# A characterization of alternatively convex or smooth Banach spaces

H. Espid R. Alizadeh*

**Abstract**

In this paper, we give a characterization of alternatively convex or smooth Banach spaces. In fact we prove that every normaloid numerical radius attaining operator on a Banach space $X$ is radialoid if and only if $X$ is alternatively convex or smooth. In addition, we show that every compact normaloid operator on $X$ is radialoid if and only if every rank one normaloid operator on X is radialoid. Finally we present some types of Banach spaces on which the compact normaloid operators are radialoid.

## 1 Introduction

The notion of normaloid operators was introduced by Wintner [12], and its features have been considered by many authors. An operator is normaloid if its norm is equal to its numerical radius and it is radialoid if its norm is equal to its spectral radius. It is obvious that every radialoid operator is normaloid. Wintner [12] proved that the converse is true for the operators which act on a Hilbert space. Lumer extended this result by showing that on every uniformly convex Banach space every normaloid operator is radialoid [2, Corollary 10.7]. Also normaloid numerical radius attaining operators on a strictly convex Banach space are radialoid [2, Theorem 10.8]. The subject of normaloid operators on a Banach space has a close relation with Daugavet and alternative Daugavet equations which are useful in approximation theory for finding the best constants in

*Corresponding author.
Received by the editors in April 17 - In revised form in August 17.
Communicated by G. Godefroy.
2010 *Mathematics Subject Classification* : 46B20, 47A12.
*Key words and phrases* : rotundity, smoothness, acs spaces, numerical range.

some inequalities [10]. The normaloid operators are important tools in studying the geometry of Banach spaces. Especially, the Banach spaces on which all operators are normaloid and those Banach spaces on which all rank one operators are normaloid are interesting topics in the geometry of Banach spaces.

Let $X$ be a real or complex Banach space, $B_X$ be its closed unit ball, $S_X$ be its unit sphere and as usual $X^*$ denote its dual. An operator $T \in B(X)$, the algebra of bounded linear operators on $X$, satisfies *Daugavet equation* [3] if $\|I+T\| = 1+\|T\|$ and it satisfies *alternative Daugavet equation*, if

$$\|I + e^{i\theta}T\| = 1 + \|T\|, \text{ for some } 0 \leq \theta < 2\pi,$$

where $I$ denotes the identity operator on $X$. It is well known that $T$ is normaloid if and only if it satisfies alternative Daugavet equation (see [7, Lemma 2.3] for an explicit proof). A Banach space $X$ has *Daugavet property* for a class $\mathbf{M}$ of operators in $B(X)$, if Daugavet equation holds for all operators in $\mathbf{M}$. If $X$ has Daugavet property for rank one operators, then it is said that $X$ has Daugavet property. For a survey of Daugavet property see [8, Chapter 6], [11] and the references therein. A Banach space $X$ has *anti-Daugavet property* for a class $\mathbf{M}$ of operators if for every $T \in \mathbf{M}$, the equivalence

$$\|I+T\| = 1+\|T\| \iff \|T\| \in \sigma(T) \tag{1.1}$$

holds. Here $\sigma(T)$ denotes the spectrum of $T$. We say that $X$ has *anti-alternative Daugavet property* for a class $\mathbf{M}$ of operators if for every $T \in \mathbf{M}$, the equivalence

$$\max_{0\leq\theta\leq2\pi} \|I + e^{i\theta}T\| = 1 + \|T\| \iff \rho(T) = \|T\| \tag{1.2}$$

holds, where $\rho(T)$ denotes the spectral radius of $T$. If $B(X) = \mathbf{M}$ in (1.1) and (1.2), it is said that $X$ has anti-Daugavet and anti-alternative Daugavet property respectively. Clearly $X$ has anti-alternative Daugavet property for a class $\mathbf{M}$ of operators if and only if every normaloid operator $T \in \mathbf{M}$ is radialoid. It is easy to see that if $X$ has anti-Daugavet property for a class $\mathbf{M}$ of operators which is closed under scalar multiplication, then $X$ has anti-alternative Daugavet property for $\mathbf{M}$. We will show that the converse is true, when $X$ is a finite dimensional Banach space and $\mathbf{M} = B(X)$. A Banach space $X$ is called *alternatively convex or smooth* [5, Definition 3.1] and [6, Definition 4.1] if for all $x, y \in S_X$ and $f \in S_{X^*}$ the implication

$$f(x) = 1,\ \|x+y\| = 2 \Rightarrow f(y) = 1$$

holds. More precisely, $X$ is alternatively convex or smooth if and only if every two points on a segment of $S_X$ have the same support functionals. Kadets [5, Theorem 3.2] proved that a finite dimensional Banach space has anti-Daugavet property if and only if it is alternatively convex or smooth. Hardtke [4] studied the absolute sums of these Banach spaces.

A pair $(x, f) \in S_X \times S_{X^*}$ is a dual pair if $f(x) = 1$. For every $x \in S_X$ and $f \in S_{X^*}$, we define

$$D(x) = \{g : (x, g) \text{ is a dual pair}\},$$

$$D(f) = \{y : (y, f) \text{ is a dual pair}\}.$$

By the Hahn-Banach theorem, $D(x)$ is not an empty set. The elements of $D(x)$ are called support functionals of $x$. For every $T$ in $B(X)$ the spatial numerical range and numerical radius of $T$ are defined as follows respectively:

$$W(T) = \{f(Tx) : (x, f) \text{ is a dual pair}\},$$

$$r(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

It is clear that $\rho(T) \leq r(T) \leq \|T\|$. An operator $T \in B(X)$ is numerical radius attaining, if there exists a dual pair $(x_0, f_0)$ such that $r(T) = |f_0(Tx_0)|$. We denote by $NRA(X)$ all operators in $B(X)$ that attain their numerical radius. $T$ is normaloid if $r(T) = \|T\|$ and it is radialoid if $\rho(T) = \|T\|$.

In this paper, we prove that $X$ has anti-alternative Daugavet property for $NRA(X)$ if and only if $X$ is alternatively convex or smooth. In addition, we show that every compact normaloid operator on $X$ is radialoid if and only if every rank one normaloid operator on X is radialoid. Finally we present some types of Banach spaces on which the compact normaloid operators are radialoid.

## 2 Anti-alternative Daugavet property for Banach spaces

As we said, normaloid numerical radius attaining operators on a strictly convex Banach space are radialoid [2, Theorem 10.8]. The following theorem gives a characterization of alternatively convex or smooth Banach spaces. Before stating this theorem, we note that in this section, $K(X)$ and $\mathbf{F}_1(X)$ are the set of all compact and rank one operators acting on $X$ respectively and for every pair $(x, f)$ in $X \times X^*$, the rank one operator $T = x \otimes f$ is defined by $T(.) = f(.)x$.

**Theorem 2.1.** Let $X$ be a Banach space. Then the following expressions are equivalent:

(i) $X$ has anti-alternative Daugavet property for $NRA(X)$.

(ii) $X$ has anti-alternative Daugavet property for $\mathbf{F}_1(X) \bigcap NRA(X)$.

(iii) $X$ is alternatively convex or smooth.

*Proof*. Clearly we have $(i) \to (ii)$.

$(ii) \to (iii)$ Suppose (iii) is not satisfied. Then there exist $x, y \in S_X$ and functionals $f, h \in S_{X^*}$ such that $f(x) = f(y) = h(x) = 1$ and $h(y) \neq 1$. Let $0 < \lambda_0 < 1$ and $g = \lambda_0 f + (1 - \lambda_0)h$. Setting $T = y \otimes g$, we have $\|T\| = 1$, $\rho(T) = |g(y)| < 1$ and $f(T(x)) = g(x)f(y) = 1$. Therefore $T$ is a normaloid operator in $NRA(X)$ which is not radialoid.

$(iii) \to (i)$ Let $T$ be a normaloid operator in $NRA(X)$. Without loss of generality we can suppose that $\|T\| = 1$. Since $T$ is numerical radius attaining, there is a dual pair $(x, f)$ such that $|f(Tx)| = 1$. Substituting $T$ with $\frac{T}{f(Tx)}$, we can suppose that $f(Tx) = 1$. If $Tx = x$ or $T^*f = f$, then $T$ will be radialoid and there is nothing to prove. Suppose that $Tx \neq x$ and $T^*f \neq f$. Then $(x, f)$, $(Tx, f)$ and $(x, T^*f)$ are distinct dual pairs. Since both $x$ and $Tx$ are in the segment $D(f)$, by our assumption, we should have $D(x) = D(T(x))$, and so $(T^*f)(Tx) = 1$. Therefore $(T^2x, f)$ is a dual pair and $\|T^2\| = 1$. Now if $(T^2)^*f = f$, then $\rho(T)^2 = \rho(T^2) = \rho((T^2)^*) = 1$, and $T$ will be radialoid. Otherwise, we have $(T^2)^*f \neq f$ and so $(x, f)$, $(Tx, f)$ and $(x, (T^2)^*f)$ are distinct dual pairs. A similar argument shows that $(T^3x, f)$ is a dual pair and $\|T^3\| = 1$. In general, either there exists a step $k$ such that $(T^k)^*(f) = f$, which implies that $T$ is radialoid or for every positive integer $k$ the equality $\|T^k\| = 1$ holds. In the second case, we have

$$\rho(T) = \lim_{k\to\infty}\|T^k\|^{\frac{1}{k}} = 1,$$

and again $T$ will be radialoid. ■

The following corollary is an immediate result of Theorem 2.1 and [5, Theorem 3.2].

**Corollary 2.2.** Let $X$ be a finite dimensional Banach space. Then the following expressions are equivalent:

(i) $X$ has anti-Daugavet property.

(ii)$X$ has anti-alternative Daugavet property.

(iii) $X$ is alternatively convex or smooth.

We should emphasize that attaining the numerical radius condition in Theorem 2.1 cannot be dropped. In the following, we give an example of a strictly convex Banach space which satisfies Kadec-Klee property and there exists a rank one normaloid operator on it which is not radialoid. We note that in [1, Example 2.8] the authors constructed a non-compact bounded linear operator $T$ on a locally uniformly convex reflexive Banach space which satisfies the Daugavet equation (and consequently is normaloid) but $T^2 = 0$ and so it is not radialoid. However, in a locally uniformly convex Banach space, the corresponding operator cannot be chosen to be compact (see Theorem 2.4).

**Example 2.3.** Consider the Banach Space $\ell^1$, the space of absolutely summable sequences and for $x = (x_1, x_2, x_3, ...) \in \ell^1$, set $x'$=$(0, x_2, x_3, ...)$. Define the equivalent norm $\|x\|_M = max(|x_1|, \|x'\|_1)$ on $\ell^1$. Let $I : \ell^1 \to \ell^2$ be the inclusion mapping and for $x \in \ell^1$ define

$$\|x\|_H = (\|x\|_M^2 + \|Ix\|_W^2)^{1/2},$$

where $\|x\|_W$ is defined on $\ell^2$ as follows

$$\|x\|_W = (\|x\|_S^2 + \|Tx\|_2^2)^{\frac{1}{2}}.$$

Here $\|x\|_S = max((|x_1|, \|x'\|_2))$, and $T$ is defined on $\ell^2$ by

$$T(x_1, x_2, x_3, x_4, ...) = (x_1, x_2, x_3/3, x_4/4, ...).$$

Then $(\ell^1, \|.\|_H)$ is a strictly convex Banach space which satisfies Kadec-Klee property [9, Example 5]. Now setting $x = (0, \sqrt{3}/3, 0, ...)$, $x_n = (\sqrt{3}/3, 0, ..., 0, \underbrace{\sqrt{3}/3}_{nth\text{ place}}, 0, ...)$ and $f = (\sqrt{3}, 0, ...)$, we have

$(i)$ $f(x) = 0$ and $f(x_n) = 1$, for $n = 1, 2, \cdots,$

$(ii)$ $\|x\|_H = 1$ and $\|x_n\|_H \to 1$,

$(iii)$ $\|x + x_n\|_H \to 2$,

$(iv)$ $\|f\| = f(\sqrt{3}/3, 0, ...) = 1$.

Setting $K = x \otimes f$, we obtain

$$2 \geq \|I + K\| \geq \frac{\|x_n + x\|_H}{\|x_n\|_H} \to 2.$$

Hence $\|I + K\| = 2$ and $K$ will be normaloid but since $\rho(K) = |f(x)| = 0$ it is not radialoid. ■

In [6, Definition 4.1] the authors defined locally uniformly alternatively convex or smooth (luacs) Banach spaces. A Banach space $X$ is luacs if for all $x_n, y \in S_X$ and $f \in S_{X^*}$ the implication

$$f(x_n) \to 1, \ \|x_n + y\| \to 2 \Rightarrow f(y) = 1$$

holds. They proved that a Banach space $X$ is luacs if and only if $X$ has the anti-Daugavet property for compact or equivalently rank one operators [6, Theorem 4.3]. Motivated by this, we define the concept of *quasi locally uniformly alternatively convex or smooth* (qluacs) Banach spaces. A Banach space $X$ is qluacs if for all $x_n, y \in S_X$ and $f \in S_{X^*}$ the implication

$$f(x_n) \to 1, \ \|x_n + y\| \to 2 \Rightarrow |f(y)| = 1$$

holds. Clearly if $X$ is luacs then it will be qluacs but the converse is an open question for us. Now we state a version of [6, Theorem 4.3] for qluacs Banach spaces.

**Theorem 2.4.** Let $X$ be a Banach space. Then the following expressions are equivalent:

(i) $X$ has anti-alternative Daugavet property for $K(X)$.

(ii) $X$ has anti-alternative Daugavet property for $\mathbf{F}_1(X)$.

(iii) $X$ is qluacs.

*Proof*. The implication $(i) \to (ii)$ is evident.

$(ii) \to (iii)$ Suppose $(iii)$ is not satisfied. Then there exist $x_n, y \in S_X$ and functional $f \in S_{X^*}$ such that $f(x_n) \to 1$, $\|x_n + y\| \to 2$ and $|f(y)| < 1$. Now the operator $T = y \otimes f$ satisfies alternative Daugavet equality. Therefore $T$ is a normaloid operator which is not radialoid.

$(iii) \to (i)$ Let $T$ be a compact normaloid operator in $B(X)$ and $\|T\| = 1$. There exists a sequence $\{x_n\}$ in $S_X$ and $0 \leq \theta < 2\pi$ such that $\|x_n + e^{i\theta}Tx_n\| \to 2$. Since $T$ is compact, passing through a subsequence we can suppose that $\|Tx_n - x\| \to 0$, for some $x \in B_X$. Hence $\|y_n + x\| \to 2$, where $y_n = e^{-i\theta}x_n$ and so $x \in S_X$. Now choose $f \in S_{X^*}$ in such a way that $f(x) = 1$ and set $g_1 = e^{i\theta}T^*f$. Then $g_1(y_n) \to 1$ and by our assumption we have $|g_1(x)| = 1$. Setting $\phi_1 = \mathrm{Arg}(g_1(x))$, $(x, f), (x, e^{-i\phi_1}g_1), (e^{i(\theta-\phi_1)}Tx, f)$ are dual pairs and $(iii)$ implies that

$$1 = |e^{-i\phi_1}g_1(e^{i(\theta-\phi_1)}Tx)| = |f(T^2x)|.$$

Therefore $\|T^2\| = 1$. Considering $g_2 = (T^*)^2f$, $\phi_2 = \mathrm{Arg}(g_2(x))$ and continuing the above argument with dual pairs $(x, f), (x, e^{-i\phi_1}g_1), (e^{-i\phi_2}T^2x, f)$ we conclude that $\|T\|^3 = 1$. In general by induction we can prove that for every positive integer $k$, the equality $\|T^k\| = 1$ holds and so $T$ is radialoid. ■

Note that weakly locally uniformly convex and uniformly Gâteaux smooth Banach spaces are luacs [4, Figs. 3 and 4] and so qluacs. In addition, by [4, Proposition 2.29] every reflexive Fréchet smooth Banach space is luacs. Also if $X^*$ is Fréchet smooth then $X$ will be luacs. To see this let $x_n, x \in S_X$, $f \in X^*$ and

$$f(x_n) \to 1,\ \|x_n + x\| \to 2.$$

Since $X^*$ is Fréchet smooth and $f(x_n) \to 1$, we conclude that there exists $y \in S_X$ such that $\|y - x_n\| \to 0$. Therefore $\|y + x\| = 2$ and strict convexity of $X$ implies that $f(x) = 1$.

We refer to [4] for various generalizations of rotundity and smoothness properties for Banach spaces and their relationships.

## Acknowledgments

The authors are very grateful to the reviewer for carefully reading the paper and for his/her constructive comments. We would also like to sincerely thank Prof. Dirk Werner for his suggestions which have improved the paper.

## References

[1] Y. A. Abramovich, C. D. Aliprantis, O. Burkinshaw, The Daugavet equation in uniformly convex Banach spaces, Journal of Functional Analysis, 97(1991) 215-230.

[2] F. F. Bonsall, J. Duncan, Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras, Cambridge university press, Cambridge (1971).

[3] I. K. Daugavet, On a property of compact operators in the space $C$, Uspekhi Mat. Nauk 18, No. 5 (1963) 157-158. [In Russian]

[4] Jan-David Hardtke, Absolute sums of Banach spaces and some geometric properties related to rotundity and smoothness, Banach J. Math. Anal. 8, No. 1 (2014) 295-334.

[5] V. M. Kadets, Some remarks concerning the Daugavet equation, Quaestiones Math. 19 (1996) 225-235.

[6] V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin, D. Werner, Banach spaces with the Daugavet property, Trans. Amer. Math. Soc. 352 (2000) 855-873.

[7] M. Martin, T. Oikhberg, *An alternative Daugavet property*, J. Math. Anal. Appl. 294 (2004) 158-180.

[8] M. Popov, B. Randrianantoanina, Narrow Operators On Function Spaces and Vector Lattices, De Gruyter Studies in Mathematics 45, Walter de Gruyter, Berlin-New York, 2013.

[9] M. A. Smith, Some Examples Concerning Rotundity in Banach Spaces, Math. Ann. 233 (1978) 155-161.

[10] S. B. Stečkin, On approximation of continuous periodic functions by Favard sums, Proc. Steklov Inst. Math. 109 (1971) 28-38.

[11] D. Werner, Recent progress on the Daugavet property, Irish Math. Soc. Bull. 46 (2001) 77-97.

[12] A. Wintner, Spectral theorie der unendlichen Matrizen: Einf*ü*hrung in den analytischen Apparat der Quantenmechanik, S. Hirzel, Leipzig, (1929) 34-37.

Department of Mathematics, Shahed University
P.O.Box:18151-159, Tehran, Iran
E-mail: alizadeh@shahed.ac.ir