{\alpha_2}$, the equation (2.6) gives that $\beta_1 = \dfrac{\lambda T^r}{P_k} + \dfrac{1}{P_k\alpha_2}$. Following (1.3) and since $\alpha_2 = \alpha_1$ and by applying the Lemma 1.1, this becomes

$$\begin{aligned}\beta_1 &= \lambda A - \frac{\lambda 2k\theta_k Q_k}{P_k} + \frac{1}{P_k\alpha_1} = [[\lambda A, -\delta_1^{-1}u_1T, \ldots, -\delta_1 u_{2k}T], P_k\alpha_1] \\ &= [\lambda A, -\delta_1^{-1}u_1T, \ldots, -\delta_1 u_{2k}T, \beta_{2k+2}]\end{aligned}$$

where

$$\beta_{2k+2} = \frac{\alpha_1}{P_k} + Q_k(\delta_1\omega_k P_k)^{-1}, \tag{2.7}$$

$\delta_1 = \lambda 2k\theta_k$ and $\omega_k = -(2k\theta_k)^{-2}$. Then $b_1 = \lambda A$, $b_2 = -\delta_1^{-1}u_1T$, …, $b_{2k+1} = -\delta_1 u_{2k}T$, and

$$\begin{aligned}\beta_{2k+2} &= \lambda A - \frac{\lambda 2k\theta_k Q_k}{P_k} - \frac{2k\theta_k Q_k}{\lambda P_k} + \frac{1}{P_k\alpha_1} \\ &= \lambda A - \delta_2\frac{Q_k}{P_k} + \frac{1}{P_k\alpha_1}\end{aligned}$$

where $\delta_2 = 2k\theta_k[\lambda, \lambda]$.
Let us define the sequence $(\delta_j)_{2\le j\le s}$ recursively by:

$$\delta_j = 2k\theta_k[\underbrace{\lambda, \ldots, \lambda}_{j}] = 2k\theta_k\lambda - (\delta_{j-1}\omega_k)^{-1},$$

and $\delta_1 = \lambda 2k\theta_k$. We have $\delta_s = 0$ by hypothesis. We prove by induction for all $2 \le j \le s-1$ that

$$\beta_{(2k+1)(j-1)+1} = \frac{\alpha_1}{P_k} + Q_k(\delta_{j-1}\omega_k P_k)^{-1}. \tag{2.8}$$

From (2.7), we have that (2.8) is true for $j = 2$. So we assume (2.8) for $j = l$ then

$$\begin{aligned}\beta_{(2k+1)(l-1)+1} &= \lambda A - \frac{\lambda 2k\theta_k Q_k}{P_k} + \frac{Q_k}{\delta_{l-1}\omega_k P_k} + \frac{1}{P_k\alpha_1} \\ &= \lambda A - \delta_l\frac{Q_k}{P_k} + \frac{1}{P_k\alpha_1} \\ &= [[\lambda A, -\delta_l^{-1}u_1T, \ldots, -\delta_l u_{2k}T], P_k\alpha_1] \\ &= [\lambda A, -\delta_l^{-1}u_1T, \ldots, -\delta_l u_{2k}T, \beta_{(2k+1)l+1}].\end{aligned}$$

where

$$\beta_{(2k+1)l+1} = \frac{\alpha_1}{P_k} + Q_k(\delta_l\omega_k P_k)^{-1}.$$

Thus (2.8) is true for $j = l+1$. By induction, we see that (2.8) holds for all $1 \le j \le s-1$. Furthermore, we get that $b_{(2k+1)(l-1)+1} = \lambda A$, $b_{(2k+1)(l-1)+1+i} =$

$-\delta_l^{(-1)^i} u_i T \quad for \;\; 1 \le i \le 2k.$
So we iterate the process until $j = s$. In this step, the equality (2.8) gives that

$$\begin{aligned} \beta_{(2k+1)(s-1)+1} &= \frac{\alpha_1}{P_k} + Q_k(\delta_{s-1}\omega_k P_k)^{-1} \\ &= \lambda A - \delta_s \frac{Q_k}{P_k} + \frac{1}{P_k \alpha_1}. \end{aligned}$$

As $\delta_s = 0$, we get that $\beta_{(2k+1)(s-1)+1} = \lambda A + \dfrac{1}{P_k \alpha_1}$. So $b_{(2k+1)(s-1)+1} = \lambda A$ and

$$\beta_{(2k+1)(s-1)+2} = P_k \alpha_1 = \lambda T^r P_k + \frac{P_k}{\alpha_1}.$$

Thus $b_{(2k+1)(s-1)+2} = \lambda T^r P_k$ and

$$\beta_{(2k+1)(s-1)+3} = \frac{\alpha_1}{P_k} \tag{2.9}$$

We see that the equation (2.9) has the same shape as (2.6), i.e $\beta_1 = \beta_{(2k+1)(s-1)+3}$ so the length of period of β divides $(2k+1)(s-1)+2$. Furthermore, as $\deg A = r - 2k$, it follows that $\deg b_t \in \{1, r-2k\}$ for all $1 \le t < (2k+1)(s-1)+2$. Since $\deg b_{(2k+1)(s-1)+2} = r + 2k > \deg b_t$ for all $1 \le t < (2k+1)(s-1)+2$, then the length of period of β is equal to $(2k+1)(s-1)+2$. ∎

Example 2.2. *Let* $\omega = [\overline{T}] \in \mathbb{F}_3((T^{-1}))$ *and* $\alpha = 1/\omega$. *Let* $\beta = (T^2-1)\alpha^{3^t}$, *with* $t \ge 1$, *then the periodic length of the continued fraction expansion of* β *is equal to* 8. *In fact, we have here* $k = 1$, $\lambda = 1$ *and since* $[1,1,1] = 0$ *in* $\mathbb{F}_3$ *then* $s = 3$.
Note that this element ω *is actually the analogue, in the formal case, of the celebrated quadratic real number* $[1, 1, \ldots, 1, \ldots] = (1+\sqrt{5})/2$.

Theorem 2.3. *Let* $\alpha = [B, \overline{\lambda_1 T^r, \lambda_2 T^r, \ldots, \lambda_n T^r, C}]$ *periodic of length* $n+1$ *with* $n \ge 2$, *where* $B \in \mathbb{F}_p[T]$, $C \in \mathbb{F}_p^*[T]$, $r = p^t$ *with* $t \ge 1$ *and* $\lambda_i \in \mathbb{F}_p^*$. *Let* β *be a quadratic power series such that*

$$\beta = P_k \alpha. \tag{2.10}$$

where $P_k(T) = (T^2-1)^k$ *and* $k \in E(r)$.

1. *Suppose that there exist* $m+1$ *integers* $n_0, n_1, \ldots, n_m$ *be such that* $n_0 = 1 < n_1 < n_2 < \ldots < n_m = n$ *with* $n_{i+1} - n_i \ge 3$ *for* $0 \le i \le m-1$, *satisfying* $[\lambda_{n_1}, \ldots, \lambda_1] = 0$, $[\lambda_{n_2}, \ldots, \lambda_{n_1+2}] = 0$, …, $[\lambda_{n_i}, \ldots, \lambda_{n_{i-1}+2}] = 0$, …, $[\lambda_n, \ldots, \lambda_{n_{m-1}+2}] = 0$. *Then the period length of* β *divides* $(2k+1)(n-2(m-1)-1)+2m$, *with equality if* $\deg C > r$.

2. *Suppose that* $\inf\{i \ge 0; [\lambda_i, \ldots, \lambda_1] = 0\} = n$, *then the period length of* β *divides* $(2k+1)(n-1)+2$, *with equality if* $\deg C > r - 4k$.

Proof: Let $\beta = [b_0, b_1, \ldots, b_n, \ldots]$. The idea of the proof of the first part of this theorem is similar to the proof of the previous one. So we resume its steps.
We have that $\alpha_1 = \alpha_{n+2}$. The equation (2.10) gives that:
$\beta = BP_k + \dfrac{P_k}{\alpha_1}$. So $b_0 = BP_k$ and

$$\beta_1 = \frac{\alpha_1}{P_k}. \tag{2.11}$$

Since $\alpha_1 = \lambda_1 T^r + \dfrac{1}{\alpha_2}$ then $\beta_1 = \dfrac{\lambda_1 T^r}{P_k} + \dfrac{1}{P_k\alpha_2}$. From (1.3) this becomes

$$\begin{aligned}\beta_1 &= \lambda_1 A - \frac{\lambda_1 2k\theta_k Q_k}{P_k} + \frac{1}{P_k\alpha_2} = [[\lambda_1 A, -\delta_1^{-1}u_1 T, \ldots, -\delta_1 u_{2k} T], P_k\alpha_2] \\ &= [\lambda_1 A, -\delta_1^{-1}u_1 T, \ldots, -\delta_1 u_{2k} T, \beta_{2k+2}]\end{aligned}$$

where

$$\beta_{2k+2} = \frac{\alpha_2}{P_k} + Q_k(\delta_1\omega_k P_k)^{-1},$$

and $\delta_1 = \lambda_1 2k\theta_k$ and $\omega_k = -(2k\theta_k)^{-2}$. Then $b_1 = \lambda_1 A$, $b_2 = -\delta_1^{-1}u_1 T$, ..., $b_{2k+1} = -\delta_1 u_{2k} T$. Put $\delta_{n_1} = 2k\theta_k[\lambda_{n_1}, \ldots, \lambda_1]$. By iteration the processus, we get that

$$\beta_{(2k+1)(n_1-1)+1} = \frac{\alpha_{n_1}}{P_k} + Q_k(\delta_{n_1-1}\omega_k P_k)^{-1} = \lambda_{n_1} A - \frac{\delta_{n_1} Q_k}{P_k} + \frac{1}{P_k \alpha_{n_1+1}}$$

So since $\delta_{n_1} = 0$, then $b_{(2k+1)(n_1-1)+1} = \lambda_{n_1} A$ and

$$\beta_{(2k+1)(n_1-1)+2} = P_k\alpha_{n_1+1} = \lambda_{n_1+1} T^r P_k + \frac{P_k}{\alpha_{n_1+2}}.$$

This gives $b_{(2k+1)(n_1-1)+2} = \lambda_{n_1+1} T^r P_k$ and $\beta_{(2k+1)(n_1-1)+3} = \dfrac{\alpha_{n_1+2}}{P_k}$. Consequently, the continued fraction expansion of β begins with:

$$\begin{aligned}\beta = \big[&BP_k, \lambda_1 A, -\delta_1^{-1}u_1 T, .., -\delta_1 u_{2k} T, .., \lambda_{n_1-1} A, \\ &-\delta_{n_1-1}^{-1} u_1 T, .., -\delta_{n_1-1} u_{2k} T, \lambda_{n_1} A, \lambda_{n_1+1} T^r P_k, \ldots\big]\end{aligned}$$

This shows a "part" of the period being equal to $(2k+1)(n_1-1)+2$.
Put $\delta_{n_i} = 2k\theta_k[\lambda_{n_i}, \ldots, \lambda_{n_{i-1}+2}]$ for $2 \le i \le m$. By recursion, for all i, the condition $\delta_{n_i} = 0$ lead us to get a new bloc of partial quotients in the continued fraction expansion of β of length $(2k+1)(n_i - n_{i-1} - 2) + 2$. So the number of partial quotients until the last step is $(2k+1)(\sum_{i=2}^{m} n_i - n_{i-1} - 2) + n_1 - 1) + 2(m-1) = (2k+1)(n - 2(m-1) - 1) + 2(m-1)$. The final equation that we will get:

$$\begin{aligned}\beta_{(2k+1)(n-2(m-1)-2)+2(m-1)+1} &= \frac{\alpha_{n_m}}{P_k} + Q_k(\delta_{n_{m-1}-1}\omega_k P_k)^{-1} \\ &= \lambda_{n_m} A - \frac{\delta_{n_m} Q_k}{P_k} + \frac{1}{P_k\alpha_{n_m+1}}\end{aligned}$$

So since $\delta_{n_m} = 0$, then $b_{(2k+1)(n-2(m-1)-1)+2(m-1)+1} = \lambda_{n_m} A = \lambda_n A$ and

$$\beta_{(2k+1)(n-2(m-1)-1)+2(m-1)+2} = P_k \alpha_{n_m+1} = CP_k + \frac{P_k}{\alpha_{n+2}}.$$

This gives $b_{(2k+1)(n-2(m-1)-1)+2(m-1)+2} = CP_k$ and

$$\beta_{(2k+1)(n-2(m-1)-1)+2(m-1)+3} = \frac{\alpha_{n+2}}{P_k}. \tag{2.12}$$

As $\alpha_{n+2} = \alpha_1$, we see that the equation (2.12) is of the same kind as the equation (2.11). So the period length of the continued fraction expansion of β divides $(2k+1)(n-2(m-1)-1)+2m$. Furthermore, if $\deg C > r$, the degree of the last partial quotient $b_{(2k+1)(n-2(m-1)-1)+2(m-1)+2}$ of the block of the period will be the greatest. In fact, we have that $\deg b_t \in \{1, r-2k, r+2k\}$ for all $1 \leq t < (2k+1)(n-2(m-1)-1)+2(m-1)+2$. So the period length of β will be equal to $(2k+1)(n-2(m-1)-1)+2m$.
The proof of the second part of the theorem can be deduced directly from the first one. In fact, suppose that $\inf\{i \geq 0; [\lambda_i, \ldots, \lambda_1] = 0\} = n$ is equivalent to taking $n_1 = n$ in the first part. So $m = 1$ and the periodic length of the continued fraction of β divides $(2k+1)(n-1)+2$. For this case, since $\deg b_{(2k+1)(n-1)+2} = \deg C + 2k$, then $\deg b_t \in \{1, r-2k, \deg C + 2k\}$ for all $1 \leq t \leq (2k+1)(n-1)+2$. So if we suppose that $\deg C + 2k > r - 2k$ then the period length of the continued fraction of β will be equal to $(2k+1)(n-1)+2$. ∎

Let $n \geq 2$. We will note by Λ_n the set of quadratic power series α of the form $[B, \overline{\lambda_1 T^r, \lambda_2 T^r, \ldots, \lambda_n T^r, C}]$, where $B \in \mathbb{F}_p[T]$, $C \in \mathbb{F}_p^*[T]$ such that the sequence of integer $\lambda_i \in \mathbb{F}_p^*$ satisfy the following condition: There exists $1 < i \leq n$ such that $[\lambda_i, \lambda_1] = 0$ and $[\lambda_n, \lambda_{i+2}] = 0$. We note such λ_i by λ_{n_m} and we will call m an "intermediate integer". Then we have the following result.

Corollary 2.4. *Let $k \in E(r)$. Then*

$$S(P_k, n) = \sup_{\alpha \in \Lambda_n} Per(P_k \alpha) = (2k+1)(n-1)+2$$

and

$$R(P_k) = \sup_{n \geq 3} \frac{S(P_k, n)}{n} = 2k+1.$$

Proof: Based the previous theorem, the periodic length of the continued fraction expansion of a power series belonging to Λ_n, which equal to $(2k+1)(n-2(m-1)-1)+2m$, depends on the number of the "intermediate integer" m. It is easily checked that we have $(2k+1)(n-2(m-1)-1)+2m \leq (2k+1)(n-1)+2$. So the greatest value of this period is obtained for $m = 1$.
Furthermore, $R(P_k) = \sup_{n \geq 3} \frac{S(P_k, n)}{n} = \sup_{n \geq 3} \frac{(2k+1)(n-1)+2}{n} = 2k+1$. ∎

Remark 2.5. *1. This work gives us infinitely many values of $Per(P_k \alpha)$ for some given periodic continued fraction α. We obtain that these values eventually depend on the degree of P_k.*

2. *Using Frobenius isomorphism, the equations* (2.5) *and* (2.10) *can be regarded as* $\beta = P_k\alpha^r$ *where* $\alpha = [B, \overline{\lambda T}]$ *and* $\alpha = [B, \overline{\lambda_1 T, \lambda_2 T, \ldots, \lambda_n T, C}]$, *and we have the same result of periodicity, simply by replacing* B *by* B^r *and* C *by* C^r *in the continued fraction of* β.

Corollary 2.6. *Let* $\alpha = [B, \overline{u_1 T^r, u_2 T^r, \ldots, u_{2k} T^r, C}]$ *and* $\beta = P_k\alpha$. *Then,*

$$\beta = [BP_k, \overline{u_1 A, -\delta_1^{-1} u_1 T, \ldots, -\delta_1 u_{2k} T, \cdots, u_{2k-1} A, -\delta_{2k-1}^{-1} T, \ldots, -\delta_{2k-1} u_{2k} T, u_{2k} A, CP_k}]$$

where the numbers $\delta_i \in \mathbb{F}_p^*$ *are defined by*

$$\delta_i = 2k\theta_k[u_i, u_{i-1}, \ldots, u_1] \quad for \ \ all \ \ 1 \leq i \leq 2k-1.$$

Proof: From the equality (1.2) and since $P_k(1) = 0$ and $Q_k(1) \neq 0$, we obtain $[u_1, \ldots, u_{2k}] = 0$ and $[u_i, \ldots, u_1] \in \mathbb{F}_p^*$ for $1 \leq i \leq 2k-1$. Then, the equality (1.4) gives that $[u_{2k}, u_{i-1}, \ldots, u_1] = 0$. Hence $\delta_{2k} = 0$ and the result is deduced from the previous Theorem. ∎

We will see how it is possible to give explicitly, up to multiplicative constants, infinitely many continuants of the continued fraction of β satisfying $\beta = P_k\alpha$ where $k \in E(r)$.

Theorem 2.7. *Let* $\alpha = [a_0, a_1, a_2, \ldots]$ *such that* $\deg a_n > 2k$ *for all* n. *Let* $\beta = P_k\alpha = [b_0, b_1, b_2, \ldots]$. *Let* $(U_n, V_n)_n$ *and* $(R_m, S_m)_m$ *be, respectively the continuants of* α *and* β.

1. *If* $V_n(\pm 1) \neq 0$, *then there exists* $m \in \mathbb{N}$, $l \in \mathbb{F}_p^*$ *such that*

$$R_m(T) = lP_k U_n, \ \ S_m(T) = lV_n(T)$$

and in this case $\deg b_{m+1} = \deg a_{n+1} - 2k$.

2. *If* $V_n(1)V_{n+1}(1) \neq 0$ *or* $V_n(-1)V_{n+1}(-1) \neq 0$, *then there exists* $m \in \mathbb{N}$, $l \in \mathbb{F}_p^*$ *such that*

$$R_m(T) = l(U_n(T)V_{n+1}(1) - U_{n+1}(T)V_n(1)),$$
$$S_m(T) = \frac{l(V_n(T)V_{n+1}(1) - V_{n+1}(T)V_n(1))}{T-1}$$

or

$$R_m(T) = l(U_n(T)V_{n+1}(-1) - U_{n+1}(T)V_n(-1)),$$
$$S_m(T) = \frac{l(V_n(T)V_{n+1}(-1) - V_{n+1}(T)V_n(-1))}{T+1}$$

and in this case $\deg b_{m+1} = 1$.

3. *If* P_k *divides* V_n, *then there exists* $m \in \mathbb{N}$, $l \in \mathbb{F}_p^*$ *such that*

$$R_m(T) = lU_n, \ \ S_m(T) = lV_n(T)/P_k$$

and in this case $\deg b_{m+1} = \deg a_{n+1} + 2k$.

Proof:

1. Let $R = lP_kU_n$ and $S = lV_n$ with $l \in \mathbb{F}_p^*$, then R and S are relatively prime polynomials. Moreover

$$|S\beta - R| = |P_kV_n\alpha - P_kU_n| = |P_k||V_n\alpha - U_n| = \\ |P_k||a_{n+1}|^{-1}|V_n|^{-1} = |P_k||a_{n+1}|^{-1}|S|^{-1}$$

As $\deg a_{n+1} > 2k$ then R/S is a convergent to β of accuracy $\deg a_{n+1} - 2k$.

2. Suppose that $V_n(1)V_{n+1}(1) \neq 0$. Let $R = l(U_n(T)V_{n+1}(1) - U_{n+1}(T)V_n(1))$ and $S = l(V_n(T)V_{n+1}(1) - V_{n+1}(T)V_n(1))/(T-1)$. Then

$$\begin{aligned} |S\beta - R| &= |(V_n(T)V_{n+1}(1) - V_{n+1}(T)V_n(1))\alpha \\ &\quad -(U_n(T)V_{n+1}(1) - U_{n+1}(T)V_n(1)| \\ &= |V_{n+1}(1)(V_n(T)\alpha - U_n(T)) - V_n(1)(V_{n+1}(T)\alpha - U_{n+1}(T))| \\ &= |V_n(T)\alpha - U_n(T)| = |V_{n+1}|^{-1} = |T|^{-1}|S|^{-1}. \end{aligned}$$

As R and S are relatively prime, then R/S is a convergent to β of accuracy 1.

3. Let $R = lU_n$ and $S = lV_n/P_k$ with $l \in \mathbb{F}_p^*$. Then R and S are relatively prime polynomials. Moreover

$$|S\beta - R| = |V_n\alpha - U_n| = |a_{n+1}|^{-1}|V_n|^{-1} = |P_k|^{-1}|a_{n+1}|^{-1}|S|^{-1}.$$

Then R/S is a convergent to β of accuracy $\deg a_{n+1} + 2k$. ■

3 On the periodic length of some square root of polynomials

Theorem 3.1. *Let $\alpha \in \mathbb{F}_p((T^{-1}))$ be the solution of strictly positive degree of the equation:*

$$\alpha^2 = (\lambda^2T^{2r} + 1)P_{2k} \tag{3.13}$$

where $\lambda \in \mathbb{F}_p^*$, $r = p^t$ with $t \geq 1$ and $k \in E(r)$. Let $s \geq 2$ be the integer such that $[\underbrace{2\lambda, 2\lambda, \ldots, 2\lambda}_{s}] = 0$. Then the periodic length of the continued fraction expansion of α is equal to $(2k+1)(s-1)+2$, and its continued fraction is:

$$\alpha = [\lambda T^rP_k, \overline{2\lambda A, -\delta_1^{-1}u_1T, \ldots, -\delta_1u_{2k}T, \ldots, 2\lambda A, -\delta_{s-1}^{-1}u_1T, \ldots, -\delta_{s-1}u_{2k}T, 2\lambda A, 2\lambda T^rP_k}], \tag{3.14}$$

where $\delta_j = 2k\theta_k[\underbrace{2\lambda, \ldots, 2\lambda}_{j}]$.

Proof: Let $\gamma = [0, \overline{2\lambda T^r}]$, then, $\gamma = \dfrac{1}{2\lambda T^r + \gamma}$. So γ satisfies the equation:

$$\gamma^2 + 2\lambda T^r \gamma - 1 = 0. \tag{3.15}$$

Let $\alpha = \lambda T^r P_k + P_k \gamma$. Then $|\alpha| > 1$. By replacing γ by $\dfrac{\alpha - \lambda T^r P_k}{P_k}$ in (3.15), we get that α satisfies the equation (3.13). As $|P_k \gamma| < 1$, then the integer part of α is $\lambda T^r P_k$, so $Per(\alpha) = Per(P_k \gamma)$. According to the Theorem 2.1, we obtain the desired result. ■

Example 3.1. *Let $\alpha \in \mathbb{F}_5((T^{-1}))$ be the solution of the equation:*

$$\alpha^2 = (T^{10} + 1)(T^2 - 1)^2.$$

Then $Per(\alpha) = 5$.

In fact, it suffices to apply the previous theorem with $k = \lambda = 1$ and $r = 5$. We give now a result related to Polynomial analogue of McMullen's Conjecture (see Conjecture **M** p. 87 in [13]).

Corollary 3.2. *Let $\lambda \in \mathbb{F}_p^*$ and $D = \lambda^2 T^{2r} + 1 \in \mathbb{F}_p[T]$ with $r = p^t$ with $t \geq 1$. Then for all $l \in E(r)$, there exists $P \in \mathbb{F}_p[T]$ such that*

$$\overline{K}(P\sqrt{D}) = r + 2l.$$

Proof: From the previous Theorem, the formal power series α satisfying the equation (3.13) can be read as $\alpha = P_k \sqrt{D}$. The continued fraction expansion of α is entirely described by (3.14). From the equality (1.3), we have that $\deg A = r - 2k$. So the largest degree of partial quotients of α is $\deg \lambda T^r P_k = r + 2k$. Since $k \in E(r)$, we deduce the desired result. ■

4 Bounded continued fraction expansion

In this paragraph, we let $W^* = a_n, a_{n-1}, \ldots, a_0$, be the word $W = a_0, a_1, \ldots, a_n$ written in reverse order. Further, for $m \in \mathbb{N}$, we write $(a_0, a_1, \ldots, a_n)^{[m]}$ for the sequence obtained by repeating the sequence $a_0, a_1, \ldots, a_n$ m times if $m \geq 1$ and the empty sequence if $m = 0$. If $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ are two such sequences we denote by $a_0, a_1, \ldots, a_n \bigoplus b_0, b_1, \ldots, b_n$ the sequence obtained by juxtaposition.

Theorem 4.1. *Let $\alpha = [\oplus_{i \geq 1}(T^3, (uT^3, (2-u)T^3)^{[\frac{r^i-1}{2}]})] \in \mathbb{F}_9((T^{-1}))$, where $u \in \mathbb{F}_9$ such that $-u^2 + 2u + 1 = 0$. Let $\beta = (T^2 - 1)\alpha$. Then then the continued fraction of β is:*

$$\beta = [\oplus_{i \geq 1}(T^3(T^2-1), (W, uT^3(T^2-1), W^*, -u^{-1}T^3(T^2-1), W)^{[(r^i-1)/8]})],$$

where $W = uT, u^{-1}T, -uT, -u^{-1}T$.

Proof: We take $k = 0$ and $q = r = 9$ in Theorem 1.1. The finite field $\mathbb{F}_9$ elements will be represented by means of a root u of the irreducible polynomial over $\mathbb{F}_3$: $P(X) = -X^2 + 2X + 1$, and then we have $\mathbb{F}_9 = \{0, u^i, 1 \leq i \leq 8\}$.
Let $\gamma = [\oplus_{i\geq 1}(T, (uT, (2-u)T)^{[\frac{r^i-1}{2}]})] \in \mathbb{F}_9((T^{-1}))$. We will apply the equality (1.3) with $k = 1$ and $r = 3$ then we have $T(T^2-1) - T^3 = -T$. Let $\beta = (T^2-1)\gamma(T^3) = (T^2-1)\alpha$. Then

$$\beta = (T^2-1)[(T^3, \underbrace{(uT^3, (2-u)T^3, \ldots, uT^3, (2-u)T^3)}_{r-1=8})^{(1)}, (T^3, \underbrace{(uT^3, (2-u)T^3, \ldots, uT^3, (2-u)T^3)}_{r^2-1=80})^{(2)}, \ldots].$$

We aim at computing the explicit continued fraction of $\beta = [b_0, b_1, \ldots]$.
$\beta = T^3(T^2-1) + \dfrac{(T^2-1)}{\alpha_1}$. So $b_0 = T^3(T^2-1)$ and

$$\beta_1 = \frac{\alpha_1}{(T^2-1)}. \tag{4.16}$$

Since $\alpha_1 = uT^3 + \dfrac{1}{\alpha_2}$ then $\beta_1 = \dfrac{uT^3}{(T^2-1)} + \dfrac{1}{(T^2-1)\alpha_2}$. So, from Lemma (1.1) we obtain

$$\begin{aligned}\beta_1 &= uT + \frac{uT}{(T^2-1)} + \frac{1}{(T^2-1)\alpha_2} = [[uT, u^{-1}T, -uT], (T^2-1)\alpha_2] \\ &= [uT, u^{-1}T, -uT, \beta_4]\end{aligned}$$

where

$$\beta_4 = \frac{\alpha_2}{(T^2-1)} + u^{-1}T(T^2-1)^{-1}.$$

Then $b_1 = uT$, $b_2 = u^{-1}T$, $b_3 = -uT$, and

$$\begin{aligned}\beta_4 &= (2-u)T + \frac{(2-u)T}{(T^2-1)} + u^{-1}T(T^2-1)^{-1} + \frac{1}{(T^2-1)\alpha_3} \\ &= (2-u)T + \delta_1\frac{T}{T^2-1} + \frac{1}{(T^2-1)\alpha_3}\end{aligned}$$

where $\delta_1 = [2-u, u]$.
Since $\delta_1 = 0$ then $b_4 = (2-u)T$, $b_5 = u(T^2-1)T^3$ and $\beta_6 = \alpha_4/(T^2-1)$.

$$\begin{aligned}\beta_6 &= \frac{a_4}{(T^2-1)} + \frac{1}{(T^2-1)\alpha_5} \\ &= \frac{(2-u)T^3}{(T^2-1)} + \frac{1}{(T^2-1)\alpha_5} \\ &= (2-u)T + \frac{(2-u)T}{(T^2-1)} + \frac{1}{(T^2-1)\alpha_5} \\ &= [[(2-u)T, (2-u)^{-1}T, -(2-u)T], (T^2-1)\alpha_5] \\ &= [(2-u)T, (2-u)^{-1}T, -(2-u)T, \beta_9]\end{aligned}$$

where

$$\beta_9 = \frac{\alpha_5}{(T^2-1)} + \frac{(2-u)^{-1}T}{(T^2-1)}.$$

So we get $b_6 = (2-u)T$, $b_7 = (2-u)^{-1}T$, $b_8 = -(2-u)T$ and we have

$$\begin{aligned}
\beta_9 &= \frac{a_5}{(T^2-1)} + \frac{1}{(T^2-1)\alpha_6} + \frac{(2-u)^{-1}T}{(T^2-1)} \\
&= \frac{uT^3}{(T^2-1)} + \frac{1}{(T^2-1)\alpha_6} + \frac{(2-u)^{-1}T}{(T^2-1)} \\
&= uT + \frac{uT}{(T^2-1)} + \frac{(2-u)^{-1}T}{(T^2-1)} + \frac{1}{(T^2-1)\alpha_6} \\
&= uT + \frac{\delta_2 T}{(T^2-1)} + \frac{1}{(T^2-1)\alpha_6}
\end{aligned}$$

where $\delta_2 = [u, 2-u] = 0$.
This gives that $b_9 = uT$ and $\beta_{10} = (T^2-1)\alpha_6 = (T^2-1)a_6 + \dfrac{(T^2-1)}{\alpha_7}$. Hence $b_{10} = (2-u)T^3(T^2-1)$,

$$\begin{aligned}
\beta_{11} &= \frac{\alpha_7}{(T^2-1)} = \frac{a_7}{(T^2-1)} + \frac{1}{(T^2-1)\alpha_8} \\
&= \frac{uT^3}{(T^2-1)} + \frac{1}{(T^2-1)\alpha_8} = uT + \frac{uT}{(T^2-1)} + \frac{1}{(T^2-1)\alpha_8} \\
&= [[uT, u^{-1}T, -uT], (T^2-1)\alpha_8] = [uT, u^{-1}T, -uT, \beta_{14}]
\end{aligned}$$

where

$$\beta_{14} = \frac{\alpha_8}{(T^2-1)} + u^{-1}T(T^2-1)^{-1}.$$

Then $b_{11} = uT$, $b_{12} = u^{-1}T$, $b_{13} = -uT$, and

$$\begin{aligned}
\beta_{14} &= (2-u)T + \frac{(2-u)T}{(T^2-1)} + u^{-1}T(T^2-1)^{-1} + \frac{1}{(T^2-1)\alpha_9} \\
&= (2-u)T + \delta_1 \frac{T}{T^2-1} + \frac{1}{(T^2-1)\alpha_9}
\end{aligned}$$

Since $\delta_1 = 0$ then $b_{14} = (2-u)T$ and $\beta_{15} = (T^2-1)\alpha_9$ which yields to $b_{15} = (T^2-1)T^3$ and $\beta_{16} = \alpha_{10}/(T^2-1)$. So the continued fraction expansion of β begin with the bloc

$$T^3(T^2-1), uT, u^{-1}T, -uT, (2-u)T, uT^3(T^2-1), (2-u)T, (2-u)^{-1}T, \quad (4.17)$$

$-(2-u)T, uT, (2-u)T^3(T^2-1), uT, u^{-1}T, -uT, (2-u)T$.
We note that this bloc of continued fraction of β is the image by product with (T^2-1) of the bloc (1) of α which is

$$T^3, uT^3, (2-u)T^3, uT^3, (2-u)T^3, uT^3, (2-u)T^3, uT^3, (2-u)T^3$$

We have that $(2-u) = 2u^{-1}$ so the bloc (4.17) can be written as

$$T^3(T^2-1), W, uT^3(T^2-1), W^*, -u^{-1}T^3(T^2-1), W$$

where $W = uT, u^{-1}T, -uT, -u^{-1}T$. Note that the bloc $W, uT^3(T^2-1), W^*, -u^{-1}T^3(T^2-1), W$ is the image of the bloc $(uT^3, (2-u)T^3)^{[4]}$, then the image of the bloc $(uT^3, (2-u)T^3)^{[40]}$ is $(W, uT^3(T^2-1), W^*, -u^{-1}T^3(T^2-1), W)^{[10]}$ and then the image of the bloc (2) is equal to $T^3(T^2-1), (W, uT^3(T^2-1), W^*, -u^{-1}T^3(T^2-1), W)^{[10=\frac{9^2-1}{8}]}$. So by recursion, we prove that the image of the bloc $T, (uT^3, (2-u)T^3)^{[(r^i-1)/2]}$ is equal to $T^3(T^2-1), (W, uT^3(T^2-1), W^*, -u^{-1}T^3(T^2-1), W)^{[(r^i-1)/8]}$. So we obtain the desired result. ∎

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