Generalized derivatives and approximation in weighted Lorentz spaces

Ramazan Akgün* Yunus Emre Yildirir†

Abstract

In the present article we prove direct, simultaneous and converse approximation theorems by trigonometric polynomials for functions f and (ψ, β)-derivatives of f in weighted Lorentz spaces.

1 Introduction

In the 1980's, the concept of (ψ, β) derivative was formed for a given function f by a given sequence (ψ_k) and numbers β [23, 24, 25]. For $r = 1, 2, ...$ the r-th derivative of a periodic function f is a particular case of the (ψ, β)-derivative for the sequence $(\psi_k) = (k^{-r})$ and $\beta = r$. For $(\psi_k) = \left(k^{-\beta}\right)$ and $\beta > 0$, we have the Weyl fractional derivative $f^{(\beta)}$ of f [28]. When we take the sequence $(\psi_k) = \left(k^{-\beta} \ln^{-\alpha} k\right)$ and $\beta, \alpha \in \mathbb{R}^+$, we obtain the power logarithmic-fractional derivative $f^{(\beta,\alpha)}$ of f [17]. In [26], some relations were established between the sequences of best approximations of continuous 2π-periodic functions f (and also $f \in L_p$) by trigonometric polynomials of order $\leq n$ and the properties of their (ψ, β)-derivatives. Thus, they extended the well known results of Stechkin and Konyushkov [16, 22] to the case of generalized (ψ, β)-derivatives. In [20, 21], for Lebesgue spaces L_p, some estimates were obtained for the norms and moduli of smoothness of transformed Fourier series which coincides up to notation

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†Corresponding author.

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with the Fourier series of the (ψ, β)-derivatives. Also there are some estimates of best approximation and modulus of smoothness in Lebesgue spaces of periodic functions with transformed Fourier series in [13]. Approximation properties of functions having (ψ, β)-derivatives in variable exponent Lebesgue spaces which is a generalization of Lebesgue spaces was investigated in the papers [1, 2, 7].

Lorentz spaces were first introduced by G. G. Lorentz in [18]. Since these spaces are the generalization of the Lebesgue spaces, many mathematicians are interested in the problems of these spaces. Also there are many results of the approximation theory obtained in these spaces. Especially, approximation by trigonometric polynomials in the weighted Lorentz spaces was considered in the papers [3, 4, 15, 29, 30]. But these papers do not have results about the approximation properties of (ψ, β)-derivatives. In this paper, we obtain some results about approximation by trigonometric polynomials of functions having (ψ, β)-derivatives in weighted Lorentz spaces.

2 Auxiliary Results

We start by giving some necessary definitions.

Let $\mathbb{T} := [-\pi, \pi]$. A measurable 2π-periodic function $\omega : \mathbb{T} \to [0, \infty]$ is called a weight function if the set $\omega^{-1}(\{0, \infty\})$ has the Lebesgue measure zero. Given a weight function ω and a measurable set e we put

$$\omega(e) = \int_e \omega(x)dx. \tag{2.1}$$

We define the decreasing rearrangement $f^*_\omega(t)$ of $f : \mathbb{T} \to \mathbb{R}$ with respect to the Borel measure (2.1) by

$$f^*_\omega(t) = \inf\left\{\tau \geq 0 : \omega\left(x \in \mathbb{T} : |f(x)| > \tau\right) \leq t\right\}.$$

The weighted Lorentz space $L^{pq}_\omega(\mathbb{T})$ is defined [10, p.20], [5, p.219] as

$$L^{pq}_\omega(\mathbb{T}) = \left\{ f \in \mathbf{M}(\mathbb{T}) : \|f\|_{pq,\omega} = \left(\int_{\mathbb{T}} \left(f^{**}(t)\right)^q t^{\frac{q}{p}} \frac{dt}{t} \right)^{1/q} < \infty,\ 1 < p, q < \infty \right\},$$

where $\mathbf{M}(\mathbb{T})$ is the set of 2π periodic integrable functions on $\mathbb{T}$ and

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*_\omega(u)du.$$

If $p = q$, $L^{pq}_\omega(\mathbb{T})$ turns into the weighted Lebesgue space $L^p_\omega(\mathbb{T})$ [10, p.20].

The generalized modulus of smoothness of a function $f \in L^{pq}_\omega(\mathbb{T})$ is defined [11] as

$$\Omega_l\left(f, \delta\right)_{pq,\omega} = \sup_{0<h_i<\delta} \left\| \prod_{i=1}^{l} \left(I - A_{h_i}\right) f \right\|_{pq,\omega}, \qquad \delta \geq 0,\ l = 1, 2, ...$$

where I is the identity operator and

$$\left(A_{h_i}f\right)(x) := \frac{1}{2h_i}\int\limits_{x-h_i}^{x+h_i} f(u)du.$$

The modulus of smoothness $\Omega_l\left(f,\delta\right)_{pq,\omega}$, $\delta \geq 0$, $l = 1,2,...$ has the following properties:

(i) $\Omega_l\left(f,\delta\right)_{pq,\omega}$ *is a non-negative, non-decreasing function of* $\delta \geq 0$ *and sub-additive in* f*,*

(ii) $\lim\limits_{\delta\to 0}\Omega_l\left(f,\delta\right)_{pq,\omega} = 0$*,*

(iii) $\Omega_l\left(f_1+f_2,\cdot\right)_{pq,\omega} \leq \Omega_l\left(f_1,\cdot\right)_{pq,\omega} + \Omega_l\left(f_2,\cdot\right)_{pq,\omega}$*.*

The weight functions ω used in the paper belong to the Muckenhoupt class $A_p(\mathbb{T})$ [19] which is defined by

$$\sup\frac{1}{|I|}\int\limits_I \omega(x)dx\left(\frac{1}{|I|}\int\limits_I \omega^{1-p'}(x)dx\right)^{p-1} < \infty, \qquad p' = \frac{p}{p-1}, \quad 1 < p < \infty$$

where the supremum is taken with respect to all the intervals I with length $\leq 2\pi$ and $|I|$ denotes the length of I.

The function $\omega(x) = |x|^{\alpha}$ can be given as an example of the weight functions, where $\omega(x) \in A_p$ if and only if $-n < \alpha < n(p-1)$, $1 < p < \infty$. More examples can be found in [9].

If $\omega \in A_p(\mathbb{T})$, $1 < p,s < \infty$, then the Hardy-Littlewood maximal function of $f \in L^{pq}_{\omega}(\mathbb{T})$ is bounded in $L^{pq}_{\omega}(\mathbb{T})$ ([8, Theorem 3]). Therefore the average $A_{h_i}f$ belongs to $L^{pq}_{\omega}(\mathbb{T})$. Thus $\Omega_l(f,\delta)_{pq,\omega}$ makes sense for $\omega \in A_p(\mathbb{T})$.

We know that the relation $L^{pq}_{\omega}(\mathbb{T}) \subset L^1(\mathbb{T})$ holds (see [15, the proof of Prop. 3.3]). For $f \in L^{pq}_{\omega}(\mathbb{T})$ we have the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx) \tag{2.2}$$

and the conjugate Fourier series

$$\tilde{f}(x) \sim \sum_{k=1}^{\infty}(a_k\sin kx - b_k\cos kx).$$

It is said that a function $f \in L^{pq}_{\omega}(\mathbb{T})$, $1 < p,q < \infty$, $\omega \in A_p$, has a (ψ,β) $-$derivative f^{β}_{ψ} if the series

$$\sum_{k=1}^{\infty}(\psi_k)^{-1}\left(a_k\cos k\left(x+\frac{\beta\pi}{2k}\right) + b_k\sin k\left(x+\frac{\beta\pi}{2k}\right)\right) \tag{2.3}$$

is the Fourier series of the function f^{β}_{ψ} for given a sequence (ψ_k), and a number $\beta \in \mathbb{R}$.

Definition 1. *A sequence of real numbers* (ψ_k) *is said to be convex downwards if*

$$\psi_k - 2\psi_{k+1} + \psi_{k+2} \geq 0.$$

We denote by Ψ *the set of convex downwards sequences* (ψ_k) *for which*

$$\lim_{k\to\infty} \psi_k = 0.$$

Let $\psi \in \Psi$. Then we denote by $\eta(t) = \eta(\psi; t)$ the function connected with ψ by the equality $\eta(t) = \psi^{-1}(\psi(t)/2)$, $t \geq 1$. The function $\mu(t)$ is defined by the equality $\mu(t) = t/(\eta(t) - t)$. We set

$$\Psi_0 := \{\psi \in \Psi : 0 < \mu(t) \leq K,\ t \geq 1\},$$

where K is a certain positive constant independent of the quantities which are parameters in the case under investigation. These classes were intensively studied in [25, 26].

By $E_n(f)_{L^{pq}_\omega}$ we denote the best approximation of $f \in L^{pq}_\omega(\mathbb{T})$ by trigonometric polynomials of degree $\leq n$, i.e.,

$$E_n(f)_{L^{pq}_\omega} = \inf_{T_n \in \mathbf{T}_n} \|f - T_n\|_{L^{pq}_\omega},$$

where $\mathbf{T}_n$ is the class of trigonometric polynomials of degree not greater than n.

Now we give the multiplier theorem for the weighted Lorentz spaces.

Lemma 1. *Let* $\lambda_0, \lambda_1, ...$ *be a sequence of real numbers such that*

$$|\lambda_l| \leq M, \quad \sum_{\nu=2^{l-1}}^{2^l-1} |\lambda_\nu - \lambda_{\nu+1}| \leq M$$

for all $l \in \mathbb{N}$. *If* $1 < p, q < \infty$, $\omega \in A_p$ *and* $f \in L^{pq}_\omega(\mathbb{T})$ *with the Fourier series*

$$\sum_{\nu=0}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x),$$

then there is a function $h \in L^{pq}_\omega(\mathbb{T})$ *such that the series*

$$\sum_{\nu=0}^{\infty} \lambda_\nu (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

is Fourier series for h and

$$\|h\|_{pq,\omega} \leq C \|f\|_{pq,\omega} \tag{2.4}$$

where C does not depend on f.

Proof. We define a linear operator

$$Tf(x) := \sum_{\nu=0}^{\infty} \lambda_\nu (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

for $f \in L^{pq}_\omega(\mathbb{T})$ which is bounded (in particular is of weak type (p, p)) in $L^p(\mathbb{T}, \omega)$ for every $p > 1$ by [6, Th. 4.4]. Therefore the hypothesis of the interpolation theorem for Lorentz spaces [5, Th. 4.13] fulfills. Applying this theorem we get the desired result (2.4). ∎

We prove a generalized Bernstein inequality in $L_{\omega}^{pq}(\mathbb{T})$.

Lemma 2. *Let* $1 < p, q < \infty$, $\omega \in A_p$, $f \in L_{\omega}^{pq}(\mathbb{T})$ *and*

$$\sup_q \sum_{k=2^q}^{2^{q+1}} \left| (\psi_{k+1}(n))^{-1} - (\psi_k(n))^{-1} \right| \le C(\psi_n)^{-1},$$

where

$$(\psi_k(n))^{-1} = \begin{cases} (\psi_k)^{-1}, & 1 \le k \le n, \\ 0, & k > n \end{cases}.$$

Then for $T_n \in \mathbf{T}_n$

$$\left\| (T_n)_{\psi}^{\beta} \right\|_{L_{\omega}^{pq}} \le c (\psi_n)^{-1} \| T_n \|_{L_{\omega}^{pq}},$$

where the constant c *is independent of* n.

Proof. We have

$$\begin{aligned}
(T_n)_{\psi}^{\beta} &= \sum_{k=1}^{n} (\psi_k)^{-1} \left(a_k \cos k \left(x + \frac{\beta\pi}{2k} \right) + b_k \sin k \left(x + \frac{\beta\pi}{2k} \right) \right) \\
&= \sum_{k=1}^{n} (\psi_k)^{-1} B_k \left(T_n, x + \frac{\beta\pi}{2k} \right) \\
&= \sum_{k=1}^{n} (\psi_k)^{-1} \left(\cos\frac{\beta\pi}{2} B_k(T_n, x) - \sin\frac{\beta\pi}{2} B_k(\tilde{T}_n, x) \right).
\end{aligned}$$

If we define the multipliers

$$\mu_k = \begin{cases} (\psi_k)^{-1} \cos\frac{\beta\pi}{2}, & 1 \le k \le n, \\ 0, & k > n,\ k = 0 \end{cases}$$

$$\tilde{\mu}_k = \begin{cases} (\psi_k)^{-1} \sin\frac{\beta\pi}{2}, & 1 \le k \le n, \\ 0, & k > n,\ k = 0, \end{cases}$$

and the operators

$$\begin{aligned}
(BT_n)(x) &= \sum_{k=1}^{n} (\psi_k)^{-1} \cos\frac{\beta\pi}{2} B_k(T_n, x), \\
(\tilde{B}\tilde{T}_n)(x) &= \sum_{k=1}^{n} (\psi_k)^{-1} \sin\frac{\beta\pi}{2} B_k(\tilde{T}_n, x),
\end{aligned}$$

then we have

$$(T_n)_{\psi}^{\beta}(\cdot) = (BT_n)(\cdot) - (\tilde{B}\tilde{T}_n)(\cdot).$$

Using the hypothesis we get

$$\sup_k |\mu_k| \le (\psi_n)^{-1}, \ \sup_k |\tilde{\mu}_k| \le (\psi_n)^{-1},$$

$$\begin{aligned}\sup_q \sum_{k=2^q}^{2^{q+1}} |\mu_{k+1}-\mu_k| &\leq C\left(\psi_n\right)^{-1},\\ \sup_q \sum_{k=2^q}^{2^{q+1}} |\tilde{\mu}_{k+1}-\tilde{\mu}_k| &\leq C\left(\psi_n\right)^{-1}.\end{aligned}$$

If we apply the multiplier theorem for the weighted Lorentz spaces we get

$$\begin{aligned}\left\|\left(T_n\right)_\psi^\beta\right\|_{L_\omega^{pq}} &= \left\|\left(BT_n\right)-\left(\tilde{B}\tilde{T}_n\right)\right\|_{L_\omega^{pq}} \leq \left\|BT_n\right\|_{L_\omega^{pq}} + \left\|\tilde{B}\tilde{T}_n\right\|_{L_\omega^{pq}}\\ &\leq C\left(\psi_n\right)^{-1}\left(\left\|\sum_{k=1}^n B_k\left(T_n,x\right)\right\|_{L_\omega^{pq}} + \left\|\sum_{k=1}^n B_k\left(\tilde{T}_n,x\right)\right\|_{L_\omega^{pq}}\right).\end{aligned}$$

The boundedness of the conjugate operator [15] implies the required inequality

$$\left\|\left(T_n\right)_\psi^\beta\right\|_{L_\omega^{pq}} \leq C\left(\psi_n\right)^{-1}\left\|\sum_{k=1}^n B_k\left(T_n,x\right)\right\|_{L_\omega^{pq}} = C\left(\psi_n\right)^{-1}\left\|T_n\right\|_{L_\omega^{pq}}. \qquad \blacksquare$$

Remark 1. *In this Lemma, one can assume that the parameter β equals zero because of the boundedness of the conjugate operator.*

Remark 2. *The condition on $\left(\psi_n\right)^{-1}$ is similar to so-called general monotonicity, see [27].*

3 Main Results

Theorem 1. *Let $1<p,q<\infty$, $\omega\in A_p(\mathbb{T})$, and $f, f_\psi^\beta \in L_\omega^{pq}(\mathbb{T})$. If (ψ_k) is an arbitrary sequence such that for every $k\in\mathbb{N}$, $\psi_k\geq 0$, $\psi_{k+1}\leq\psi_k$ and $(\psi_k)\to 0$ as $k\to\infty$, then for $n=0,1,2,...$ the inequality*

$$\left\|f-S_n(f)\right\|_{L_w^{pq}} \leq c\psi_{n+1}\left\|f_\psi^\beta - S_n\left(\cdot, f_\psi^\beta\right)\right\|_{L_w^{pq}}, \quad n\in\mathbb{N}$$

holds with a constant $c>0$ independent of n, where $S_n(f)$ denotes the $n-th$ partial sum of the Fourier series (2.2) of f.

Corollary 1. *Under the conditions of Theorem 1, there is a constant $c>0$ independent of n such that the inequality*

$$E_n\left(f\right)_{L_\omega^{pq}} \leq c\psi_{n+1}E_n\left(f_\psi^\beta\right)_{L_\omega^{pq}}$$

holds.

Using corollary 1 and Theorem 2 of [3] we get the following Jackson type direct Theorem.

Theorem 2. *Let $1 < p,q < \infty$, $\omega \in A_p$, and f, $f_\alpha^\psi \in L_\omega^{pq}(\mathbb{T})$. If (ψ_k) is an arbitrary sequence such that for every $k \in \mathbb{N}$, $\psi_k \geq 0$, $\psi_{k+1} \leq \psi_k$ and $(\psi_k) \to 0$ as $k \to \infty$, then for every $n = 1,2,3,\ldots$ there is a constant $c > 0$ independent of n such that*

$$E_n(f)_{L_\omega^{pq}} \leq c\psi_{n+1}\Omega_r\left(f_\psi^\beta, \frac{1}{n}\right)_{L_\omega^{pq}}.$$

Theorem 3. *Let $1 < p,q < \infty$, $\omega \in A_p$, $f \in L_\omega^{pq}(\mathbb{T})$, $\psi \in \Psi_0$. Assume that*

$$\sum_{k=1}^{\infty} (k\psi_k)^{-1} E_k(f)_{L_\omega^{pq}} < \infty,$$

then $f_\psi^\beta \in L_\omega^{pq}(\mathbb{T})$ and for $n = 0,1,2,\ldots$ the estimate

$$E_n(f_\psi^\beta)_{L_\omega^{pq}} \leq c\left\{(\psi_n)^{-1} E_n(f)_{L_\omega^{pq}} + \sum_{k=n+1}^{\infty} (k\psi_k)^{-1} E_k(f)_{L_\omega^{pq}}\right\}$$

holds with a constant $c > 0$ independent of n and f.

Corollary 2. *Under the conditions of Theorem 3 if $r \in \mathbb{N}$ and*

$$\sum_{\nu=1}^{\infty} (\nu\psi(v))^{-1} E_\nu(f)_{L_\omega^{pq}} < \infty,$$

there are the constants $c_1, c_2 > 0$ independent of n and f such that the inequality

$$\Omega_r\left(f_\psi^\beta, \frac{1}{n}\right)_{L_\omega^{pq}} \leq \frac{c_1}{n^{2r}} \sum_{\nu=0}^{n} \nu^{2r-1} (\psi_\nu)^{-1} E_\nu(f)_{L_\omega^{pq}} + c_2 \sum_{\nu=n+1}^{\infty} (\nu\psi_\nu)^{-1} E_\nu(f)_{L_\omega^{pq}}$$

holds.

Theorem 4. *Let $1 < p,q < \infty$, $\omega \in A_p$, f, $f_\alpha^\psi \in L_\omega^{pq}(\mathbb{T})$, $\beta \in [0,\infty)$ and $\psi \in \Psi_0$. Assume that (ψ_k) is an arbitrary non-increasing sequence of nonnegative numbers that $(\psi_k) \to 0$ as $k \to \infty$. Then there is a $T \in \mathbf{T}_n$, $n = 1,2,3,\ldots$ and a constant $C > 0$ independent of n and f such that*

$$\left\|f_\beta^\psi - T_\beta^\psi\right\|_{L_\omega^{pq}} \leq CE_n\left(f_\beta^\psi\right)_{L_\omega^{pq}}.$$

Particularly, in the case $\psi_k = k^{-\beta}\ln^{-\alpha} k$, $k = 1,2,\ldots,\beta$, $\alpha \in \mathbb{R}^+$, we get the following new results for the power logarithmic-fractional derivatives $f^{(\beta,\alpha)}$ of f.

Theorem 5. *Let $1 < p,q < \infty$, $\omega \in A_p(\mathbb{T})$, $\alpha, \beta \in \mathbb{R}$ and f, $f^{(\beta,\alpha)} \in L_\omega^{pq}(\mathbb{T})$.Then for every $n = 1,2,3,\ldots$ there is a constant $c > 0$ independent of n such that the estimate*

$$\|f - S_n(f)\|_{L_w^{pq}} \leq \frac{c}{n^\beta \ln^\alpha(n+1)} \left\|f^{(\beta,\alpha)} - S_n\left(\cdot, f^{(\beta,\alpha)}\right)\right\|_{L_w^{pq}}, \quad n \in \mathbb{N}$$

holds.

Corollary 3. *Under the conditions of Theorem 5 we have the inequality*

$$E_n(f)_{L^{pq}_\omega} \leq \frac{c}{n^\beta \ln^\alpha (n+1)} E_n\left(f^{(\beta,\alpha)}\right)_{L^{pq}_\omega}$$

with a constant $c > 0$ independent of n.

Theorem 6. *Let $1 < p, q < \infty$, $\omega \in A_p$, $\alpha, \beta \in \mathbb{R}$ and $f, f^{(\beta,\alpha)} \in L^{pq}_\omega(\mathbb{T})$. Then for every $n = 1, 2, 3, \ldots$ and $r \in \mathbb{N}$, there is a constant $c > 0$ independent of n such that*

$$E_n(f)_{L^{pq}_\omega} \leq \frac{c}{n^\beta \ln^\alpha (n+1)} \Omega_r\left(f^{(\beta,\alpha)}, \frac{1}{n}\right)_{L^{pq}_\omega}.$$

Theorem 7. *Let $1 < p, q < \infty$, $\omega \in A_p$, $f \in L^{pq}_\omega(\mathbb{T})$, $\beta \in \mathbb{R}$ and*

$$\sum_{\nu=1}^{\infty} \nu^{\beta-1} \ln^\alpha \nu E_\nu(f)_{L^{pq}_\omega} < \infty.$$

Then $f^{(\beta,\alpha)} \in L^{pq}_\omega(\mathbb{T})$ and we have

$$E_n\left(f^{(\beta,\alpha)}\right)_{L^{pq}_\omega} \leq c\left(n^\beta \ln^\alpha n E_n(f)_{L^{pq}_\omega} + \sum_{\nu=n+1}^{\infty} \nu^{\beta-1} \ln^\alpha \nu E_\nu(f)_{L^{pq}_\omega}\right),$$

where the constant $c > 0$ independent of n and f.

Corollary 4. *Under the conditions of Theorem 7 if $r \in \mathbb{N}$ and*

$$\sum_{\nu=1}^{\infty} \nu^{\beta-1} \ln^\alpha \nu E_\nu(f)_{L^{pq}_\omega} < \infty,$$

there are the constants $c_1, c_2 > 0$ independent of n and f such that

$$\Omega_r\left(f^{(\beta,\alpha)}, \frac{1}{n}\right)_{L^{pq}_\omega} \leq \frac{c_1}{n^r} \sum_{\nu=1}^{n} \nu^{r+\beta-1} \ln^\alpha \nu E_\nu(f)_{L^{pq}_\omega} + c_2 \sum_{\nu=n+1}^{\infty} \nu^{\beta-1} \ln^\alpha \nu E_\nu(f)_{L^{pq}_\omega}.$$

Theorem 8. *Let $1 < p, q < \infty$, $\omega \in A_p$, $f, f^\psi_\alpha \in L^{pq}_\omega(\mathbb{T})$ and $\beta \in [0, \infty)$. Then there is a $T \in \mathbf{T}_n$, $n = 1, 2, 3, \ldots$ and a constant $c > 0$ independent of n and f such that*

$$\left\| f^{(\beta,\alpha)} - T^{(\beta,\alpha)} \right\|_{L^{pq}_\omega} \leq c E_n\left(f^{(\beta,\alpha)}\right)_{L^{pq}_\omega}.$$

Theorem 7 and Corollary 4 were proved in L^p ($\omega \equiv 1$, constant $p \in (1, \infty)$) in [21].

Proof of Theorem 1. Let

$$A_k(f, x) := a_k(f) \cos kx + b_k(f) \sin kx,$$

where $a_k(f)$, $b_k(f)$, $k = 1, 2, \ldots$ are Fourier coefficients of f. We know that the relation $L^{pq}_\omega(\mathbb{T}) \subset L^1(\mathbb{T})$ holds [15]. Let $S_n(f)$ be the n.th partial sum of Fourier series of f. The inequalities

$$\|S_n(f)\|_{L^{pq}_w} \lesssim \|f\|_{L^{pq}_w}, \quad \left\|\tilde{f}\right\|_{L^{pq}_w} \lesssim \|f\|_{L^{pq}_w}, \tag{3.1}$$

hold (see [14, Theorem 6.6.2], [15]). By [25, p. 120] we have

$$f(x)-S_n(x,f)=\sum_{k=n+1}^{\infty}\frac{\psi_k}{\pi}\int_{\mathbb{T}}\left(f_{\psi}^{\beta}(t)-S_n\left(t,f_{\psi}^{\beta}\right)\right)\cos\left(k(x-t)-\frac{\beta\pi}{2}\right)dt.$$

Then

$$\begin{aligned}f(\cdot)-S_n(\cdot,f)=\cos\frac{\beta\pi}{2}\sum_{k=n+1}^{\infty}\psi_k A_k\left(f_{\psi}^{\beta}-S_n\left(f_{\psi}^{\beta}\right),\cdot\right)+&\\ \sin\frac{\beta\pi}{2}\sum_{k=n+1}^{\infty}\psi_k A_k\left(\tilde{f}_{\psi}^{\beta}-S_n\left(\tilde{f}_{\psi}^{\beta}\right),\cdot\right).&\end{aligned}$$

By (3.1) and the equalities

$$\begin{aligned}&\sum_{k=n+1}^{\infty}\psi_k A_k\left(f_{\psi}^{\beta}-S_n\left(f_{\psi}^{\beta}\right),\cdot\right)\\ &\quad=\sum_{k=n+1}^{\infty}(\psi_k-\psi_{k+1})S_k\left(\cdot,f_{\psi}^{\beta}-S_n\left(f_{\psi}^{\beta}\right)\right)-\psi_{n+1}S_n\left(\cdot,f_{\psi}^{\beta}-S_n\left(f_{\psi}^{\beta}\right)\right),\\ &\quad\sum_{k=n+1}^{\infty}\psi_k A_k\left(\tilde{f}_{\psi}^{\beta}-S_n\left(\tilde{f}_{\psi}^{\beta}\right),\cdot\right)\\ &\quad=\sum_{k=n+1}^{\infty}(\psi_k-\psi_{k+1})S_k\left(\cdot,\tilde{f}_{\psi}^{\beta}-S_n\left(\tilde{f}_{\psi}^{\beta}\right)\right)-\psi_{n+1}S_n\left(\cdot,\tilde{f}_{\psi}^{\beta}-S_n\left(\tilde{f}_{\psi}^{\beta}\right)\right)\end{aligned}$$

we obtain

$$\begin{aligned}&\|f(\cdot)-S_n(\cdot,f)\|_{L_{\omega}^{pq}}\\ &\quad\leq\sum_{k=n+1}^{\infty}(\psi_k-\psi_{k+1})\left\|S_k\left(\cdot,f_{\psi}^{\beta}-S_n\left(f_{\psi}^{\beta}\right)\right)\right\|+\psi_{n+1}\left\|S_n\left(\cdot,f_{\psi}^{\beta}-S_n\left(f_{\psi}^{\beta}\right)\right)\right\|\\ &\qquad+\sum_{k=n+1}^{\infty}(\psi_k-\psi_{k+1})\left\|S_k\left(\cdot,\tilde{f}_{\psi}^{\beta}-S_n\left(\tilde{f}_{\psi}^{\beta}\right)\right)\right\|+\psi_{n+1}\left\|S_n\left(\cdot,\tilde{f}_{\psi}^{\beta}-S_n\left(\tilde{f}_{\psi}^{\beta}\right)\right)\right\|\\ &\quad\preceq\sum_{k=n+1}^{\infty}(\psi_k-\psi_{k+1})\left\|f_{\psi}^{\beta}-S_n\left(f_{\psi}^{\beta}\right)\right\|+\psi_{n+1}\left\|f_{\psi}^{\beta}-S_n\left(f_{\psi}^{\beta}\right)\right\|+\\ &\qquad+\sum_{k=n+1}^{\infty}(\psi_k-\psi_{k+1})\left\|\tilde{f}_{\psi}^{\beta}-S_n\left(\tilde{f}_{\psi}^{\beta}\right)\right\|+\psi_{n+1}\left\|\tilde{f}_{\psi}^{\beta}-S_n\left(\tilde{f}_{\psi}^{\beta}\right)\right\|\\ &\quad\preceq\sum_{k=n+1}^{\infty}((\psi_k-\psi_{k+1})+\psi_{n+1})\left(\left\|f_{\psi}^{\beta}-S_n\left(f_{\psi}^{\beta}\right)\right\|+\left\|\tilde{f}_{\psi}^{\beta}-S_n\left(\tilde{f}_{\psi}^{\beta}\right)\right\|\right)\\ &\quad\preceq\psi_{n+1}\left\|f_{\psi}^{\beta}-S_n\left(f_{\psi}^{\beta}\right)\right\|.\end{aligned}$$

Theorem 1 is proved. ∎

Proof of Theorem 3. Let T_n be the best approximating polynomial for $f \in L_\omega^{pq}$. We set $n_0 = n, n_1 := [\eta(n)] + 1, \ldots, n_k := [\eta(n_{k-1})] + 1, \ldots$, here $[\eta(n)]$ denotes the integer part of the nonnegative real number $\eta(n)$. In this case the series

$$T_{n_0}(\cdot) + \sum_{k=1}^{\infty} \left(T_{n_k}(\cdot) - T_{n_{k-1}}(\cdot)\right)$$

converges to f in norm in L_ω^{pq}. We consider the series

$$(T_{n_0}(\cdot))_\psi^\beta + \sum_{k=1}^{\infty} \left(T_{n_k}(\cdot) - T_{n_{k-1}}(\cdot)\right)_\psi^\beta. \tag{3.2}$$

Applying generalized Bernstein inequality for the difference $u_k(\cdot) := T_{n_k}(\cdot) - T_{n_{k-1}}(\cdot)$ we get

$$\left\| (u_k)_\psi^\beta \right\|_{L_\omega^{pq}} \leq c E_{n_{k-1}+1}(f)_{L_\omega^{pq}} (\psi(n_k))^{-1}.$$

Hence

$$\sum_{k=1}^{\infty} \left\| (u_k)_\psi^\beta \right\|_{L_\omega^{pq}} \leq c \left(E_{n+1}(f)_{L_\omega^{pq}} (\psi(n))^{-1} + \sum_{k=1}^{\infty} E_{n_k+1}(f)_{L_\omega^{pq}} (\psi(n_k))^{-1} \right).$$

Since $\psi \in \Psi_0$, we have $\psi(\tau) \geq \psi(\eta(t)) = \psi(\tau)/2$ for any $\tau \in [t, \eta(t)]$, $\tau \geq \eta(1)$. Without loss of generality one can assume $\eta(t) - t > 1$. In this case we get

$$\frac{E_{n_k+1}(f)_{L_\omega^{pq}}}{\psi(n_k)} \leq \sum_{v=n_{k-1}}^{n_k - 1} \frac{E_{v+1}(f)_{L_\omega^{pq}}}{v\psi(v)}.$$

Therefore

$$\sum_{k=1}^{\infty} \left\| (u_k)_\psi^\beta \right\|_{L_\omega^{pq}} \leq c \left(E_{n+1}(f)_{L_\omega^{pq}} (\psi(n))^{-1} + \sum_{v=n+1}^{\infty} E_v(f)_{L_\omega^{pq}} (v\psi(v))^{-1} \right).$$

Right hand side of last inequality converges and hence the series (3.2) is converges in norm to some function $g(\cdot)$ from L_ω^{pq}. It is easily seen that the Fourier series of g is of the form (2.3). This means that the function f has a (ψ, β)-derivative f_ψ^β of class L_ω^{pq} and

$$f_\psi^\beta = (T_n)_\psi^\beta + \sum_{k=1}^{\infty} (u_k)_\psi^\beta \tag{3.3}$$

holds in norm in $L_\omega^{p(\cdot)}$. Therefore from (3.3)

$$E_n\left(f_\psi^\beta\right)_{L_\omega^{pq}} \leq c \left((\psi(n))^{-1} E_n(f)_{L_\omega^{pq}} + \sum_{\nu=n+1}^{\infty} (\nu\psi(v))^{-1} E_\nu(f)_{L_\omega^{pq}} \right). \qquad \blacksquare$$

Proof of Corollary 2. We note that the sharp inverse inequality to the Jackson-Stechkin type inequality was proved in [15, Th. 1]. In the sequel we use a weak version of inverse estimate: *Let $1 < p, q < \infty$ and let $\omega \in A_p(\mathbf{T})$. Then there exists a positive constant c such that*

$$\Omega_l(f,\delta)_{L^{pq}_\omega} \leq \frac{c}{n^{2l}}\sum_{k=1}^{n} k^{2l-1}E_{k-1}(f)_{L^{pq}_\omega}$$

for an arbitrary $f \in L^{pq}_\omega(\mathbf{T})$ and every natural n [15, Prop. 4.1]. Using Theorem 3 we have

$$\begin{aligned}
&\Omega_r\left(f^\beta_\psi, \frac{1}{n}\right)_{L^{pq}_\omega} \leq \frac{c}{n^{2r}}\sum_{\nu=1}^{n}\nu^{2r-1}E_\nu\left(f^\beta_\psi\right)_{L^{pq}_\omega} \\
&\leq \frac{c}{n^{2r}}\left\{\sum_{\nu=1}^{n}\nu^{2r-1}\left(\psi\left(v\right)\right)^{-1}E_v\left(f\right)_{L^{pq}_\omega} + \sum_{\nu=1}^{n}\nu^{2r-1}\sum_{m=v+1}^{\infty}\left(m\psi\left(m\right)\right)^{-1}E_m\left(f\right)_{L^{pq}_\omega}\right\} \\
&\leq \frac{c}{n^{2r}}\sum_{\nu=0}^{n}\nu^{2r-1}\left(\psi\left(v\right)\right)^{-1}E_\nu\left(f\right)_{L^{pq}_\omega} + C\sum_{\nu=n+1}^{\infty}\left(\nu\psi\left(v\right)\right)^{-1}E_\nu\left(f\right)_{L^{pq}_\omega}. \qquad \blacksquare
\end{aligned}$$

Proof of Theorem 4. We define $W_n(f) := W_n(\cdot, f) := \frac{1}{n+1}\sum_{\nu=n}^{2n} S_\nu(\cdot, f)$ for $n = 0, 1, 2, \ldots$. Since

$$W_n(\cdot, f^\beta_\psi) = (W_n(\cdot, f))^\beta_\psi$$

we obtain that

$$\begin{aligned}
&\left\|f^\beta_\psi(\cdot) - (S_n(\cdot, f))^\beta_\psi\right\|_{L^{pq}_\omega} \\
&\quad\leq \left\|f^\beta_\psi(\cdot) - W_n(\cdot, f^\beta_\psi)\right\|_{L^{pq}_\omega} + \left\|(S_n(\cdot, W_n(f)))^\beta_\psi - (S_n(\cdot, f))^\beta_\psi\right\|_{L^{pq}_\omega} \\
&\quad\quad + \left\|(W_n(\cdot, f))^\beta_\psi - (S_n(\cdot, W_n(f)))^\beta_\psi\right\|_{L^{pq}_\omega} = I_1 + I_2 + I_3.
\end{aligned}$$

In this case, the boundedness of the operator S_n in L^{pq}_ω implies the boundedness of operator W_n in L^{pq}_ω and we get

$$\begin{aligned}
I_1 &\leq \left\|f^\beta_\psi(\cdot) - S_n(\cdot, f^\beta_\psi)\right\|_{L^{pq}_\omega} + \left\|S_n(\cdot, f^\beta_\psi) - W_n(\cdot, f^\beta_\psi)\right\|_{L^{pq}_\omega} \\
&\leq cE_n\left(f^\beta_\psi\right)_{L^{pq}_\omega} + \left\|W_n(\cdot, S_n(f^\beta_\psi) - f^\beta_\psi)\right\|_{L^{pq}_\omega} \leq cE_n\left(f^\beta_\psi\right)_{L^{pq}_\omega}.
\end{aligned}$$

Using Lemma 2 we obtain

$$I_2 \leq c\left(\psi\left(n\right)\right)^{-1}\left\|S_n(\cdot, W_n(f)) - S_n(\cdot, f)\right\|_{L^{pq}_\omega}$$

and

$$I_3 \leq c\left(\psi\left(n\right)\right)^{-1}\left\|W_n(\cdot, f) - S_n(\cdot, W_n(f))\right\|_{L^{pq}_\omega} \leq c\left(\psi\left(n\right)\right)^{-1}E_n\left(W_n(f)\right)_{L^{pq}_\omega}.$$

Now we have

$$
\begin{aligned}
&\|S_n(\cdot, W_n(f)) - S_n(\cdot, f)\|_{L^{pq}_\omega} \\
&\leq \|S_n(\cdot, W_n(f)) - W_n(\cdot, f)\|_{L^{pq}_\omega} + \|W_n(\cdot, f) - f(\cdot)\|_{L^{pq}_\omega} + \|f(\cdot) - S_n(\cdot, f)\|_{L^{pq}_\omega} \\
&\leq cE_n(W_n(f))_{L^{pq}_\omega} + cE_n(f)_{L^{pq}_\omega} + cE_n(f)_{L^{pq}_\omega}.
\end{aligned}
$$

Since

$$E_n(W_n(f))_{L^{pq}_\omega} \leq cE_n(f)_{L^{pq}_\omega}$$

we obtain

$$
\begin{aligned}
\left\| f^\beta_\psi(\cdot) - (S_n(\cdot, f))^\beta_\psi \right\|_{L^{pq}_\omega} &\leq cE_n\left(f^\beta_\psi\right)_{L^{pq}_\omega} + c(\psi(n))^{-1} E_n(W_n(f))_{L^{pq}_\omega} + cE_n(f)_{L^{pq}_\omega} \\
&\leq cE_n\left(f^\beta_\psi\right)_{L^{pq}_\omega} + c(\psi(n))^{-1} E_n(f)_{L^{pq}_\omega}.
\end{aligned}
$$

Since by Theorem 1

$$E_n(f)_{L^{pq}_\omega} \leq c\psi(n+1) E_n\left(f^\beta_\psi\right)_{L^{pq}_\omega}$$

we get

$$\left\| f^\beta_\psi(\cdot) - (S_n(\cdot, f))^\beta_\psi \right\|_{L^{pq}_\omega} \leq cE_n\left(f^\beta_\psi\right)_{L^{pq}_\omega}$$

and the proof is completed. ∎

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Balikesir University, Faculty of Art and Science,
Department of Mathematics,
10145, Balikesir, Turkey
email: rakgun@balikesir.edu.tr

Balikesir University, Faculty of Education,
Department of Mathematics,
10145, Balikesir, Turkey
email: yildirir@balikesir.edu.tr