Dynamics of linear operators on non-Archimedean vector spaces

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Abstract

In the present paper we study dynamics of linear operators defined on topological vector space over non-Archimedean valued fields. We give sufficient and necessary conditions of hypercyclicity (resp. supercyclicity) of linear operators on separable F-spaces. It is proven that a linear operator T on topological vector space X is hypercyclic (supercyclic) if it satisfies Hypercyclicity (resp. Supercyclicity) Criterion. We consider backward shifts on $c_0(\mathbb{Z})$ and $c_0(\mathbb{N})$, respectively, and characterize hypercyclicity and supercyclicity of such kinds of shifts. Finally, we study hypercyclicity, supercyclicity of operators $\lambda I + \mu B$, where I is identity and B is backward shift. We note that there are essential differences between the non-Archimedean and real cases.

1 Introduction

Linear dynamics is a young and rapidly evolving branch of functional analysis, which was probably born in 1982 with the Toronto Ph.D. thesis of C. Kitai [10]. It has become rather popular, thanks to the efforts of many mathematicians (see [5, 6]). In particular, hypercyclicity and supercyclicity of weighted bilateral shifts were characterized by Salas [16, 17]. In [18] Shkarin proved the existence of a bounded linear operator T satisfying the Kitai Criterion on each separable infinite-dimensional Banach space. For more detailed information about cyclic, hypercyclic linear operators we refer to [1].

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We stress that all investigations on dynamics of linear operators were considered over the field of the real or complex numbers. On the other hand, non-Archimedean functional analysis is well-established discipline, which was developed in Monna's series of works in 1943. Last decades there have been published a lot of books devoted to the non-Archimedean functional analysis (see for example [14, 19]). In [12] a Non-Archimedean spectral theorem has been recently developed for normal operator linear operators on non-Archimedean Banach spaces.

In the present paper, we are going to study dynamics of linear operators defined on topological vector space over non-Archimedean valued fields. In section 3, we will show that there does not exist any hypercyclic operator on a finite

dimensional space. Moreover, we give sufficient and necessary conditions of hypercyclicity (supercyclicity) of linear operators on separable F-spaces Theorem 3.2 (resp. Theorem 3.12). We will show that a linear operator T on topological vector space X is hypercyclic (supercyclic) if it satisfies Hypercyclic (resp. Supercyclic) Criterion. Note that the shift operators have many applications in many branches of modern mathematics (in real setting). In [13] the p-adic counterpart of the shift operator has been studied and established that some properties of these operators are parallel to the classical patterns, others are quite different. For example, the lattice of invariant subspaces of the non-Archimedean unilateral shift is indexed by polynomials (in the classical Beurling theorem this role is played by inner functions), the operator itself can be seen also as an analog of the Volterra integration operator whose properties are (classically) very far from those of the unilateral shift. Moreover, the non-Archimedean shift operators have certain applications in p-adic dynamical systems [8, 9]. These investigations motivate us to consider weighted shifts (which are more general). Therefore, in section 4 we study weighted backward shifts on $c_0(\mathbb{Z})$ and $c_0(\mathbb{N})$ spaces, respectively, and characterize hypercyclicity and supercyclicity of such kinds of operators. In section 5, we will consider an operator $\lambda I + \mu B$, where I is the identity operator and B is the backward shift. We prove that if $|\mu| \leq 1$ then the operator $I + \mu B$ cannot be hypercyclic while in the real case this operator is hypercyclic when $\mu \neq 0$ (see [18]). This is an essential difference between the non-Archimedean and real cases. Our investigations will open further investigations of non-Archimedean analogous of Volterra integration operators.

2 Definitions and preliminary results

All fields appearing in this paper are commutative. A valuation on a field $\mathbb{K}$ is a map $|\cdot| : \mathbb{K} \to [0, +\infty)$ such that:

(i) $|\lambda| = 0$ if and only if $\lambda = 0$,

(ii) $|\lambda\mu| = |\lambda| \cdot |\mu|$ (multiplicativity),

(iii) $|\lambda + \mu| \leq |\lambda| + |\mu|$ (triangle inequality), for all $\lambda, \mu \in \mathbb{K}$.

The pair $(\mathbb{K}, |\cdot|)$ is called a *valued field*. We often write $\mathbb{K}$ instead of $(\mathbb{K}, |\cdot|)$.

Definition 2.1. *Let $\mathbb{K} = (\mathbb{K}, |\cdot|)$ be a valued field. If $|\cdot|$ satisfies the strong triangle inequality: (iii') $|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$, for all $\lambda, \mu \in \mathbb{K}$, then $|\cdot|$ is called non-Archimedean, and $\mathbb{K}$ is called a non-Archimedean valued field*

Remark 2.1. In what follows, we always assume that a norm in non-Archimedean valued field is nontrivial.

From the strong triangle inequality we get the following useful property of non-Archimedean value: If $|\lambda| \neq |\mu|$ then $|\lambda \pm \mu| = \max\{|\lambda|, |\mu|\}$. We frequently use this property, and call it as *the non-Archimedean norm's property*. A non-Archimedean valued field $\mathbb{K}$ is a metric space and it is called *ultrametric space*.

Let $a \in \mathbb{K}$ and $r > 0$. The set

$$B(a, r) := \{x \in \mathbb{K} : |x - a| \leq r\}$$

is called the *closed ball with radius r about a*. (Indeed, $B(a, r)$ is closed in the induced topology). Similarly,

$$B(a, r^-) := \{x \in \mathbb{K} : |x - a| < r\}$$

is called the *open ball with radius r about a*.

We set $|\mathbb{K}| := \{|\lambda| : \lambda \in \mathbb{K}\}$ and $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$, the *multiplicative group* of $\mathbb{K}$. Also, $|\mathbb{K}^\times| := \{|\lambda| : \lambda \in \mathbb{K}^\times\}$ is a multiplicative group of positive real numbers, the *value group* of $\mathbb{K}$.

Lemma 2.2. (*Lemma 1.4 [19]*) *Let $\mathbb{K}$ be a non-Archimedean valued field. Then the value group of $\mathbb{K}$ either is dense or is discrete; in the latter case there is a real number $0 < r < 1$ such that $|\mathbb{K}^\times| = \{r^s : s \in \mathbb{Z}\}$.*

Example 2.1. Let us provide some examples of non-Archimedean fields (we refer a reader to [15] for more information).

1. Let $\mathbb{Q}$ be the field of rational numbers. For a fixed prime number p, every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, and n and m are relatively prime with p: $(p, n) = 1$, $(p, m) = 1$. The p-adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is denoted by $\mathbb{Q}_p$, and it is called the *field of p-adic numbers*. One can see that $|\mathbb{Q}_p^\times| = \{p^s : s \in \mathbb{Z}\}$.

2. Let $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$ with respect to the extension of the absolute value $|\cdot|_p$. For this field $|\mathbb{C}_p^\times|$ is dense in $\mathbb{R}$. The defined field is called *p-adic complex field*.

Definition 2.3. *A pair $(E, \|\cdot\|)$ is called a $\mathbb{K}$-normed space over $\mathbb{K}$, if E is a $\mathbb{K}$-vector space and $\|\cdot\|$: $E \to [0, +\infty)$ is a non-Archimedean norm, i.e.*
(i) $\| \boldsymbol{x} \| = \boldsymbol{0}$ if and only if $\boldsymbol{x} = \boldsymbol{0}$,
(ii) $\| \lambda \boldsymbol{x} \| = |\lambda| \, \| \boldsymbol{x} \|$,
(iii) $\| \boldsymbol{x} + \boldsymbol{y} \| \leq \max\{\| \boldsymbol{x} \|, \| \boldsymbol{y} \|\}$, for all $\boldsymbol{x}, \boldsymbol{y} \in E$, $\lambda \in \mathbb{K}$.

We frequently write E instead of $(E, \|\cdot\|)$. Moreover, E is called a $\mathbb{K}$-Banach space or a Banach space over $\mathbb{K}$ if it is complete with respect to the induced ultrametric $d(\mathbf{x}, \mathbf{y}) = \| \mathbf{x} - \mathbf{y} \|$.

Example 2.2. Let $\mathbb{K}$ be a non-Archimedean valued field; then

$$l_\infty := \text{all bounded sequences on } \mathbb{K}$$

with pointwise addition and scalar multiplication and the norm

$$\| \mathbf{x} \|_\infty := \sup_n |x_n|$$

is a $\mathbb{K}$-Banach space.

Remark 2.2. From now on we often drop the prefix "$\mathbb{K}$"- and write vector space, normed space, Banach space instead of $\mathbb{K}$-vector space, $\mathbb{K}$-normed space, $\mathbb{K}$-Banach space, respectively.

In what follows, we need the following auxiliary fact.

Lemma 2.4. *Let E be a normed space over a non-Archimedean valued field $\mathbb{K}$. Then for each pair of sequences $(\boldsymbol{x}_n)$ and $(\boldsymbol{y}_n)$ in E such that $\| \boldsymbol{x}_n \| \cdot \| \boldsymbol{y}_n \| \to 0$ as $n \to \infty$ there exists a sequence $(\lambda_n) \subset \mathbb{K}^\times$ such that*

$$\lambda_n \boldsymbol{x}_n \to \boldsymbol{0} \quad \textit{and} \quad \lambda_n^{-1} \boldsymbol{y}_n \to \boldsymbol{0}, \quad \textit{as } \ n \to \infty. \tag{1}$$

Proof. First, we will prove the lemma for the case when the value group of $\mathbb{K}$ is a discrete. Then according to Lemma 2.2 there exists a real number $r \in (0,1)$ such that $|\mathbb{K}^\times| = \{r^s : s \in \mathbb{Z}\}$. Let (n_k) and (m_k) be the increasing subsequences of $\mathbb{N}$ with $(n_k) \cup (m_k) = \mathbb{N}$ such that

$$\|\mathbf{x}_{n_k}\| \cdot \|\mathbf{y}_{n_k}\| = \mathbf{0}, \ \ \|\mathbf{x}_{m_k}\| \cdot \|\mathbf{y}_{m_k}\| \neq \mathbf{0}, \ \forall k$$

Let us define $\nu_{n_k} \in \mathbb{K}$ as follows

$$|\nu_{n_k}| = \begin{cases} 1, & \text{if } \ \mathbf{x}_{n_k} = \mathbf{y}_{n_k} = \mathbf{0}; \\ \frac{\|\mathbf{y}_{n_k}\|}{r^{n_k}}, & \text{if } \ \mathbf{x}_{n_k} = \mathbf{0}, \ \mathbf{y}_{n_k} \neq \mathbf{0}; \\ \frac{r^{n_k}}{\|\mathbf{x}_{n_k}\|}, & \text{if } \ \mathbf{x}_{n_k} \neq \mathbf{0}, \ \mathbf{y}_{n_k} = \mathbf{0}, \end{cases}$$

Since $0 < r < 1$, for any $\varepsilon > 0$ there exists a positive integer k' such that $\|\nu_{n_k} \mathbf{x}_{n_k}\| < \varepsilon$ and $\|\nu_{n_k}^{-1} \mathbf{y}_{n_k}\| < \varepsilon$ for any $k > k'$.

On the other hand, according to Lemma 2.2, there exists a sequence of integer numbers (α_{m_k}) such that

$$r^{2\alpha_{m_k}} \leq \frac{\| \mathbf{y}_{m_k} \|}{\| \mathbf{x}_{m_k} \|} \leq r^{2\alpha_{m_k} - 2}. \tag{2}$$

For any $k \geq 1$ we take $\mu_{m_k} \in \mathbb{K}^\times$ such that $|\mu_{m_k}| = r^{\alpha_{m_k}}$. Then from (2) we get

$$\begin{aligned} &\| \mu_{m_k} \mathbf{x}_{m_k} \| = r^{\alpha_{m_k}} \| \mathbf{x}_{m_k} \| \leq \| \mathbf{x}_{m_k} \|^{\frac{1}{2}} \cdot \| \mathbf{y}_{m_k} \|^{\frac{1}{2}}, \\ &\| \mu_{m_k}^{-1} \mathbf{y}_{m_k} \| = r^{-\alpha_{m_k}} \| \mathbf{y}_{m_k} \| \leq r^{-1} \| \mathbf{x}_{m_k} \|^{\frac{1}{2}} \cdot \| \mathbf{y}_{m_k} \|^{\frac{1}{2}}, \end{aligned}$$

Since $\| \mathbf{x}_{m_k} \| \cdot \| \mathbf{y}_{m_k} \| \to 0$, for any $\varepsilon > 0$ there exists a positive integer $k'' > 0$ such that $\|\mu_{m_k}\mathbf{x}_{m_k}\| < \varepsilon$ and $\|\mu_{m_k}^{-1}\mathbf{y}_{m_k}\| < \varepsilon$ for any $k > k''$.

Finally, we define a sequence $\{\lambda_n\}$ as follows:

$$\lambda_n = \begin{cases} \nu_n, & if \ n \in (n_k) \\ \mu_n, & if \ n \in (m_k) \end{cases}.$$

Then for any $\varepsilon > 0$ one has $\|\lambda_n\mathbf{x}_n\| < \varepsilon$ and $\|\lambda_n^{-1}\mathbf{y}_n\| < \varepsilon$ for any $n > \max\{n_{k'}, m_{k''}\}$.

Now, we suppose that value group of $\mathbb{K}$ is dense. Then we can find sequences $(\mathbf{x}'_n)$ and $(\mathbf{y}'_n)$ such that

$$\|\mathbf{x}'_n\| > \|\mathbf{x}_n\|, \ \ \|\mathbf{y}'_n\| > \|\mathbf{y}_n\|, \ \|\mathbf{x}'_n\| \cdot \|\mathbf{y}'_n\| < \|\mathbf{x}_n\| \cdot \|\mathbf{y}_n\| + \frac{1}{n}$$

It is clear that $\|\mathbf{x}'_n\| \cdot \|\mathbf{y}'_n\| \to 0$ as $n \to \infty$. Fix a $a \in \mathbb{K}^\times$ with $|a| > 1$ Then there exists a sequence (β_n) such that

$$|a|^{\beta_n} \leq \sqrt{\frac{\| \mathbf{y}'_n \|}{\| \mathbf{x}'_n \|}} \leq |a|^{\beta_n+1}$$

Define a sequence $\lambda_n := a^{\beta_n}$. Then we have

$$\|\lambda_n\mathbf{x}_n\| < \|\lambda_n\mathbf{x}'_n\| = |a|^{\beta_n}\|\mathbf{x}'_n\| \leq \sqrt{\|\mathbf{x}'_n\| \cdot \|\mathbf{y}'_n\|}$$
$$\|\lambda_n^{-1}\mathbf{y}_n\| < \|\lambda_n^{-1}\mathbf{y}'_n\| = |a|^{-\beta_n}\|\mathbf{y}'_n\| \leq |a|\sqrt{\|\mathbf{x}'_n\| \cdot \|\mathbf{y}'_n\|}$$

Since $\|\mathbf{x}'_n\| \cdot \|\mathbf{y}'_n\| \to 0$ we get (1). This completes the proof. ∎

Let X and Y be topological vector spaces over non-Archimedean valued field $\mathbb{K}$. By $L(X,Y)$ we denote the set of all continuous linear operators from X to Y. If $X = Y$ then $L(X,Y)$ is denoted by $L(X)$. In what follows, we use the following terminology: T is a linear continuous operator on X means that $T \in L(X)$. The *T-orbit* of a vector $\mathbf{x} \in X$, for some operator $T \in L(X)$, is the set

$$O(\mathbf{x}, T) := \{T^n(\mathbf{x}); n \in \mathbb{Z}_+\}.$$

An operator $T \in L(X)$ is called *hypercyclic* if there exists some vector $\mathbf{x} \in X$ such that its T-orbit is dense in X. The corresponding vector $\mathbf{x}$ is called *T-hypercyclic*, and the set of all T-hypercyclic vectors is denoted by $HC(T)$. Similarly, T is called *supercyclic* if there exists a vector $\mathbf{x} \in X$ such that whose projective orbit

$$\mathbb{K} \cdot O(\mathbf{x}, T) := \{\lambda T^n(\mathbf{x}); n \in \mathbb{Z}_+, \ \lambda \in \mathbb{K}\}$$

is dense in X. The set of all T-supercyclic vectors is denoted by $SC(T)$. Finally, we recall that T is called *cyclic* if there exists $\mathbf{x} \in X$ such that

$$\mathbb{K}[T]\mathbf{x} := \mathrm{span}O(\mathbf{x}, T) = \{P(T)x; P \text{ polynomial}\}$$

is dense in X. The set of all T-cyclic vectors is denoted by $CC(T)$.

Remark 2.3. We stress that the notion of hypercyclicity makes sense only if the space X is separable. Note that one has

$$HC(T) \subset SC(T) \subset CC(T).$$

Remark 2.4. Note that if T is a hypercyclic operator on a Banach space then $\|T\| > 1$.

3 Hypercyclicity and supercyclicity of linear operators

In this section we find sufficient and necessary conditions to hypercyclicity of linear operators on F-spaces. In what follows, by *F-space* we mean a topological vector space X which is metrizable and complete over a non-Archimedean field. Basically, this section mostly repeats the well-known facts from the dynamics of linear operators [1]. But for the sake of completeness, we are going to prove them (with taking into account non-Archimedeanness of the space). In this section, a main approach is based on the Baire category theorem.

We start with the well-known equivalence between hypercyclicity and topological transitivity: an operator T acting on some separable completely metrizable space X is hypercyclic iff for each pair of non-empty open sets $(U,V) \in X$, one can find $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \varnothing$; in this case, there is in fact a residual set of hypercyclic vectors. From this, one gets immediately the so-called Hypercyclicity Criterion, a set of sufficient conditions for hypercyclicity with a remarkably wide range of applications. The analogous Supercyclicity Criterion is proved along the same lines.

Now we show that hypercyclicity turns out to be a purely infinite-dimensional phenomenon.

Proposition 3.1. *Let $X \neq \{0\}$ be a finite-dimensional space. Then each operator $T \in L(X)$ is not hypercyclic.*

Proof. Without loss of generality, we may assume that $X = \mathbb{K}^m$ for some $m \geq 1$. Now we are going to prove that each operator $T \in L(\mathbb{K}^m)$ is not hypercyclic. Suppose that a linear operator T on $\mathbb{K}^m$ is hypercyclic. Take $\mathbf{x} \in HC(T)$. The density of $O(\mathbf{x}, T)$ in $\mathbb{K}^m$ implies that the family $\{\mathbf{x}, T(\mathbf{x}), \ldots, T^{m-1}(\mathbf{x})\}$ forms a linearly independent system. Hence, this collection is a basis of $\mathbb{K}^m$. For any $\alpha \in \mathbb{K} \setminus \{0\}$, one can find a sequence of integers (n_k) such that $T^{n_k}(\mathbf{x}) \to \alpha\mathbf{x}$. Then $T^{n_k}(T^i\mathbf{x}) = T^i(T^{n_k}\mathbf{x}) \to \alpha T^i(\mathbf{x})$ for each $i < m$. Hence for any $\mathbf{y} \in \mathbb{K}^m$ we obtain $T^{n_k}(\mathbf{y}) \to \alpha\mathbf{y}$ which yields that $\det(T^{n_k}) \to \alpha^m$, i.e. $\det(T)^{n_k} \to \alpha^m$. Thus, putting $a := \det(T)$, we have the set $\{a^n; n \in \mathbb{N}\}$ is dense in $\mathbb{K} \setminus \{0\}$, but it is impossible. Indeed, it is clear that $|a^n - z| > 1$ for any $z \in \mathbb{K} \setminus B(0,1)$ if $|a| \leq 1$ and $|a^n - w| > 1$ for any $w \in B(0,1)$ if $|a| > 1$. ∎

Our first characterization of hypercyclicity is a direct application of the Baire category theorem.

Theorem 3.2. *(cp. [3])* (TRANSITIVITY THEOREM) *Let X be a separable F-space and $T \in L(X)$. The following statements are equivalent:*

(i) T is hypercyclic;

(ii) T is **topologically transitive***; that is, for each pair of non-empty open sets $(U,V) \subset X$ there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \varnothing$.*

Proof. (i) Assume that $HC(T) \neq \varnothing$. Since X has no isolated points, for any $k \in \mathbb{N}$ it is easy to see that $T^k(\mathbf{x}) \in HC(T)$ if and only if $\mathbf{x} \in HC(T)$. Let U, V be open sets in X. Take $\mathbf{x} \in U \cap HC(T)$. Then there exists a number $n \in \mathbb{N}$ such that $T^n(\mathbf{x}) \in V$. This means that T is topologically transitive.

(ii) Let T be topologically transitive and $\{V_k\}_{k\in\mathbb{N}}$ be a countable basis of open sets on X (this kind of basis exists since X is a separable F-space). Then from the topological transitivity of T, for any $k \geq 1$ and non-empty open set $U \subset X$ there exists an n such that $U \cap T^{-n}(V_k) \neq \varnothing$. This means that each open set $\bigcup_{n\geq 0} T^{-n}(V_k)$ is dense, hence one gets the density of $\bigcap_{k\geq 1}\bigcup_{n\geq 0} T^{-n}(V_k)$. On the other hand, we have

$$HC(T) = \bigcap_{k\geq 1}\bigcup_{n\geq 0} T^{-n}(V_k). \tag{3}$$

Consequently, $HC(T) \neq \varnothing$. This completes the proof. ∎

Corollary 3.3. *Let X be a separable F-space and $T \in L(X)$. If T is hypercyclic then $HC(T)$ is a dense G_δ-set.*

Proof. According to Theorem 3.2 the hypercyclicity of T implies its topological transitivity. From (3) one easily sees that $HC(T)$ is a G_δ-set. ∎

Definition 3.4. *[2] Let X be a topological vector space, and let $T \in L(X)$. It is said that T satisfies the* **Hypercyclicity Criterion** *if there exist an increasing sequence of integers (n_k), two dense sets $\mathcal{D}_1, \mathcal{D}_2 \subset X$ and a sequence of maps $S_{n_k} : \mathcal{D}_2 \to X$ such that:*

(1) $T^{n_k}(\boldsymbol{x}) \to \boldsymbol{0}$ for any $\boldsymbol{x} \in \mathcal{D}_1$;

(2) $S_{n_k}(\boldsymbol{y}) \to \boldsymbol{0}$ for any $\boldsymbol{y} \in \mathcal{D}_2$;

(3) $T^{n_k}S_{n_k}(\boldsymbol{y}) \to \boldsymbol{y}$ for any $\boldsymbol{y} \in \mathcal{D}_2$.

Note that in the above definition the maps S_{n_k} are not assumed to be continuous or linear. We will sometimes say that T satisfies the Hypercyclicity Criterion with respect to the sequence (n_k). When it is possible to take $n_k = k$ and $\mathcal{D}_1 = \mathcal{D}_2$, it is usually said that T satisfies *Kitai's Criterion* [10].

Theorem 3.5. *Let $T \in L(X)$, where X is a separable F-space. Assume that T satisfies the Hypercyclicity Criterion. Then the operator T is hypercyclic.*

Proof. According to the Transitivity Theorem it is enough to show that T is topologically transitive. Let U, V be two non-empty open subsets of X. Take $\mathbf{x} \in \mathcal{D}_1 \cap U$, $y \in \mathcal{D}_2 \cap V$. Then $\mathbf{x} + S_{n_k}(\mathbf{y}) \to \mathbf{x} \in U$ as $k \to \infty$. Due to the linearity and the continuity of T^{n_k} we obtain $T^{n_k}(\mathbf{x} + S_{n_k}(\mathbf{y})) = T^{n_k}(\mathbf{x}) + T^{n_k}S_{n_k}(\mathbf{y}) \to \mathbf{y} \in V$. Hence, for sufficiently large k one gets $T^{n_k}(U) \cap V \neq \varnothing$. The proof is complete. ∎

Definition 3.6. *Let $T_0 : X_0 \to X_0$ and $T : X \to X$ be two continuous maps acting on topological spaces X_0 and X. The map T is said to be a* **quasi-factor** *of T_0 if there exists a continuous map with dense range $J : X_0 \to X$ such that $TJ = JT_0$. When this can be achieved with a homeomorphism $J : X_0 \to X$, we say that T_0 and T are* **topological conjugate**. *Finally, when $T_0 \in L(X_0)$ and $T \in L(X)$ and the factoring map (resp. the homeomorphism) J can be taken as linear, we say that T is a* **linear quasi-factor** *of T_0 (resp. that T_0 and T are* **linearly conjugate***).*

The usefulness and importance of these definitions can be seen in the following

Lemma 3.7. *Let $T_0 \in L(X_0)$ and $T \in L(X)$. Assume that there exists a continuous map with dense range $J : X_0 \to X$ such that $TJ = JT_0$. Then the following statements are satisfied:*

(1) Hypercyclicity of T_0 implies hypercyclicity of T;

(2) Let J be a homeomorphism and T_0 satisfies Hypercyclicity Criterion then T satisfies Hypercyclicity Criterion;

(3) Let J be a linear homeomorphism then T is hypercyclic iff T_0 is hypercyclic.

Proof. (1) Due to $TJ = JT_0$ one can see that $O(J(\mathbf{x}_0), T) = J(O(\mathbf{x}_0, T_0))$ for any $\mathbf{x}_0 \in X_0$. This with the density of $\mathrm{Ran}(J)$ implies $J(\mathbf{x}) \in HC(T)$ if $\mathbf{x} \in HC(T_0)$.

(2) Now we assume that T_0 satisfies Hypercyclicity Criterion. Then $J(\mathcal{D}_1)$ and $J(\mathcal{D}_2)$ are both dense sets in X, since J has a dense range. For all $\mathbf{x} = J(\mathbf{x}_0) \in J(\mathcal{D}_1)$ we have

$$T^{n_k}(\mathbf{x}) = T^{n_k}J(\mathbf{x}_0) = JT_0^{n_k}(\mathbf{x}_0).$$

The continuity of J implies that $T^{n_k}(\mathbf{x}) \to \mathbf{0}$. Denoting by $\tilde{S}_{n_k} := JS_{n_k}J^{-1}$, for every $\mathbf{y} \in J(\mathcal{D}_2)$ one finds

$$T^{n_k}\tilde{S}_{n_k}(\mathbf{y}) = JT_0^{n_k}S_{n_k}J^{-1}(\mathbf{y}) \to JJ^{-1}(\mathbf{y}) = \mathbf{y}$$

and

$$\tilde{S}_{n_k}(\mathbf{y}) = JS_{n_k}J^{-1}(\mathbf{y}) \to \mathbf{0}.$$

Thus, we have shown that T satisfies Hypercyclicity Criterion.

The proof of (3) is obvious. ∎

Remark 3.1. Note that if $T \in L(X)$ is hypercyclic and if $J \in L(X)$ has a dense range and $JT = TJ$ then $HC(T)$ is invariant under J.

We have already observed that if T is a hypercyclic operator on some F-space X then $HC(T)$ is a dense G_δ-set in X. It shows that the set $HC(T)$ is large in a topological sense. This implies largeness in an algebraic sense.

Proposition 3.8. *Let $T \in L(X)$ be hypercyclic on the separable F-space X. Then for every $\boldsymbol{x} \in X$ there exist $\boldsymbol{y}, \boldsymbol{z} \in HC(T)$ such that $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$.*

Proof. According to Corollary 3.3 $HC(T)$ is a dense G_δ-set, therefore, $X \setminus HC(T)$ and $X \setminus (\mathbf{x} - HC(T))$ are the first category sets. Then by the Baire category Theorem, we infer that $HC(T)$ and $x - HC(T)$ have non-empty intersection. This completes the proof. ∎

We say that a linear subspace $E \subset X$ is a *hypercyclic manifold* for T if $E \setminus \{\mathbf{0}\}$ consists entirely of hypercyclic vectors.

Lemma 3.9. *Let $T \in L(X)$ and $E \subset X$ be a closed T-invariant subspace. Then either $E = X$ or E has infinite codimension in X.*

Proof. Assume that $\dim(X/E) < \infty$. Let $q : X \to X/E$ be the canonical quotient map. By T-invariance of E we get $\mathrm{Ker}(q) \subset \mathrm{Ker}(qT)$. Therefore, one can find an operator $A \in L(X/E)$ such that $Aq = qT$. Since q is continuous onto, the operator A is a quasi-factor of T. According to Lemma 3.7 the operator A is hypercyclic on X/E. Due to $\dim(X/E) < \infty$, by Proposition 3.1, it follows that $X/E = \{\mathbf{0}\}$, i.e. $E = X$. ∎

Lemma 3.10. *Let $T \in L(X)$ be hypercyclic. For any non-zero polynomial P, the operator $P(T)$ has a dense range.*

Proof. Let P be a non-zero polynomial and $E := \overline{\mathrm{Ran}(P(T))}$. For any $\mathbf{x} \in E$ there exists a sequence $(\mathbf{x}_n) \subset X$ such that $P(T)\mathbf{x}_n \to \mathbf{x}$. Then from $P(T)T(\mathbf{x}_n) = TP(T)\mathbf{x}_n \to T(\mathbf{x}) \in E$ we conclude that E is T-invariant. Hence, by Lemma 3.9 it is enough to show that $\dim(X/E) < \infty$.

Let $\mathbf{x} \in HC(T)$ and $q : X \to X/E$ be the canonical quotient map. By the division algorithm and the commutativity of the algebra $\mathbb{K}[T]$, one can easily see that

$$\mathbb{K}[T]\mathbf{x} \subset \mathrm{Ran}(P(T)) + \mathrm{span}\{T^i(\mathbf{x}) : i < \deg(P)\}.$$

From this we conclude that $q(\mathbb{K}[T]\mathbf{x})$ is finite-dimensional. By the cyclicity of $\mathbf{x}$ one finds the finite dimensionality of $X/E = q(X)$. ∎

Theorem 3.11. *[4, 7] Let X be a topological vector space, and $T \in L(X)$ be hypercyclic. If $\boldsymbol{x} \in HC(T)$, then $\mathbb{K}[T]\boldsymbol{x}$ is a hypercyclic manifold for T. In particular, T admits a dense hypercyclic manifold.*

Proof. Let $\mathbf{x} \in HC(T)$ and P be a non-zero polynomial. According to Lemma 3.10 the operator $P(T)$ has a dense range and it commutes with T. By Lemma 3.7 one gets $P(T)\mathbf{x} \in HC(T)$. This means that $\mathbb{K}[T]$ is a hypercyclic manifold for T. The density of $\mathbb{K}[T]$ follows from $O(\mathbf{x}, T) \subset \mathbb{K}[T]$. ∎

We now turn to the supercyclic analogues of Theorems 3.2 and 3.5.

Theorem 3.12. *Let X be a separable F-space, and $T \in L(X)$. The following statements are equivalent:*

(i) *T is supercyclic;*

(ii) *For each pair of non-empty open sets $(U, V) \subset X$ there exist $n \in \mathbb{N}$ and $\lambda \in \mathbb{K}$ such that $\lambda T^n(U) \cap V \neq \varnothing$.*

The proof is similar to the proof of Theorem 3.2.

Definition 3.13. *[17] Let X be a topological vector space, and let $T \in L(X)$. We say that T satisfies the* **Supercyclic Criterion** *if there exist an increasing sequence of integers (n_k), two dense sets $\mathcal{D}_1, \mathcal{D}_2 \subset X$ and a sequence of maps $S_{n_k} : \mathcal{D}_2 \to X$ such that:*

(1) $\| T^{n_k}(\boldsymbol{x}) \| \| S_{n_k}(\boldsymbol{y}) \| \to 0$ *for any $\boldsymbol{x} \in \mathcal{D}_1$ and any $\boldsymbol{y} \in \mathcal{D}_2$;*

(2) $T^{n_k}S_{n_k}(\boldsymbol{y}) \to \boldsymbol{y}$ *for any* $\boldsymbol{y} \in \mathcal{D}_2$.

Theorem 3.14. *Let $T \in L(X)$, where X is a separable Banach space. Assume that T satisfies the Supercyclic Criterion. Then T is supercyclic.*

Proof. Let U and V be two non-empty open subsets of X. Take $\mathbf{x} \in \mathcal{D}_1 \cap U$ and $\mathbf{y} \in \mathcal{D}_2 \cap V$. It follows from part (1) of Definition 3.13 and by Lemma 2.4, we can find a sequence of non-zero scalars (λ_k) such that $\lambda_k T^{n_k}(\mathbf{x}) \to \mathbf{0}$ and $\lambda_k^{-1}S_{n_k}(\mathbf{y}) \to \mathbf{0}$. Then, for large enough k, the vector $\mathbf{z} = \mathbf{x} + \lambda_k^{-1}S_{n_k}(\mathbf{y})$ belongs to U and $\lambda_k T^{n_k}(\mathbf{z})$ belongs to V. By Theorem 3.12 we infer that T is supercyclic. ∎

Lemma 3.15. *Let X_0 and X be Banach spaces over the field $\mathbb{K}$ and $T_0 \in L(X_0)$, $T \in L(X)$ be such that there exists a $J \in L(X_0, X)$ which has a dense range and satisfying $TJ = JT_0$. Then the supercyclicity (resp. cyclicity) of T_0 implies the supercyclicity (resp. cyclicity) of T.*

Proof. Observe that

$$\{\lambda(T^nJ)(\mathbf{x}_0) : n \in \mathbb{Z}_+,\ \lambda \in \mathbb{K}\} = J(\{\lambda T_0^n(\mathbf{x}_0) : n \in \mathbb{Z}_+,\ \lambda \in \mathbb{K}\}),$$
$$\text{span}\{(T^nJ)(\mathbf{x}_0) : n \in \mathbb{Z}_+\} = J(\text{span}\{T_0^n(\mathbf{x}_0) : n \in \mathbb{Z}_+\})$$

for any $\mathbf{x}_0 \in X_0$. Hence, $J(\mathbf{x}_0)$ is a supercyclic (resp. cyclic) vector for T for each $\mathbf{x}_0 \in SC(T_0)$ (resp. $\mathbf{x}_0 \in C(T)$). ∎

4 Backward shifts on c_0

In the present section, we are going to study the backward shifts on c_0. We notice that similar to results in the archimedean case have been investigated in [6, 16, 17]. Here, as usual, c_0 stands for the set of all sequences which tend to zero equipped with a norm

$$\| \mathbf{x} \| := \sup_n\{|x_n|\}, \quad \mathbf{x} \in c_0.$$

It is clear that c_0 is a Banach space. For convenience, we denote

$$c_0(\mathbb{Z}) := \{(x_n)_{n\in\mathbb{Z}} : x_n \in \mathbb{K}, |x_{\pm n}| \to 0 \text{ as } n \to +\infty\}$$

and

$$c_0(\mathbb{N}) := \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{K}, |x_n| \to 0 \text{ as } n \to +\infty\}$$

In what follows, we always assume that c_0 is a separable space. Note that the separability of c_0 is equivalent to the separability of $\mathbb{K}$. Let K be a countable dense subset of $\mathbb{K}$. Then the countable set

$$c_{00}(\mathbb{Z}) := \{\lambda_{-n}\mathbf{e}_{-n} + \lambda_{-n+1}\mathbf{e}_{-n+1} + \cdots + \lambda_n\mathbf{e}_n : \lambda_{\pm j} \in K,\ 0 \le j \le n, \forall n \in \mathbb{N}\}$$

is dense in $c_0(\mathbb{Z})$, where $\mathbf{e}_n$ is an unit vector such that only n-th coordinate equals to 1 and others are zero.

Let $\mathbf{a} = (a_n)_{n\in\mathbb{Z}}$ be a bounded sequence of non-zero numbers of $\mathbb{K}$. An operator $B_\mathbf{a}$ on $c_0(\mathbb{Z})$ defined by $B_\mathbf{a}(\mathbf{e}_n) = a_n\mathbf{e}_{n-1}$ is called *bilateral weighted backward* *shift* if $a_i \neq 1$ for some $i \in \mathbb{Z}$, otherwise it is called *bilateral unweighted backward shift* and we denote it by B. In general, the (unweighted) backward shift B is considered as a weighted shift and is thus not excluded from the family of weighted shifts. The operator B is an example of weighted shifts where each weight is equal to 1.

Theorem 4.1. *Let $B_{\mathbf{a}}$ be a bilateral weighted backward shift operator on $c_0(\mathbb{Z})$. Then the following statements hold:*

(i) $B_{\mathbf{a}}$ is hypercyclic if and only if, for any $q \in \mathbb{N}$,

$$\liminf_{n\to+\infty} \max\left\{\prod_{i=1}^{n+q}|a_i^{-1}|, \prod_{j=1}^{n-q}|a_{-j+1}|\right\} = 0. \tag{4}$$

(ii) $B_{\mathbf{a}}$ is supercyclic if and only if, for any $q \in \mathbb{N}$,

$$\liminf_{n\to+\infty} \prod_{i=1}^{n+q}|a_i^{-1}| \times \prod_{j=1}^{n-q}|a_{-j+1}| = 0. \tag{5}$$

Proof. For any weight $\mathbf{b} \in l_\infty(\mathbb{Z})$ with $b_n \neq 0$, $n = 0, \pm1, \pm2, \dots$ we introduce the weighted space

$$c_0(\mathbb{Z}, \mathbf{b}) := \left\{\mathbf{x} \in c_0(\mathbb{Z}) : \| \mathbf{x} \|_{\mathbf{b}} = \sup_n |b_n x_n|\right\}.$$

Take a weight sequence $\mathbf{b} = (b_n)_{n\in\mathbb{Z}}$ as follows $b_0 = 1$ and $b_n b_{n+1}^{-1} = a_{n+1}$. Let B be the bilateral backward shift on $c_0(\mathbb{Z}, \mathbf{b})$. Then $B_{\mathbf{a}}$ is linearly conjugate to B. Indeed, the operator $J : c_0(\mathbb{Z}) \to c_0(\mathbb{Z}, \mathbf{b})$ defined by $(J\mathbf{x})_n = b_n^{-1} x_n$ is a linear homeomorphism and $J(c_0(\mathbb{Z})) = c_0(\mathbb{Z}, \mathbf{b})$, $JB_{\mathbf{a}} = BJ$. According to Lemma 3.7 (resp. Lemma 3.15) hypercyclicity (supercyclicity) of $B_{\mathbf{a}}$ is equivalent to the hypercycility (resp. supercyclicity) of B.

Assume that B is hypercyclic and fix $q \in \mathbb{N}$. Due to the density of $O(\mathbf{x}, B)$ (for all $\mathbf{x} \in HC(B)$), for an arbitrary $\varepsilon > 0$ one can find $\mathbf{x} \in HC(B)$ and an integer $n > 2q$ such that

$$\| \mathbf{x} - \mathbf{e}_q \|_{\mathbf{b}} < \varepsilon \quad \text{and} \quad \| B^n(\mathbf{x}) - \mathbf{e}_q \|_{\mathbf{b}} < \varepsilon.$$

These inequalities imply that

$$|b_q(x_q - 1)| < \varepsilon, \quad |b_{n+q} x_{n+q}| < \varepsilon, \tag{6}$$

$$|b_q(x_{n+q} - 1)| < \varepsilon, \quad |b_{-n+q} x_q| < \varepsilon. \tag{7}$$

We assume that $\varepsilon < |b_q|$. Then from the first inequalities of (6) and (7) we obtain $|x_q - 1| < 1$ and $|x_{n+q} - 1| < 1$. Hence, by the non-Archimedean norm's property, one gets $|x_q| = |x_{n+q}| = 1$. Putting it into the second inequalities of (6) and (7) one finds $|b_{\pm n+q}| < \varepsilon$, which is equivalent to

$$\liminf_{n\to+\infty} |b_{\pm n+q}| = 0. \tag{8}$$

Now let us assume that (8) holds for any $q \in \mathbb{N}$. We will show that B satisfies the Hypercyclicity Criterion. Take some positive number M such that

$$M > \max\left\{1, \sup_n \frac{|b_n|}{|b_{n+1|}}\right\}.$$

By (8), one can find an increasing sequence of positive integers $\{n_k\}$ such that

$$|b_{\pm n_k+k}| \le M^{-3k} \quad \text{for all } \; k \in \mathbb{N}.$$

Let i be a fixed integer and $k > |i|$. Then $\left|b_{\pm n_k+i}\right| < M^{i+k}\left|b_{\pm n_k+k}\right| \le M^{-2k+i} < M^{-k}$. It follows that $b_{n_k+i} \to 0$ as $k \to \infty$ for any $i \in \mathbb{Z}$. Now, let $\mathcal{D}_1 = \mathcal{D}_2 := c_{00}(\mathbb{Z})$ and let S be the forward shift, defined on $\mathcal{D}_2$ by $S(\mathbf{e}_i) = \mathbf{e}_{i+1}$. Due to the linearity of B and S, it is enough to show that $B^{n_k}(\mathbf{e}_i) \to 0$ and $S^{n_k}(\mathbf{e}_i) \to 0$ for any $i \in \mathbb{Z}$, but this is clear, since

$$\| B^{n_k}(\mathbf{e}_i) \|_{\mathbf{b}} = |b_{-n_k+i}| \quad \text{and} \quad \| S^{n_k}(\mathbf{e}_i) \|_{\mathbf{b}} = |b_{n_k+i}|.$$

Thus, we have shown that $B_{\mathbf{a}}$ is hypercyclic if and only if for any $q \in \mathbb{N}$ holds (8). According to

$$b_n = \prod_{i=1}^{n} a_i^{-1} \quad \text{and} \quad b_{-n} = \prod_{j=1}^{n} a_{-j+1} \quad \text{for all } n \in \mathbb{N}$$

one can see that (8) and (4) are equivalent.

Now we turn to the supercyclic case. Suppose that B is supercyclic and $q \in \mathbb{N}$. Let $\varepsilon > 0$ be an arbitrary number. Then the density of supercyclic vectors implies the existence of $\mathbf{x} \in c_0(\mathbb{Z}, \mathbf{b})$, $\lambda \in \mathbb{K}^{\times}$ and $n > 2q$ such that

$$\| \mathbf{x} - \mathbf{e}_q \|_{\mathbf{b}} < \varepsilon \quad \text{and} \quad \| \lambda B^n(\mathbf{x}) - \mathbf{e}_q \|_{\mathbf{b}} < \varepsilon.$$

As above, we obtain

$$\begin{aligned} &|b_q(x_q - 1)| < \varepsilon, && |b_{n+q}x_{n+q}| < \varepsilon, \\ &|b_q(\lambda x_{n+q} - 1)| < \varepsilon, && |\lambda b_{-n+q}x_q| < \varepsilon. \end{aligned}$$

Assuming $\varepsilon < |b_q|$ and using the non-Archimedean norm's property one finds

$$|b_{-n+q}| < \frac{\varepsilon}{|\lambda|} \quad \text{and} \quad |b_{n+q}| < \varepsilon|\lambda|.$$

Hence, $|b_{-n+q}b_{n+q}| < \varepsilon^2$ which yields

$$\liminf_{n\to+\infty} |b_{n+q}b_{-n+q}| = 0. \tag{9}$$

Note that (9) and (5) are equivalent.

If the condition (9) holds then we can find as above an increasing sequence (n_k) such that, for any $i, j \in \mathbb{Z}$,

$$b_{n_k+i}b_{-n_k+j} \to 0, \quad \text{as } k \to +\infty.$$

Hence, from

$$\| B^{n_k}(\mathbf{e}_j) \|_{\mathbf{b}} \cdot \| S_{n_k}(\mathbf{e}_i) \|_{\mathbf{b}} = |b_{n_k+i}b_{-n_k+j}|.$$

we infer the Supercyclic Criterion is satisfied for $\mathcal{D}_1 = \mathcal{D}_2 := c_{00}(\mathbb{Z})$ and the forward shift S. ∎

From this theorem we immediately find the following facts.

Corollary 4.2. *Let $B_{\mathbf{a}}$ be a bilateral weighted backward shift on $c_0(\mathbb{Z})$. Then the following statements hold:*

(i) if $B_{\mathbf{a}}$ is supercyclic then $\lambda B_{\mathbf{a}}$ is supercyclic for any $\lambda \in \mathbb{K}^{\times}$;

(ii) if the weight sequence $\mathbf{a} = (a_n)_{n\in\mathbb{Z}}$ is symmetrical to the norm, i.e. $|a_n| = |a_{-n}|$, $n = 1, 2, \ldots$ then $B_{\mathbf{a}}$ is not supercyclic.

Corollary 4.3. *Let B be the bilateral unweighted backward shift on $c_0(\mathbb{Z})$. Then B is not supercyclic. Moreover, λB is not supercyclic for any $\lambda \in \mathbb{K}$.*

Corollary 4.4. *Let $\mathbf{a}$ and $\mathbf{b}$ be weighted sequences such that $|a_n| > |b_n|$ for any $n \in \mathbb{Z}$. Then $B_{\mathbf{a}+\mathbf{b}}$ is hypercyclic (resp. supercyclic) if and only if $B_{\mathbf{a}}$ is hypercyclic (resp. supercyclic).*

Proof. By the non-Archimedean norm's property we have $|a_n + b_n| = |a_n|$ for any $n \in \mathbb{Z}$. Using it to (4) (resp. (5)) we can conclude that hypercyclicity (supercyclicity) of $B_{\mathbf{a}}$ and $B_{\mathbf{a}+\mathbf{b}}$ are equivalent. ∎

Remark 4.1. We first notice that Theorem 4.1 remains the same in the real setting, but the valuation should be replaced with the usual absolute value. However, in the real case, Corollary 4.4 is not true. Indeed, for the weights $\mathbf{a}$ and $\mathbf{b}$ defined by

$$a_n = \begin{cases} n, & \text{if } n \geq 1, \\ -\frac{1}{n-1}, & \text{if } n < 1. \end{cases}$$

$$b_n = \begin{cases} -n + \frac{1}{n+1}, & \text{if } n \geq 1, \\ \frac{1}{n-1} - \frac{1}{n-2}, & \text{if } n < 1. \end{cases}$$

the operators $B_{\mathbf{a}}$ and $B_{\mathbf{b}}$ are hypercyclic. But, the weight $\mathbf{a} + \mathbf{b}$ does not satisfy (4). Consequently, according to Theorem 4.1 the operator $B_{\mathbf{a}+\mathbf{b}}$ can not be hypercyclic.

Now let us consider a unilateral weighted backward shifts on $c_0(\mathbb{N})$. Recall that the operator defined as $B_{\mathbf{a}}(\mathbf{e}_1) = 0$ and $B_{\mathbf{a}}(\mathbf{e}_n) = a_{n-1}\mathbf{e}_{n-1}$ if $n \geq 2$, is called *unilateral* weighted backward shift. Here $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ be a bounded sequence of non-zero numbers of $\mathbb{K}$. The operator $B_{\mathbf{a}}$ is called unilateral unweighted backward shift if $a_n = 1$ for all $n \geq 1$. We denote by B a unilateral unweighted backward shift operator.

In [13] various functional models of the unilateral shift operator B has been given. For the sake of completeness, let us provide one an illustrative example.

Example 4.1. Let $\mathbb{Z}_p$ be the unit ball in $\mathbb{Q}_p$. By $C(\mathbb{Z}_p, \mathbb{C}_p)$ we denote the space of all continuous functions on $\mathbb{Z}_p$ with values in $\mathbb{C}_p$ endowed with "sup"-norm. Consider a linear operator $T : C(\mathbb{Z}_p, \mathbb{C}_p) \to C(\mathbb{Z}_p, \mathbb{C}_p)$ defined by

$$(Tf)(x) = f(x+1) - f(x), \ \ (x \in \mathbb{Z}_p), \ \ f \in C(\mathbb{Z}_p, \mathbb{C}_p).$$

We note (see [11]) that the operator T can be interpreted as the annihilation operators in a p-adic representation of the canonical commutation relations of quantum mechanics.

It is well known [15] that the Mahler polynomials
$$P_n(x) = \frac{x(x-1)\cdots(x-n+1)}{n!}, \quad n \in \mathbb{N}; \ P_0(x) = 1,$$
form an orthonormal basis in $C(\mathbb{Z}_p, \mathbb{C}_p)$. The operator T acts on the Mahler polynomials as follows:
$$TP_n = P_{n-1}, \ n \in \mathbb{N}; \ TP_0 = 0.$$
It is known that the spaces $C(\mathbb{Z}_p, \mathbb{C}_p)$ and $c_0(\mathbb{N})$ are isomorphic via the isomorphism
$$\sum_{n=0}^{\infty} x_n P_n \to (x_0, x_1, \dots, x_n, \dots)$$
therefore, the operator T is transformed to B.

Theorem 4.5. *Any unilateral weighted backward shift $B_{\mathbf{a}}$ on $c_0(\mathbb{N})$ is supercyclic. Moreover, $B_{\mathbf{a}}$ is hypercyclic iff*
$$\limsup_{n\to\infty} \prod_{i=1}^{n} |a_i| = \infty. \tag{10}$$

Proof. Let $B_{\mathbf{a}}$ be a unilateral weighted backward shift. Let $\mathcal{D}_1 = \mathcal{D}_2 := c_{00}(\mathbb{N})$ be the set of all finitely supported sequences. Let $S_{\mathbf{a}}$ be the linear map defined on $\mathcal{D}_2$ by $S_{\mathbf{a}}(\mathbf{e}_n) = a_n^{-1}\mathbf{e}_{n+1}$ and, for each $k \in \mathbb{N}$, set $S_k := S_{\mathbf{a}}^k$. Then, the Supercyclicity Criterion is satisfied with respect to k because $\| B_{\mathbf{a}}^k(\mathbf{x}) \| = 0$ for large enough k on $\mathcal{D}_1$ and $B_{\mathbf{a}}^k S_k = I$ on $\mathcal{D}_2$. According to Theorem 3.14 operator $B_{\mathbf{a}}$ is supercyclic.

Now we are going to establish that the hypercyclicity of $B_{\mathbf{a}}$ is equivalent to (10). First we suppose that (10) holds, and let us show that $B_{\mathbf{a}}$ satisfies the Hypercyclicity Criterion. It is enough to show that $S_k(\mathbf{x}) \to \mathbf{0}$ as $k \to \infty$ for all $\mathbf{x} \in c_{00}(\mathbb{N})$. Let $\mathbf{x} \in c_{00}(\mathbb{N}) \setminus \{\mathbf{0}\}$. Then there exists a positive integer q such that $x_q \neq 0$ and $x_m = 0$ for all $m > q$. Denote $x_j^{(k)} := (S_k\mathbf{x})_j$, $j = 1, 2, 3, \dots$. We have $x_j^{(k)} = 0$ if $1 \le j \le k$ or $j > q + k$, and
$$x_{j+k}^k = \frac{x_j}{\prod\limits_{i=1}^{k} |a_{j+i-1}|}, \quad 1 \le j \le q.$$
From (10), we obtain that x_{j+k}^k tends to 0 along the subsequence.

Let us assume that $B_{\mathbf{a}}$ is hypercyclic, and take an arbitrary number $\varepsilon > 0$. Then the density of hypercyclic vectors implies the existence of $\mathbf{x} \in c_0(\mathbb{N})$ and an integer $k > 2$ such that
$$\| \mathbf{x} - \mathbf{e}_1 \| < \varepsilon \quad \text{and} \quad \| B_{\mathbf{a}}^k(\mathbf{x}) - \mathbf{e}_1 \| < \varepsilon.$$
From these relations, we obtain $|x_{k+1}| < \varepsilon$ and $\left| \prod_{i=1}^{k} a_i x_{k+1} - 1 \right| < \varepsilon$. Again using the non-Archimedean norm's property from the last inequalities one finds
$$\prod_{i=1}^{k} |a_i| = \frac{1}{|x_{k+1}|} > \frac{1}{\varepsilon}.$$
The arbitrariness of ε yields (10). The proof is complete. ∎

Corollary 4.6. *Let B be an unilateral unweighted backward shift on $c_0(\mathbb{N})$. Then the following assertions hold:*

(i) The operator λB is supercyclic for any $\lambda \in \mathbb{K}^{\times}$;

(ii) λB is hypercyclic iff $|\lambda| > 1$.

5 λI + μB operators on c_0

In this section, we are going to consider the following operator

$$T_{\lambda,\mu} = \lambda I + \mu B,$$

where I is a identity and B is the unweighted backward shift. We will show that there does not exist pair of (λ, μ) such that $T_{\lambda,\mu}$ can be supercyclic on $c_0(\mathbb{Z})$. But, for any pair of (λ, μ) with $|\lambda| < |\mu|$ an operator $T_{\lambda,\mu}$ is supercyclic on $c_0(\mathbb{N})$. Moreover, we will prove that the condition $|\lambda| < |\mu|$ is necessary for the supercyclicity of $T_{\lambda,\mu}$ on $c_0(\mathbb{N})$.

Theorem 5.1. *The operator $T_{\lambda,\mu}$ on $c_0(\mathbb{Z})$ is not supercyclic for all $\lambda, \mu \in \mathbb{K}$.*

Proof. First, we consider the case $|\lambda| \geq |\mu|$. Take $\mathbf{x} \in c_0(\mathbb{Z}) \setminus \{\mathbf{0}\}$. Then there exists a number $k \in \mathbb{N}$ such that $|x_k| > |x_m|$ for all $m > k$.

Denote

$$x_i^{(n)} := \left(T^n_{\lambda,\mu}\mathbf{x}\right)_i, \quad i = 0, \pm 1, \pm 2, \ldots$$

It is easy to get the following recurrence formula

$$x_i^{(n)} = \lambda^n \sum_{j=0}^{n} \binom{n}{j} \mu^j \lambda^{-j} x_{i+j}, \quad i = 0, \pm 1, \pm 2, \ldots$$

Due to $|\binom{n}{j}| \leq 1,\ j = \overline{1,n}$ and $|\mu| \leq |\lambda|$, and using the non-Archimedean norm's property one gets

$$\left|x_k^{(n)}\right| = |\lambda^n x_k|$$

and

$$\left|x_{k+1}^{(n)}\right| < |\lambda^n x_k|.$$

Then for any $\alpha \in \mathbb{K}^{\times}$ we get

$$\left|\alpha x_k^{(n)}\right| > \left|\alpha x_{k+1}^{(n)}\right|.$$

This with the non-Archimedean norm's property yields that

$$\| \alpha T^n_{\lambda,\mu}(\mathbf{x}) - \mathbf{e}_{k+1} \| \geq 1,$$

which means that $O(\mathbf{x}, \alpha T_{\lambda,\mu}) \cap B(\mathbf{e}_{k+1}, 1) = \varnothing$. The arbitrariness of α implies that $c_0(\mathbb{Z}) \setminus \overline{\mathbb{K} \cdot O(\mathbf{x}, T_{\lambda,\mu})} \neq \varnothing$. Since $\mathbf{x}$ is an arbitrary vector we conclude that $T_{\lambda,\mu}$ can not be supercyclic if $|\lambda| \geq |\mu|$.

Now we assume that $|\mu| > |\lambda|$. Pick a non-zero vector $\mathbf{y}$. We can take an integer number ℓ such that $|y_\ell| \geq |y_i|$ for all $i > \ell$ and $|y_\ell| > |y_j|$ for all $j < \ell$. Then for any $k \in \mathbb{Z}$ we have

$$y_{k-n}^{(n)} = \mu^n \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} \mu^{j-n} y_{k-n+j}. \tag{11}$$

Using the strong triangle inequality one gets

$$\left| y_{\ell-n}^{(n)} \right| = |\mu^n y_\ell|$$

Pick an integer number m such that $|y_i| < |y_\ell|$ for all $i > m$. Then from (11) for any $i > m$ we obtain

$$\left| y_i^{(n)} \right| < |\mu^n y_\ell| .$$

Due to $\left| y_{\ell-n}^{(n)} \right| > \left| y_{m+1}^{(n)} \right|$ and the inequality $\left| \beta y_{\ell-n}^{(n)} \right| < 1$, for any $\beta \in \mathbb{K}$, we obtain $\left| \beta y_{m+1}^{(n)} - 1 \right| = 1$. It yields that $O(\mathbf{y}, \beta T_{\lambda,\mu}) \cap B(\mathbf{e}_{m+1}, 1) = \varnothing$ for any $\beta \in \mathbb{K}$. The arbitrariness of $\mathbf{y}$ implies that $T_{\lambda,\mu}$ can not be supercyclic on $c_0(\mathbb{Z})$ if $|\lambda| < |\mu|$. This completes the proof. ∎

From Remark 2.3 we obtain the following

Corollary 5.2. *The operator $T_{\lambda,\mu}$ on $c_0(\mathbb{Z})$ is not hypercyclic for all $\lambda, \mu \in \mathbb{K}$.*

Now we consider the operator $T_{\lambda,\mu}$ on $c_0(\mathbb{N})$. We will show that hypercyclicity of $T_{\lambda,\mu}$ is equivalent to the Hypercyclicity Criterion.

Theorem 5.3. *For the the operator $T_{\lambda,\mu}$ acting on $c_0(\mathbb{N})$ the following statements are equivalent:*

(i) *$T_{\lambda,\mu}$ satisfies Hypercyclicity Criterion;*

(ii) *$T_{\lambda,\mu}$ is hypercyclic;*

(iii) $|\lambda| < |\mu|$ *and* $|\mu| > 1$.

To prove the theorem we first prove three auxiliary lemmas.

Lemma 5.4. *If the operator $T_{\lambda,\mu}$ acting on $c_0(\mathbb{N})$ is hypercyclic then $|\mu| > |\lambda|$ and $|\mu| > 1$.*

Proof. Assume that $T_{\lambda,\mu}$ is hypercyclic. We immediately get that $\| T_{\lambda,\mu} \| > 1$. Using the non-archimedean norm's property one finds $\max\{|\lambda|, |\mu|\} > 1$. Let us suppose that $|\mu| \leq |\lambda|$. Take $\mathbf{x} \in HC(T_{\lambda,\mu})$. Since the vector $\mathbf{x}$ is not zero, then there exists a number $k \in \mathbb{N}$ such that $|x_k| > |x_m|$ for all $m > k$.

It is easy to get the following recurrence formula

$$x_i^{(n)} = \lambda^n \sum_{j=0}^{n} \binom{n}{j} \left(\frac{\mu}{\lambda} \right)^j x_{i+j}, \quad i = 1, 2, 3, \ldots$$

From $|\binom{n}{j}| \leq 1$, $j = \overline{1,n}$ and $|\mu| \leq |\lambda|$, by means of the non-Archimedean norm's property one gets

$$\left|x_k^{(n)}\right| = |\lambda^n x_k|\,.$$

From $|\lambda| > 1$ we get $\left|x_k^{(n)}\right| > |x_k|$. Hence,

$$\| T^n_{\lambda,\mu}(\mathbf{x}) \| > |x_k| > 0.$$

Then $O(\mathbf{x}, T_{\lambda,\mu}) \cap B(0,\varepsilon) = \varnothing$ for any positive $\varepsilon < |x_k|$. This means that $\mathbf{x} \notin HC(T_{\lambda,\mu})$. Thus, we have shown that $T_{\lambda,\mu}$ cannot be hypercyclic if $|\mu| \leq |\lambda|$. From this fact and $\max\{|\lambda|, |\mu|\} > 1$ we get $|\mu| > 1$. ∎

Lemma 5.5. *Let $|\mu| > 1$. If $|\lambda| < 1$, then $T_{\lambda,\mu}$ acting on $c_0(\mathbb{N})$ is hypercyclic.*

Proof. Let $|\lambda| < 1 < |\mu|$. We define the operator $S_{\mu,\lambda}$ as follows

$$\begin{aligned} &\left(S_{\mu,\lambda}\mathbf{x}\right)_1 = 0 \\ &\left(S_{\mu,\lambda}\mathbf{x}\right)_i = \tfrac{1}{\mu}\left(\sum_{j=1}^{i-1}\left(\tfrac{-\lambda}{\mu}\right)^{j-1} x_{i-j}\right), \quad i = 2,3,4,\ldots \end{aligned} \tag{12}$$

Then one has $T_{\lambda,\mu}S_{\mu,\lambda} = I$. Let $\mathbf{x} \in c_{00}$. It is clear that $T^n_{\lambda,\mu}(\mathbf{x}) \to 0$ as $n \to \infty$. It follows from the strong triangle inequality that

$$\| S^n_{\mu,\lambda}(\mathbf{x}) \| \leq \frac{1}{|\mu^n|} \| \mathbf{x} \|\,.$$

Since $|\mu| > 1$ we obtain $S^n_{\mu,\lambda}(\mathbf{x}) \to 0$ as $n \to \infty$. Hence, the operator $T_{\lambda,\mu}$ satisfies the Hypercyclicity Criterion, therefore, Theorem 3.5 implies that $T_{\lambda,\mu}$ is a hypercyclic. ∎

Proposition 5.6. *If $1 \leq |\lambda| < |\mu|$ then $\bigcup_{n\geq 1}^{\infty} T^{-n}_{\lambda,\mu}(\boldsymbol{0})$ is a dense set in $c_0(\mathbb{N})$.*

Proof. First we show that $T^n_{\lambda,\mu}(\mathbf{x}) = \mathbf{0}$ has a solution for any $n \geq 1$. Note that $T^n_{\lambda,\mu}(\mathbf{x}) = \mathbf{0}$ is equivalent to

$$\sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}\mu^j x_{k+j} = 0, \;\; k = 1,2,3,\ldots,$$

where $x_m \to 0$ as $m \to \infty$.

Dividing by λ^n and denoting $\tilde{\mu} = \frac{\mu}{\lambda}$ from the last equality, we obtain

$$\sum_{j=0}^{n}\binom{n}{j}\tilde{\mu}^j x_{k+j} = 0, \; k = 1,2,3,\ldots \tag{13}$$

Then for any $(a_1, a_2, \ldots, a_n) \in \mathbb{K}^n$ the sequence $(b_m)_{m\geq 1}$ defined by

$$\begin{cases} b_m = a_m, & \text{if } m \leq n, \\ b_m = -\frac{\sum_{j=0}^{n}\binom{n}{j}\tilde{\mu}^j b_{m-n+j}}{\tilde{\mu}^n}, & \text{if } m > n. \end{cases} \tag{14}$$

is a solution of the system (13). Using the strong triangle inequality from (14) we get

$$|b_{n+k}| \leq \max_{0\leq j\leq n-1}\left|\frac{b_{k+j}}{\tilde{\mu}^{n-j}}\right|, \quad k=1,2,3\ldots$$

Hence,

$$|b_{n+k}| \leq \frac{1}{|\tilde{\mu}^{k-1}|}\max_{1\leq j\leq n}\left|\frac{b_j}{\tilde{\mu}^{n-j+1}}\right|, \quad k=1,2,3\ldots$$

From $|\tilde{\mu}| > 1$ one has $b_m \to 0$ as $m \to \infty$. Thus, we have shown that $\mathbf{y} = (y_1, y_2, \dots) \in c_0(\mathbb{N})$ is a solution of $T^n_{\lambda,\mu}(\mathbf{x}) = \mathbf{0}$ if and only if the sequence $(y_m)_{m\geq 1}$ satisfies (14).

Now, let us show that $\overline{\bigcup_{n\geq 1}^{\infty} T^{-n}_{\lambda,\mu}(\mathbf{0})} = c_0(\mathbb{N})$. Indeed, pick up any $\mathbf{x} \in c_0(\mathbb{N})$. Then for any $\varepsilon > 0$ there exists a positive integer n_0 such that $|x_m| < \varepsilon$ for all $m > n_0$. We can find an integer number $N > n_0$ such that

$$\frac{\max\limits_{1\leq j\leq n_0}\{|x_j|\}}{|\tilde{\mu}^{N-n_0+1}|} < \varepsilon. \tag{15}$$

Then for the vector $\mathbf{y} = (y_1, y_2, y_3, \dots)$ defined by

$$y_m = \begin{cases} x_m, & \text{if } m \leq N, \\ -\frac{\sum_{j=0}^{N}\binom{N}{j}\tilde{\mu}^j y_{m-N+j}}{\tilde{\mu}^N}, & \text{if } m > N. \end{cases} \tag{16}$$

we have $T^N_{\lambda,\mu}(\mathbf{y}) = \mathbf{0}$. Hence, $\mathbf{y} \in \bigcup_{n\geq 1}^{\infty} T^{-n}_{\lambda,\mu}(\mathbf{0})$. Noting (15), using the strong triangle inequality from (16) one finds $|y_m| < \varepsilon$ for every $m > N$. Hence, $\| x - y \| < \varepsilon$. The arbitrariness of $\mathbf{x}$ implies that the density of $\bigcup_{n\geq 1}^{\infty} T^{-n}_{\lambda,\mu}(\mathbf{0})$ in $c_0(\mathbb{N})$. ∎

Lemma 5.7. *Let $1 \leq |\lambda| < |\mu|$. Then the operator $T_{\lambda,\mu}$ satisfies the Hypercyclicity Criterion.*

Proof. Let $1 \leq |\lambda| < |\mu|$. We denote $\mathcal{D}_1 = \bigcup_{n\geq 1}^{\infty} T^{-n}_{\lambda,\mu}(\mathbf{0})$ and $\mathcal{D}_2 = c_{00}$. According to Proposition 5.6 the set $\mathcal{D}_1$ is dense. Let $\mathbf{x} \in \mathcal{D}_1$. Then there exists an integer $k \geq 1$ such that $T^k_{\lambda,\mu}(\mathbf{y}) = \mathbf{0}$. Hence, for any $n \geq k$ we have $T^n_{\lambda,\mu}(\mathbf{y}) = \mathbf{0}$.

For the linear operator $S_{\mu,\lambda}$ defined by (12), we can easily check that $S^n_{\mu,\lambda}(\mathbf{y}) \to 0$ for every $\mathbf{y} \in c_{00}$, here we have used $1 \leq |\lambda| < |\mu|$. From $T_{\lambda,\mu}S_{\mu,\lambda} = I$, one gets $T^n_{\lambda,\mu}S^n_{\mu,\lambda}(\mathbf{y}) \to \mathbf{y}$ for every $\mathbf{y} \in c_{00}$. Hence, we have shown that the operator $T_{\lambda,\mu}$ satisfies Hypercyclicity Criterion. ∎

Proof of Theorem 5.3 The implication (i)⇒(ii) follows from Theorem 3.5. By Lemma 5.4 we obtain the implication (ii)⇒(iii). Finally, (iii)⇒(i) follows from Lemma 5.5 and Lemma 5.7. This completes the proof.

Remark 5.1. According to Theorem 5.3 an operator $I + \mu B$ on $c_0(\mathbb{N})$ can not be hypercyclic for any $\mu \in \mathbb{K}$. But, in real case [18], it is hypercyclic for $\mu \neq 0$.

Now we will study supercyclicity of $T_{\lambda,\mu}$. Similarly to the hypercyclic case we have the following

Theorem 5.8. *For the operator $T_{\lambda,\mu}$ acting on $c_0(\mathbb{N})$ the following statements are equivalent:*

(i) $T_{\lambda,\mu}$ satisfies Supercyclicity Criterion;

(ii) $T_{\lambda,\mu}$ is supercyclic;

(iii) $|\lambda| < |\mu|$.

Proof. The implication (i)⇒(ii) follows from Theorem 3.14. We will establish the implications (ii)⇒(iii)⇒(i).

(ii)⇒(iii) Let us assume that $T_{\lambda,\mu}$ is a supercyclic. Suppose that $|\mu| \le |\lambda|$. Let $\mathbf{x} \in SC(T_{\lambda,\mu})$. Due to $\mathbf{x} \neq \mathbf{0}$ there exists a positive integer k such that $|x_k| > |x_m|$ for all $m > k$. Denote $x_i^{(n)} := \left(T_{\lambda,\mu}^n \mathbf{x}\right)_i$. Then for any $n \ge 1$ using the non-Archimedean norm's property we get

$$\left|x_k^{(n)}\right| = |\lambda^n x_k| \tag{17}$$

and

$$\left|x_{k+1}^{(n)}\right| < |\lambda^n x_k| \tag{18}$$

Since $\mathbf{x}$ is a supercyclic vector, there exist $n \in \mathbb{N}$ and $\alpha \in \mathbb{K}$ such that

$$\| \alpha T_{\lambda,\mu}^n(\mathbf{x}) - \mathbf{e}_{k+1} \| < 1.$$

It follows that

$$\left|\alpha x_k^{(n)}\right| < 1, \quad \left|\alpha x_{k+1}^{(n)} - 1\right| < 1 \tag{19}$$

On the other hand, from (17) and (18) we obtain

$$\left|\alpha x_k^{(n)}\right| = |\alpha \lambda^n x_k|$$

and

$$\left|\alpha x_{k+1}^{(n)}\right| < |\alpha \lambda^n x_k|.$$

The last ones with the non-Archimedean norm's property imply

$$\left|\alpha x_k^{(n)}\right| < 1, \quad \left|\alpha x_{k+1}^{(n)} - 1\right| = 1.$$

It is a contradiction to (19). This yields that $T_{\lambda,\mu}$ can not be supercyclic.

(iii)⇒(i) Let $|\lambda| < |\mu|$. Take an arbitrary vector $\mathbf{x} \in c_{00}(\mathbb{N})$. Then there exists a number $\ell \in \mathbb{N}$ such that $x_\ell \neq 0$ and $x_j = 0$ for all $j > \ell$. It is clear that $x_j^{(n)} = 0$, $j > \ell$ for any $n \ge 1$. For a given $n \ge \ell$ we have

$$x_j^{(n)} = \sum_{i=0}^{\ell-j} \binom{n}{i} \lambda^{n-i} \mu^i x_{j+i}, \qquad 1 \le j \le \ell.$$

which yields

$$\left|x_j^{(n)}\right| \le \left|\lambda^{n-\ell+j} \mu^{\ell-j}\right| \cdot |x_l|.$$

Hence,

$$\| T^n_{\lambda,\mu}(\mathbf{x}) \| \leq \left|\lambda^{n-l}\mu^l\right| \cdot \| \mathbf{x} \| \tag{20}$$

Now let us take an arbitrary vector $\mathbf{y} \in c_{00}(\mathbb{N})$ and compute the norm of $S^n_{\mu,\lambda}(\mathbf{y})$, where the operator $S_{\mu,\lambda}$ is defined by (12). From (12) and using the non-Archimedean norm's property one finds

$$\| S^n_{\mu,\lambda}(\mathbf{y}) \| \leq \left|\mu^{-n}\right| \cdot \| \mathbf{y} \| . \tag{21}$$

Multiplying (20) and (21) we obtain

$$\| T^n_{\lambda,\mu}(\mathbf{x}) \| \cdot \| S^n_{\mu,\lambda}(\mathbf{y}) \| \leq \left(\frac{|\lambda|}{|\mu|}\right)^{n-l} \| \mathbf{x} \| \cdot \| \mathbf{y} \| .$$

Due to $|\lambda| < |\mu|$ one has $\| T^n_{\lambda,\mu}(\mathbf{x}) \| \cdot \| S^n_{\mu,\lambda}(\mathbf{y}) \| \to 0$ as $n \to \infty$. Hence, $T_{\lambda,\mu}$ satisfies the Supercyclicity Criterion. This completes the proof. ■

Remark 5.2. We stress that all operators on c_0 considered above are hypercyclic (resp. supercyclic) if they satisfy Hypercyclic (reps. Supercyclic) Criterion. It is natural to ask: does there exist a hypercyclic (resp. supercyclic) linear operator on c_0 which does not satisfy HC (SC)? We conjecture that such kind of linear operators on c_0 do not exist.

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