Existence of multiple nontrivial solutions for a class of quasilinear Schrödinger equations on $\mathbb{R}^N$ *

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Abstract

This paper is concerned with the following fourth-order elliptic equations

$$\triangle^2 u - \Delta u + V(x)u - \frac{\kappa}{2}\Delta(u^2)u = f(x,u), \quad \text{in } \mathbb{R}^N,$$

where $N \leq 6$, $\kappa \geq 0$. Under some appropriate assumptions on $V(x)$ and $f(x,u)$, we prove the existence and multiplicity of solutions for the above equations via variational methods. Recent results from the literature are extended.

1 Introduction

Consider the following fourth-order elliptic equations of the form

$$\alpha\triangle^2 u - \Delta u + V(x)u - \frac{\kappa}{2}\Delta(u^2)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $\triangle^2 := \triangle(\triangle)$ is the biharmonic operator, $\alpha,\ \kappa \in \mathbb{R}$.

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When $\alpha = 1, \kappa = 0$, (1.1) becomes the following fourth-order elliptic equation

$$\triangle^2 u - \Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Many authors studied Eq. (1.2) on a bounded domain as follows

$$\begin{cases} \triangle^2 u - \Delta u = f(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In [1], An and Liu used the Mountain Pass Theorem to get the existence results for Eq. (1.3). In [34], when the nonlinearity $f(x,t)$ is odd in t and satisfies some additional conditions, Zhou and Wu got infinitely many sign-changing solutions via variational methods. While without symmetry, Wang and Shen in [22] obtained the multiplicity result by perturbation theory. In [32], Zhang and Wei obtained the existence of infinitely many solutions via variant fountain theorem established in Zou [35] when the nonlinearity $f(x,u)$ involves a combination of superlinear and asymptotically linear terms.

Fourth-order elliptic equation on unbounded domains also attract a lot of attention. For instance, see [2, 3, 24, 25, 26, 27, 28] and the references therein. In [28], by using the Mountain Pass Theorem and the Symmetric Mountain Pass Theorem, Yin and Wu obtained infinitely many high energy solutions for problem (1.2) under the condition that $f(x,u)$ is superlinear at infinity in u. However, for the whole space $\mathbb{R}^N$ case, the main difficulty of this problem is the lack of compactness for the Sobolev's embedding theorem. In order to overcome this difficulty, they assumed that the potential $V(x)$ satisfies

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf\limits_{x\in\mathbb{R}^N} V(x) \geq a > 0$, where $a > 0$ is a constant. Moreover, for any $M > 0$, $meas\{x \in \mathbb{R}^N : V(x) \leq M\} < \infty$, where meas denotes the Lebesgue measure in $\mathbb{R}^N$.

Later, under the condition (V_1), when $f(x,u)$ satisfies more general conditions, Ye and Tang [27] obtained the existence of infinitely many large-energy and small-energy solutions, which unified and generalized the results in [28], besides, the sublinear case was also considered by them.

Eq. (1.1) with $\alpha = 0$ is a quasilinear Schrödinger equation (also called modified nonlinear Schrödinger equation), whose solutions are related to the existence of solitary wave solutions for the following quasilinear Schrodinger equation

$$i\frac{\partial\psi}{\partial t} = -\triangle\psi + V(x)\psi - \kappa\triangle(\rho(|\psi|^2))\rho'(|\psi|^2) - f(x,\psi), \quad x \in \mathbb{R}^N, \tag{1.4}$$

where $V(x)$ is a given potential, κ is a real constant, ρ and f are real functions. We would like to mention that quasilinear equation of the form (1.4) arises in various branches of mathematical physics and has been derived as models of several physics phenomenon corresponding to various types of nonlinear terms ρ, see [6, 7, 12].

The semilinear case ($\kappa = 0$) has been studied extensively in recent years with a huge variety of conditions on the potential $V(x)$ and the nonlinearity f, see for example [14, 20, 33] and the references therein. Compared to the semilinear problem, the quasilinear case ($\kappa \neq 0$) becomes more complicated since the effects of the quasilinear and non-convex term $\triangle(u^2)u$. One of the main difficulties of the quasilinear problem is that there is no suitable space on which the energy functional is well defined and belongs to C^1-class except for $N = 1$ (see [13]). There has been several ideas and approaches used in recent years to overcome the difficulties such as by minimizations [11, 13], the Nehari or Pohozaev manifold [10, 16] and change of variables [29, 30].

On the other hand, Morse theory and local linking theorem are powerful tools in modern nonlinear analysis [4, 5, 17, 19], especially for the problems with resonance [8, 18]. However, to the best of our knowledge, there are no papers dealing with the existence of solutions for modified nonlinear fourth-order elliptic equations by using Morse theory.

Inspired by the above facts, the aim of this paper is to study the existence of multiple nontrivial solutions for problem (1.1) with $\alpha = 1$. On the one hand, we prove problem (1.1) has at least two nontrivial solutions by using Morse theory and local linking arguments. On the other hand, by using the Clark theorem, the existence results of at least k distinct pairs of nontrivial solutions is obtained.

We assume that $V(x)$ satisfies (V_1) and $f(x,u)$ satisfies the following hypotheses.

(f_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and there exist $1 < \alpha_1 < \alpha_2 < 2$ and positive functions $c_1 \in L^{\frac{2}{2-\alpha_1}}(\mathbb{R}^N, \mathbb{R})$, $c_2 \in L^{\frac{2}{2-\alpha_2}}(\mathbb{R}^N, \mathbb{R})$ such that

$$|f(x,u)| \le \alpha_1 c_1(x)|u|^{\alpha_1 - 1} + \alpha_2 c_2(x)|u|^{\alpha_2 - 1}, \ \ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

(f_2) There exist $c_1 > 0$, $0 < c_2 < \frac{1}{2S_2^2}$, $1 < \gamma < 2$ and small constants $0 < r < r_0$, such that

$$c_1|u|^\gamma < F(x,u) \le c_2|u|^2, \ \ r \le |u| \le r_0, \ \ a.e. \ \ x \in \mathbb{R}^N,$$

where S_2 is the the best Sobolev constant from the working space E into $L^2(\mathbb{R}^N)$ and $F(x,u) = \int_0^u f(x,s)ds$.

(f_3) $f(x,u) = -f(x,-u)$, *for all* $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.

Now, we state our main results.

Theorem 1.1. *Assume conditions (V_1) and $(f_1) - (f_2)$ hold, then problem (1.1) has at least two nontrivial solutions.*

Theorem 1.2. *Assume conditions (V_1) and $(f_1) - (f_3)$ hold, then problem (1.1) has at least k distinct pairs of nontrivial solutions, where $k \in \mathbb{N}$.*

Remark 1.1. It is well known that for the quasilinear Schrödinger equation problem (1.1), we must overcome the difficulty that the energy functional is not well defined due to the non-convex term $\triangle(u^2)u$, while in this paper, under the assumptions (V_1) and $N \le 6$, we prove $\int_{\mathbb{R}^N} |\nabla u|^2 u^2 dx < \infty$, which implies the energy functional of problem (1.1) is well defined on our working space.

Notation 1.1. Throughout this paper, we shall denote by $\|\cdot\|_r$ the L^r-norm and C various positive generic constants, which may vary from line to line. $2_* = +\infty$ for $N \le 4$ and $2_* = \frac{2N}{N-4}$ for $N \ge 5$, is the critical Sobolev exponent. Also if we take a subsequence of a sequence $\{u_n\}$ we shall denote it again by $\{u_n\}$.

2 Variational setting and preliminaries

Let

$$L^r(\mathbb{R}^N)=\{u:\mathbb{R}^N\to\mathbb{R}:\ \text{u is measurable and } \int_{\mathbb{R}^N}|u|^r dx<\infty\},\ 1\le r<\infty,$$

with the norm

$$\|u\|_r=(\int_{\mathbb{R}^N}|u|^r dx)^{\frac{1}{r}}.$$

$$H^2(\mathbb{R}^N):=\{u\in L^2(\mathbb{R}^N):\nabla u,\triangle u\in L^2(\mathbb{R}^N)\}.$$

$$E:=\{u\in H^2(\mathbb{R}^N)|\int_{\mathbb{R}^N}V(x)u^2dx<+\infty\}.$$

Then, under the conditions (V_1), E is a Hilbert space with the following inner product and norm

$$\langle u,v\rangle=\int_{\mathbb{R}^N}(\Delta u\Delta v+\nabla u\nabla v+V(x)uv)dx,$$

$$\|u\|=(\int_{\mathbb{R}^N}(|\Delta u|^2+|\nabla u|^2+V(x)|u|^2)dx)^{\frac{1}{2}}.$$

Moreover, we have the following compactness lemma from [3].

Lemma 2.1.*([[3], Lemma 2.1]) Under the assumption (V_1), the embedding $E\hookrightarrow L^r(\mathbb{R}^N)$ is continuous for $2\le r\le 2_*$ and $E\hookrightarrow L^r(\mathbb{R}^N)$ is compact for $2\le r<2_*$.*

Lemma 2.2. *Under assumption (V_1), (f_1) and $N\le 6$, the functional $I:E\to\mathbb{R}$ defined by*

$$I(u)=\frac{1}{2}\|u\|^2+\frac{\kappa}{2}\int_{\mathbb{R}^N}u^2|\nabla u|^2dx-\int_{\mathbb{R}^N}F(x,u)dx \tag{2.1}$$

is well defined and of class $C^1(E,\mathbb{R})$ and

$$\langle I'(u),v\rangle=(u,v)+\kappa\int_{\mathbb{R}^N}(uv|\nabla u|^2+u^2\nabla u\nabla v)dx-\int_{\mathbb{R}^N}f(x,u)vdx. \tag{2.2}$$

Moreover, the critical points of I in E are solutions of problem (1.1).
Proof . From (f_1), one has

$$|F(x,u)|\le c_1(x)|u|^{\alpha_1}+c_2(x)|u|^{\alpha_2},\ \forall(x,u)\in\mathbb{R}^N\times\mathbb{R}. \tag{2.3}$$

Then, for any $u\in E$, it follows from (V_1), (2.3) and the Hölder inequality that

$$\begin{aligned}\int_{\mathbb{R}^N}|F(x,u)|dx&\le\int_{\mathbb{R}^N}\left[c_1(x)|u|^{\alpha_1}+c_2(x)|u|^{\alpha_2}\right]dx\\&\le\sum_{i=1}^{2}a^{\frac{-\alpha_i}{2}}\Big(\int_{\mathbb{R}^N}|c_i(x)|^{\frac{2}{2-\alpha_i}}dx\Big)^{\frac{2-\alpha_i}{2}}\Big(\int_{\mathbb{R}^N}V(x)|u|^2dx\Big)^{\frac{\alpha_i}{2}}\\&\le\sum_{i=1}^{2}a^{\frac{-\alpha_i}{2}}||c_i||_{\frac{2}{2-\alpha_i}}||u||^{\alpha_i}.\end{aligned} \tag{2.4}$$

Next, we prove $\int_{\mathbb{R}^N} u^2|\nabla u|^2dx < +\infty$ for every $u \in E$. Firstly, we choose two numbers $p = 3$ and $t = \frac{p}{p-1}$. Then $\frac{1}{p} + \frac{1}{t} = 1$, $2 \le 2p \le 2_*$ and $2 \le 2t \le 2^*$ for $N \le 6$. Then by Lemma 2.1 and the assumption of (V_1), we have

$$\|u\|_{H^2}^2 = \int_{\mathbb{R}^N}(|\Delta u|^2 + |\nabla u|^2 + |u|^2)dx \le C\int_{\mathbb{R}^N}(|\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2)dx = C\|u\|^2,$$

where $C = max\{1, \frac{1}{a}\}$.

Since $H^2(\mathbb{R}^N) = W^{2,2}(\mathbb{R}^N) \hookrightarrow W^{1,r}(\mathbb{R}^N)$, $2 \le r \le 2^*$ and $H^2(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$, $2 \le r \le 2_*$, we have

$$\int_{\mathbb{R}^N} u^{2p}dx < +\infty, \quad \int_{\mathbb{R}^N} |\nabla u|^{2t}dx < +\infty.$$

By Holder inequality and Lemma 2.1, we have

$$\int_{\mathbb{R}^N} u^2|\nabla u|^2dx \le \Big(\int_{\mathbb{R}^N} u^{2p}dx\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{R}^N} |\nabla u|^{2t}dx\Big)^{\frac{1}{t}} < +\infty, \tag{2.5}$$

It follows from (2.4) and (2.5) that I is well defined on E.

Now, we prove that $I \in C^1(E, \mathbb{R})$. Set

$$\Phi_1(u) := \frac{1}{2}\int_{\mathbb{R}^N} u^2|\nabla u|^2dx, \quad \Phi_2(u) := \int_{\mathbb{R}^N} F(x,u)dx.$$

Then $I(u) = \frac{1}{2}\|u\|^2 + \kappa\Phi_1(u) - \Phi_2(u)$. In order to prove $I \in C^1(E, \mathbb{R})$, we only have to prove that $\Phi_i \in C^1(E, \mathbb{R})$, i=1,2. By the proof of Lemma 2.2 in [3], it is easy to verify that $\Phi_1 \in C^1(E, \mathbb{R})$. Next, we prove (2.2) and $\Phi_2 \in C^1(E, \mathbb{R})$.

For any function $\theta : \mathbb{R}^N \to (0, 1)$, by (f_1) and the Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \max_{t\in[0,1]} |f(x, u(x) + t\theta(x)v(x))v(x)|dx \\
&\quad= \int_{\mathbb{R}^N} \max_{t\in[0,1]} |f(x, u(x) + t\theta(x)v(x))||v(x)|dx \\
&\quad\le \sum_{i=1}^{2} \alpha_i \int_{\mathbb{R}^N}(c_i(x)|u(x) + t\theta(x)v(x)|^{\alpha_i-1})|v(x)|dx \\
&\quad\le \sum_{i=1}^{2} \alpha_i \int_{\mathbb{R}^N}(c_i(x)(|u(x)|^{\alpha_i-1} + |v(x)|^{\alpha_i-1})|v(x)|dx \\
&\quad\le \sum_{i=1}^{2} \alpha_i a^{\frac{-\alpha_i}{2}}\Big(\int_{\mathbb{R}^N} |c_i(x)|^{\frac{2}{2-\alpha_i}}dx\Big)^{\frac{2-\alpha_i}{2}}\Big(\int_{\mathbb{R}^N} V(x)|u(x)|^2dx\Big)^{\frac{\alpha_i-1}{2}} \\
&\qquad\times\Big(\int_{\mathbb{R}^N} V(x)|v(x)|^2dx\Big)^{\frac{1}{2}} \\
&\qquad+ \sum_{i=1}^{2} \alpha_i a^{\frac{-\alpha_i}{2}}\Big(\int_{\mathbb{R}^N} |c_i(x)|^{\frac{2}{2-\alpha_i}}dx\Big)^{\frac{2-\alpha_i}{2}}\Big(\int_{\mathbb{R}^N} V(x)|v(x)|^2dx\Big)^{\frac{\alpha_i}{2}} \\
&\quad\le \sum_{i=1}^{2} \alpha_i a^{\frac{-\alpha_i}{2}}||c_i||_{\frac{2}{2-\alpha_i}}(||u||^{\alpha_i-1} + ||v||^{\alpha_i-1})||v|| \\
&\quad< +\infty.
\end{aligned} \tag{2.6}$$

Then, by (2.1), (2.6) and Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned}\langle I'(u),v\rangle &= \lim_{t\to 0}\frac{I(u+tv)-I(u)}{t}\\ &= \lim_{t\to 0}\Big[(u,v)+\frac{t}{2}||v||^2+\frac{\kappa}{2}\int_{\mathbb{R}^N}\big(t^3v^2|\nabla v|^2+2t^2v^2\nabla u\nabla v+2t^2uv|\nabla v|^2\\ &\quad +4tuv\nabla u\nabla v+tu^2|\nabla v|^2+tv^2|\nabla u|^2+2u^2\nabla u\nabla v+2uv|\nabla u|^2\big)\\ &\quad -\int_{\mathbb{R}^N}f(x,u+\theta tv)vdx\Big]\\ &= (u,v)+\kappa\int_{\mathbb{R}^N}(uv|\nabla u|^2+u^2\nabla u\nabla v)dx-\int_{\mathbb{R}^N}f(x,u)vdx.\end{aligned} \tag{2.7}$$

Therefore, it follows from Proposition 1.3 in [23] and (2.7) that (2.2) holds. Now, we show that $\Phi_2\in C^1(E,\mathbb{R})$. Let $u_n\to u$ in E, then $u_n\to u$ in $L^2(\mathbb{R}^N)$ and

$$\lim_{n\to\infty}u_n=u\ a.e.\ x\in\ \mathbb{R}^N. \tag{2.8}$$

Now, we claim that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|f(x,u_n)-f(x,u)|^2dx=0. \tag{2.9}$$

Otherwise, there exist a constant $\varepsilon_0>0$ and a sequence $\{u_{ni}\}$ such that

$$\int_{\mathbb{R}^N}|f(x,u_{ni})-f(x,u)|^2dx\geq\varepsilon_0,\ \ \forall i\in\mathbb{N}. \tag{2.10}$$

In fact, since $u_n\to u$ in $L^2(\mathbb{R}^N)$, passing to a subsequence if necessary, it can be assumed that $\sum_{i=1}^{\infty}||u_{ni}-u||_2^2<+\infty$. Set $\omega(x)=(\sum_{i=1}^{\infty}|u_{ni}(x)-u(x)|^2)^{\frac{1}{2}}$, then $\omega(x)\in L^2(\mathbb{R}^N)$. Evidently

$$\begin{aligned}&|f(x,u_{ni})-f(x,u)|^2\\ &\qquad\leq 2|f(x,u_{ni})|^2+2|f(x,u)|^2\\ &\qquad\leq 4\alpha_1^2|c_1(x)|^2\big[|u_{ni}|^{2(\alpha_1-1)}+|u|^{2(\alpha_1-1)}\big]\\ &\qquad\quad +4\alpha_2^2|c_2(x)|^2\big[|u_{ni}|^{2(\alpha_2-1)}+|u|^{2(\alpha_2-1)}\big]\\ &\qquad\leq\sum_{j=1}^{2}(4^{\alpha_j}+4)\alpha_j^2|c_j(x)|^2\big[|u_{ni}-u|^{2(\alpha_j-1)}+|u|^{2(\alpha_j-1)}\big]\\ &\qquad\leq\sum_{j=1}^{2}(4^{\alpha_j}+4)\alpha_j^2|c_j(x)|^2\big[|\omega(x)|^{2(\alpha_j-1)}+|u|^{2(\alpha_j-1)}\big]\\ &\qquad:=h(x),\ \ \forall\, i\in\ \mathbb{N},\ \ x\in\mathbb{R}^N\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N}h(x)dx&=\sum_{j=1}^{2}(4^{\alpha_j}+4)\alpha_j^2\int_{\mathbb{R}^N}|c_j(x)|^2\big[|\omega(x)|^{2(\alpha_j-1)}+|u|^{2(\alpha_j-1)}\big]dx\\ &\leq\sum_{j=1}^{2}(4^{\alpha_j}+4)\alpha_j^2||c_j||^2_{\frac{2}{2-\alpha_j}}\big(||\omega||_2^{2(\alpha_j-1)}+||u||_2^{2(\alpha_j-1)}\big)\\ &<+\infty.\end{aligned} \tag{2.12}$$

It follows from (2.11), (2.12) and the Lebesgue's Dominated Convergence Theorem, we have

$$\int_{\mathbb{R}^N} |f(x, u_{ni}) - f(x, u)|^2 dx = 0,$$

which is a contradiction with (2.10). Hence (2.9) holds. Then, by (2.2), (2.9) and $\Phi_1 \in C^1(E, \mathbb{R})$, we have

$$\begin{aligned}
\left|\left\langle I'(u_n) - I'(u), v\right\rangle\right| &= \Big|(u_n - u, v) + \kappa \int_{\mathbb{R}^N} \left(|u_n|^2 \nabla u_n - |u|^2 \nabla u\right) \cdot \nabla v dx \\
&\quad + \kappa \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 u_n - |\nabla u|^2 u\right) \cdot v dx \\
&\quad - \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u)] v dx \Big| \\
&\leq ||u_n - u|| ||v|| + |\kappa \int_{\mathbb{R}^N} \left(|u_n|^2 \nabla u_n - |u|^2 \nabla u\right) \cdot \nabla v dx \\
&\quad + \kappa \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 u_n - |\nabla u|^2 u\right) v dx| \\
&\quad + a^{-\frac{1}{2}} \Big(\int_{\mathbb{R}^N} |f(x, u_n) - f(x, u)|^2 dx\Big)^{\frac{1}{2}} ||v|| \\
&\to 0, \quad \textit{as } n \to \infty,
\end{aligned}$$

which implies that $I \in C^1(E, \mathbb{R})$. Moreover, by a standard argument, it is easy to verify that the critical points of I in E are solutions of problem (1.1). The proof is complete. ∎

We will use Morse theory in combination with local linking arguments to obtain the critical points of I, so we recall the following definitions and results.

Definition 2.1. Let E be a real reflexive Banach space. We say that I satisfies the (PS)-condition, i.e. every sequence $\{u_n\} \subset E$ satisfying $I(u_n)$ bounded and $\lim_{n\to\infty} I'(u_n) = 0$ contains a convergent subsequence.

Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$. $K = \{u \in E : I'(u) = 0\}$, then the qth critical group of I at an isolated critical point $u \in K$ with $I(u) = c$ is defined by

$$C_q(I, u) := H_q(I^c \cap U, I^c \cap U \setminus \{u\}), \quad q \in \mathbb{N} := \{0, 1, 2, \cdots\},$$

where $I^c = \{u \in E : I(u) \leq c\}$, U is a neighborhood of u, containing the unique critical point, H_* is the singular relative homology with coefficient in an Abelian group G.

We say that $u \in E$ is a homological nontrivial critical point of I if at least one of its critical groups is nontrivial.

Now, we present the following propositions that will be used later.

Theorem 2.1 ([9], Theorem 2.1). *Assume that I has a critical point $u = 0$ with $I(0) = 0$. Suppose that I has a local linking at 0 with respect to $E = V \oplus W$, $k = dim\ V < \infty$, that is, there exists $\rho > 0$ small such that*

$$\begin{cases} I(u) \leq 0, & u \in V, \quad \|u\| \leq \rho; \\ I(u) > 0, & u \in W, \quad 0 < \|u\| \leq \rho. \end{cases}$$

Then $C_k(I,0) \ncong 0$, hence 0 is a homological nontrivial critical point of I.

Theorem 2.2 ([9], Theorem 2.1)**.** *Let E be a real Banach space and let $I \in C^1(E,\mathbb{R})$ satisfy the (PS)-condition and is bounded from below. If I has a critical point that is homological nontrivial and is not a minimizer of I, then I has at least three critical points.*

Theorem 2.3 ([15], Theorem 9.1)**.** *Let E be a real Banach space, $I \in C^1(E,\mathbb{R})$ with I even, bounded from below, and satisfying (PS)-condition. Suppose $I(0) = 0$, there is a set $K \subset E$ such that K is homeomorphic to S^{j-1} by an odd map, and $\sup_K I < 0$. Then I possesses at least j distinct pairs of critical points.*

3 Proofs of main results

In this section, we will prove Theorem 1.1 and Theorem 1.2. To complete the proof, we need the following lemmas.

Lemma 3.1. *Assume that (V_1), (f_1) and $N \leq 6$ hold, then I is bounded from below and satisfies the (PS) condition.*
Proof . By Lemma 2.1, (f_1), the Sobolev embedding theorem and the Hölder inequality, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{\kappa}{2}\int_{\mathbb{R}^N} u^2|\nabla u|^2 dx - \int_{\mathbb{R}^N} F(x,u)dx \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)dx \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} c_1(x)|u|^{\alpha_1}dx - \int_{\mathbb{R}^N} c_2(x)|u|^{\alpha_2}dx \\ &\geq \frac{1}{2}\|u\|^2 - \sum_{i=1}^{2} a^{\frac{-\alpha_i}{2}}||c_i||_{\frac{2}{2-\alpha_i}}||u||^{\alpha_i}, \end{aligned} \tag{3.1}$$

which implies that $I(u) \to +\infty$, *as* $n \to \infty$, since α_1, $\alpha_2 \in (1,2)$. Consequently, I is bounded from below.

Next, we prove that I satisfies the (PS) condition. Assume that $\{u_n\}$ is a (PS) sequence of I such that $I(u_n)$ is bounded and $||I'(u_n)|| \to 0$, *as* $n \to \infty$. Then, it follows from (3.1) that there exists a constant $C > 0$ such that

$$||u_n||_2 \leq a^{\frac{-1}{2}}||u_n|| \leq C, \;\; n \in \mathbb{N}. \tag{3.2}$$

Then by Lemma 2.1, there exists $u \in E$ such that

$$\begin{aligned} &u_n \rightharpoonup u \;\; in \; E, \\ &u_n \to u \;\; in \; L^s(\mathbb{R}^N), \;\; s \in [2, 2_*), \\ &u_n \to u \;\; a.e. \;\; \mathbb{R}^N. \end{aligned} \tag{3.3}$$

Therefore

$$\int_{\mathbb{R}^N} \left(|u_n|^2\nabla u_n - |u|^2\nabla u\right)\cdot\nabla(u_n-u)dx$$

$$\begin{aligned}
&= \int_{\mathbb{R}^N} \left(|u_n|^2 - |u|^2\right)\nabla u_n \nabla(u_n-u)dx + \int_{\mathbb{R}^N} |u|^2|\nabla(u_n-u)|^2dx \\
&\geq \int_{\mathbb{R}^N} \left(|u_n|^2 - |u|^2\right)\nabla u_n \nabla(u_n-u)dx \\
&\geq -\int_{\mathbb{R}^N} (|u_n-u|(|u_n|+|u|)|\nabla u_n||\nabla(u_n-u)|)dx \\
&\geq -\Big(\int_{\mathbb{R}^N} |u_n-u|^6dx\Big)^{\frac{1}{6}}\Big(\int_{\mathbb{R}^N}(|u_n|+|u|)^6dx\Big)^{\frac{1}{6}} \\
&\qquad\qquad \times\Big(\int_{\mathbb{R}^N}|\nabla u_n|^3dx\Big)^{\frac{1}{3}}\Big(\int_{\mathbb{R}^N}|\nabla(u_n-u)|^3dx\Big)^{\frac{1}{3}} \\
&\qquad\qquad \geq -C||u_n-u||_6 \to 0,\ \ n\to\infty.
\end{aligned} \tag{3.4}$$

Analogously, we have

$$\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 u_n - |\nabla u|^2 u\right)\cdot(u_n-u)dx$$

$$\begin{aligned}
&= \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 - |\nabla u|^2\right)u(u_n-u)dx + \int_{\mathbb{R}^N} |\nabla u_n|^2|(u_n-u)|^2dx \\
&\geq -\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + |\nabla u|^2\right)|u_n||u_n-u|dx \\
&\geq -\Big(\int_{\mathbb{R}^N} |u_n-u|^6dx\Big)^{\frac{1}{6}}\Big(\int_{\mathbb{R}^N} |u|^6dx\Big)^{\frac{1}{6}}\Big(\int_{\mathbb{R}^N} |\nabla u_n|^3dx\Big)^{\frac{1}{3}} \\
&\quad -\Big(\int_{\mathbb{R}^N} |u_n-u|^6dx\Big)^{\frac{1}{6}}\Big(\int_{\mathbb{R}^N} |u|^6dx\Big)^{\frac{1}{6}}\Big(\int_{\mathbb{R}^N} |\nabla u|^3dx\Big)^{\frac{1}{3}} \\
&\geq -C||u_n-u||_6 \to 0,\ \ n\to\infty.
\end{aligned} \tag{3.5}$$

On the other hand, for any given $\varepsilon > 0$, by (f_1) , we can choose $R_\varepsilon > 0$ such that

$$\Big(\int_{|x|>R_\varepsilon} |c_i(x)|^{\frac{2}{2-\alpha_i}}dx\Big)^{\frac{2-\alpha_i}{2}} < \varepsilon,\ \ i=1,2. \tag{3.6}$$

It follows from (3.3) that there exists $n_0 > 0$ such that

$$\int_{|x|\leq R_\varepsilon} |u_n-u|^2dx < \varepsilon^2,\ \ for\ n\geq n_0. \tag{3.7}$$

Therefore, by (f_1), (3.2), (3.7) and the Hölder inequality, for any $n \geq n_0$, one has

$$\begin{aligned}
&\int_{|x|\leq R_\varepsilon} |f(x,u_n) - f(x,u)||u_n - u|dx \\
&\qquad \leq \Big(\int_{|x|\leq R_\varepsilon} |f(x,u_n) - f(x,u)|^2 dx\Big)^{\frac{1}{2}} \Big(\int_{|x|\leq R_\varepsilon} |u_n - u|^2 dx\Big)^{\frac{1}{2}} \\
&\qquad \leq \varepsilon \Big[\int_{|x|\leq R_\varepsilon} 2(|f(x,u_n)|^2 + |f(x,u)|^2) dx\Big]^{\frac{1}{2}} \\
&\qquad \leq \varepsilon \Big[4 \sum_{i=1}^{2} \alpha_i^2 \int_{|x|\leq R_\varepsilon} |c_i(x)|^2 (|u_n|^{2(\alpha_i-1)} + |u|^{2(\alpha_i-1)}) dx\Big]^{\frac{1}{2}} \\
&\qquad \leq C\varepsilon \Big[\sum_{i=1}^{2} \alpha_i^2 ||c_i||^2_{\frac{2}{2-\alpha_i}} \big(||u_n||_2^{2(\alpha_i-1)} + ||u||_2^{2(\alpha_i-1)} \big)\Big]^{\frac{1}{2}} \\
&\qquad \leq C\varepsilon \Big[\sum_{i=1}^{2} \alpha_i^2 ||c_i||^2_{\frac{2}{2-\alpha_i}} \big(C^{2(\alpha_i-1)} + ||u||_2^{2(\alpha_i-1)} \big)\Big]^{\frac{1}{2}}.
\end{aligned} \tag{3.8}$$

For another, for $n \in \mathbb{N}$, it follows from (f_1), (3.2), (3.6) and Hölder inequality that

$$\begin{aligned}
&\int_{|x|> R_\varepsilon} |f(x,u_n) - f(x,u)||u_n - u|dx \\
&\qquad \leq \sum_{i=1}^{2} \alpha_i \int_{|x|> R_\varepsilon} |c_i(x)| \big(|u_n|^{\alpha_i - 1} + |u|^{\alpha_i-1} \big) \big(|u_n| + |u| \big) dx \\
&\qquad \leq 2 \sum_{i=1}^{2} \alpha_i \int_{|x|> R_\varepsilon} |c_i(x)| \big(|u_n|^{\alpha_i} + |u|^{\alpha_i} \big) dx \\
&\qquad \leq 2 \sum_{i=1}^{2} \alpha_i \Big(\int_{|x|> R_\varepsilon} |c_i(x)|^{\frac{2}{2-\alpha_i}} dx\Big)^{\frac{2-\alpha_i}{2}} \big(||u_n||_2^{\alpha_i} + ||u||_2^{\alpha_i} \big) \\
&\qquad \leq 2 \sum_{i=1}^{2} \alpha_i \Big(\int_{|x|> R_\varepsilon} |c_i(x)|^{\frac{2}{2-\alpha_i}} dx\Big)^{\frac{2-\alpha_i}{2}} \big(C^{\alpha_i} + ||u||_2^{\alpha_i} \big) \\
&\qquad \leq 2\varepsilon \sum_{i=1}^{2} \alpha_i \big(C^{\alpha_i} + ||u||_2^{\alpha_i} \big).
\end{aligned} \tag{3.9}$$

Since ε is arbitrary, combining (3.8) and (3.9), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))(u_n - u) dx = 0. \tag{3.10}$$

Then by (2.2), (3.4), (3.5), (3.10) and the weak convergence of $\{u_n\}$, one has

$$\begin{aligned}
o_n(1) &= \langle I'(u_n) - I'(u), u_n - u \rangle \\
&= \int_{\mathbb{R}^N} |\triangle(u_n - u)|^2 dx + \int_{\mathbb{R}^N} |\nabla(u_n - u)|^2 dx + \int_{\mathbb{R}^N} V(x)(u_n - u)^2 dx \\
&\quad + \kappa \int_{\mathbb{R}^N} \left(|u_n|^2 \nabla u_n - |u|^2 \nabla u\right) \cdot \nabla(u_n - u) dx \\
&\quad + \kappa \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 u_n - |\nabla u|^2 u\right) \cdot (u_n - u) dx \\
&\quad - \int_{\mathbb{R}^N} (f(x, u_n) - f(x, u))(u_n - u) dx \\
&\geq ||u_n - u||^2 + \kappa \int_{\mathbb{R}^N} (u_n^2 - u^2) \nabla u \nabla(u_n - u) dx + \\
&\quad \kappa \int_{\mathbb{R}^N} (|\nabla u_n|^2 - |\nabla u|^2) u(u_n - u) dx \\
&\quad - \int_{\mathbb{R}^N} (f(x, u_n) - f(x, u))(u_n - u) dx \\
&= ||u_n - u||^2 + o_n(1),
\end{aligned}$$

which implies that $u_n \to u$ in E. Therefore, I satisfies the (PS) condition. The proof is complete. ∎

We choose an orthogonal basis $\{e_j\}$ of E and define $X_j := \operatorname{span}\{e_j\}$, $j = 1, 2, \cdots$, $Y_k := \oplus_{j=1}^{k} X_j$, $Z_k = \overline{\oplus_{j=k+1}^{\infty} X_j}$, then $E = Y_k \oplus Z_k$.

Lemma 3.2. *Suppose that the conditions of Theorem 1.1 are satisfied, then $C_k(I, 0) \not\cong 0$.*

Proof. It follows from (f_1) that the zero function is a critical point of I. So we only need to prove that I has a local linking at 0 with respect to $E = Y_k \oplus Z_k$.

Step 1: Take $u \in Y_k$, since Y_k is finite dimensional, we have that for given r_0, there exists $0 < \rho < 1$ small such that

$$u \in Y_k, \quad \|u\| \leq \rho \Rightarrow |u| < r_0, \quad x \in \mathbb{R}^N.$$

For $0 < r < r_0$, let $\Omega_1 = \{x \in \mathbb{R}^N : |u(x)| < r\}$, $\Omega_2 = \{x \in \mathbb{R}^N : r \leq |u(x)| \leq r_0\}$, $\Omega_3 = \{x \in \mathbb{R}^N : |u(x)| > r_0\}$, then $\mathbb{R}^N = \bigcup_{i=1}^{3} \Omega_i$. For the sake of simplicity, let $G(x, u) = F(x, u) - c_1|u|^\gamma$. Therefore, from (f_2) it follows that

$$\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 + \frac{\kappa}{2} \int_{\mathbb{R}^N} u^2 |\nabla u|^2 dx - \int_{\mathbb{R}^N} c_1 |u|^\gamma dx - \left(\int_{\Omega_1} + \int_{\Omega_2} + \int_{\Omega_3} \right) G(x, u) dx \\
&\leq \frac{1}{2}\|u\|^2 + C\|u\|^4 - \int_{\mathbb{R}^N} c_1 |u|^\gamma dx - \int_{\Omega_1} G(x, u) dx.
\end{aligned}$$

Note that the norms on Y_k are equivalent to each other, $\|u\|_\gamma$ is equivalent to $\|u\|$ and $\int_{\Omega_1} G(x, u) dx \to 0$ as $r \to 0$. Since $0 < \gamma < 2$, then $I(u) \leq 0$, for all $u \in Y_k$ with $\|u\| \leq \rho$.

Step 2: Take $u \in Z_k$, since the embedding $E \hookrightarrow L^p$ is continuous, we have that for given r_0, there exists $0 < \rho < 1$ small such that

$$u \in Z_k, \quad \|u\| \le \rho \Rightarrow |u| < r_0, \quad x \in \mathbb{R}^N.$$

Therefore, it follows from (f_2) that

$$\begin{aligned} I(u) &\ge \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} c_2|u|^2 dx \\ &> \frac{1}{2}\|u\|^2 - \frac{1}{2}\|u\|^2 = 0. \end{aligned}$$

Therefore, we complete the proof due to Theorem 2.1. ■

Proof of Theorem 1.1. By Lemma 3.1, I satisfies the (PS)-condition and is bounded from below. By Lemma 3.2 and Theorem 2.1, the trivial solution $u = 0$ is homological nontrivial and is not a minimizer. Then Theorem 1.1 follows immediately from Theorem 2.2. ■

Proof of Theorem 1.2. By (f_3), we can easily check that the functional I is even. Lemma 3.1 shows that I satisfies the (PS)-condition and is bounded from below. For $\rho > 0$, let $K = S_\rho = \{u \in Y_k : \|u\| = \rho\}$. Thus, just as shown in the proof of Lemma 3.2, if $\rho > 0$ is small enough, we have that

$$\sup_K I(u) < 0.$$

By the definition of Y_k, we have $\dim Y_k = k$, then by Theorem 2.3, we have that I has at least k distinct pairs of critical points. Therefore, problem (1.1) has at least k distinct pairs of solutions. ■

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