A new analytical technique for solving Lane - Emden type equations arising in astrophysics

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Abstract

Lane - Emden type equations are nonlinear differential equations which represent many scientific phenomena in astrophysics and mathematical physics. In this study, a new analytic approximate technique for addressing nonlinear problems, namely the optimal perturbation iteration method, is introduced and implemented to singular initial value Lane-Emden type problems to test the effectiveness and performance of the method. This technique provides us to adjust the convergence regions when necessary.Comparing different methods reveals that the proposed method is highly accurate and has great potential to be a new kind of powerful analytical tool for Lane-Emden type equations.

1 Introduction

Many problems of science and engineering lead to different types of differential equations and it is still very hard to solve them in the presence of strong nonlinearity. Many numerical methods have been dealt with in order to solve these equations. Alternatively, there is great interest in discovering methods for analytic approximate solutions. Recently, there has been much attention devoted to investigate better and more efficient analytical techniques such as homotopy

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decomposition method [1], auxiliary equation method [2], homotopy perturbation method [3, 4], Taylor collocation method [5], Sumudu transform method [6] and Adomian decomposition method [7, 8].

Emden-Fowler equation is one of the most important differential equations of mathematical physics [9, 10]. It distinctively characterizes many scientific phenomena. The generalized Emden-Fowler equation is defined in the following form

$$y'' + \alpha(x)y' + \beta(x)\gamma(y) = 0 \tag{1.1}$$

subject to conditions

$$y(0) = A, y'(0) = B$$

where A, B are constants and $\alpha(x), \beta(x), \gamma(y)$ are some arbitrary functions. For different $\gamma(y)$, the Eq. (1.1) has been subject of many studies in the literature such as the theory of stellar structure, thermionic currents and isothermal gas spheres [11]. When $\alpha(x) = \frac{k}{x}, \beta(x) = \beta_0 x^r, \gamma(y) = y^s$ (k and β_0 are constants, s and r are real numbers), Eq.(1.1) reduces to the classic Emden-Fowler equation:

$$y'' + \frac{k}{x}y' + \beta_0 x^r y^s = 0; \quad y(0) = A, y'(0) = B \tag{1.2}$$

Furthermore, by choosing $r = 0$ and $k = 2$, we get the standard Lane-Emden equation

$$y'' + \frac{2}{x}y' + \beta_0 y^s = 0 \tag{1.3}$$

with supplementary conditions

$$y(0) = A, y'(0) = 0$$

which arises in astrophysics. Eq. (1.2) is also used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules [12]. Many analytical techniques have been considered by various researchers to obtain the approximate solutions for these types of equations [13–16].

In this study, we derive a new effective technique namely optimal perturbation iteration method (OPIM) to get a new approximate solutions for nonlinear problems. Main idea of this method is essentially based on optimal homotopy asymptotic method [17, 18] and perturbation iteration technique [19]. Both of these methods have been recently developed and they have been successfully implemented to some strongly nonlinear systems [20–24]. We apply OPIM to obtain more reliable approximate solutions to the Lane-Emden equations

$$y'' + \frac{2}{x}y' + \beta(x)\gamma(y) = 0; \quad y(0) = A, y'(0) = 0 \tag{1.4}$$

with different choices of $\beta(x), \gamma(y)$. It is also indicated that this method enables us to control the convergence of solution series for the given illustrations.

2 Optimal Perturbation Iteration Algorithm

In this section, the following formulation is given to explain the basic concept of OPIM for second order differential equations.

(a) Write the governing differential equation as:

$$F(y'', y', y, \varepsilon) = A + g(x) = 0 \tag{2.1}$$

where ε is the auxiliary perturbation parameter, $y = y(x)$ is the unknown function and $g(x)$ is the source term. Furthermore, (2.1) can be decomposed into $A = L + N$ where L is the linear simpler part, which can be easily managed and N is the remaining part which is more crucial for algorithms of OPIM. Here we have a great freedom to choose linear part L.

(b) Approximate solution is taken as

$$y_{n+1} = y_n + \varepsilon (y_c)_n \tag{2.2}$$

with one correction term in the perturbation expansion. Inserting (2.2) into the (2.1) and expanding the remaining part (N) in a Taylor series with first derivatives yields optimal perturbation iteration algorithm (OPIA):

$$N({y_n}'', {y_n}', y_n, 0) + N_y (y_c)_n \varepsilon + N_{y'} (y'_c)_n \varepsilon + N_{y''} (y''_c)_n \varepsilon + N_\varepsilon \varepsilon = -L - g(x). \tag{2.3}$$

It should be noted that all derivatives and functions are calculated at $\varepsilon = 0$. To describe the iterative scheme, first correction term $(y_c)_0$ can be computed from the algorithm (2.3) by using a first guess y_0 and initial condition(s).
(2.3) may seem complicated at first, but it should not be forgotten that we use the general form of the differential equations of second order to illustrate the proposed method. Actually, most differential equations in literature contain only some of the nonlinear terms y, y', y''. So, the algorithm (2.3) reduces to some simple mathematical expressions in many cases.

c) Use the following equation

$$y_{n+1} = y_n + S_n (y_c)_n \tag{2.4}$$

to increase the accuracy of the results and effectiveness of the method. Here $S_0, S_1, S_2, \ldots$ are convergence control parameters which provide us to adjust the convergence.
Proceeding for $n = 0, 1, \ldots$, approximate solutions are found as:

$$\begin{aligned} &y_1 = y(x, S_0) = y_0 + S_0 (y_c)_0 \\ &y_2(x, S_0, S_1) = y_1 + S_1 (y_c)_1 \\ &\quad \vdots \\ &y_m(x, S_0, \ldots, S_{m-1}) = y_{m-1} + S_{m-1} (y_c)_{m-1} \end{aligned} \tag{2.5}$$

d) Substitute the approximate solution y_m into the Eq.(2.1) and the general problem results in the following residual:

$$R(x, S_0, \ldots, S_{m-1}) = A\left(y_m(x, S_0, \ldots, S_{m-1})\right) + g(x). \tag{2.6}$$

Obviously, when $R(x, S_0, \ldots, S_{m-1}) = 0$ then the approximation $y_m(x, S_0, \ldots, S_{m-1})$ will be the exact solution. Generally it doesn't happen in nonlinear equations, but the functional can be minimized as:

$$J(S_0, \ldots, S_{m-1}) = \int_a^b R^2(x, S_0, \ldots, S_{m-1})dx \tag{2.7}$$

where a and b are selected from the domain of the problem. Optimum values of $S_0, S_1, \ldots$ can be obtained from the conditions

$$\frac{\partial J}{\partial S_0} = \frac{\partial J}{\partial S_1} = \ldots = \frac{\partial J}{\partial S_{m-1}} = 0. \tag{2.8}$$

The constants $S_0, S_1, \ldots$ can also be defined from

$$R(x_0, S_i) = R(x_1, S_i) = \cdots = R(x_{m-1}, S_i) = 0, \;\; i = 0, 1, \ldots, m-1 \tag{2.9}$$

where $x_i \in (a, b)$. Putting these constants into the last one of the Eqs. (2.5), the approximate solution of order m is determined.For much more information about finding these constants please see [25].

3 Applications

In this section, we apply OPIA to solve the Lane-Emden type problems. Obtained results show that the new algorithm gives better results when compared with many other methods in literature.

Example 3.1. *Consider the following homogeneous Lane-Emden equation of the first kind:*

$$y'' + \frac{2}{x}y' + y^5 = 0, \;\; y(0) = 1, y'(0) = 0, \;\; x \geq 0. \tag{3.1}$$

is a basic equation in the theory of stellar structure [26, 27]. In [28], exact solution of (3.1) is given by

$$y = \left(1 + \frac{x^2}{3}\right)^{-1/2}. \tag{3.2}$$

Perturbation parameter ε can be inserted into (3.1) as:

$$F(y'', y', y, \varepsilon) = y'' + \varepsilon\frac{2}{x}y' + \varepsilon y^5 = 0 = L + N = 0 \tag{3.3}$$

where

$$L = y'' \;\text{ and }\; N = \varepsilon\left(\frac{2}{x}y' + y^5\right). \tag{3.4}$$

For this problem, we do not have the term y'' in the remaining part N. Therefore, Eq. (2.3) is simplified to:

$$N + N_y(y_c)_n\varepsilon + N_{y'}(y_c{}')_n\varepsilon + N_\varepsilon\varepsilon = -L \tag{3.5}$$

Using the Eqs. (2.2), (3.4), (3.5) and setting $\varepsilon = 1$ yields the following algorithm

$$(y_c)''_n = -(y_n)'' - \frac{2}{x}(y_n)' - (y_n)^5. \tag{3.6}$$

One may select $y_0 = 1$ as a starting guess which satisfies the given initial conditions. Substituting y_0 into the Eq. (3.6), yields first order problem:

$$(y_c)''_n = -1, \;\; y(0) = y'(0) = 0. \tag{3.7}$$

Solving the (3.7) gives first correction term:

$$(y_c)_0 = -\frac{x^2}{2} \tag{3.8}$$

Now, Eq.(3.8) is inserted into Eq.(2.4) to obtain first approximate solution:

$$y_1 = 1 - \frac{S_0 x^2}{2} \tag{3.9}$$

By using the procedure mentioned in Section 2, one obtains the following approximate solutions:

$$\begin{aligned} y_2 = 1 - \frac{S_0 x^2}{2} + \frac{S_1}{88704}\Big[&21S_0^5x^{12} - 308S_0^4x^{10} + 1980S_0^3x^8 - 7392S_0^2x^6 + \\ &18480S_0x^4 + 133056S_0x^2 - 44352x^2\Big] \end{aligned} \tag{3.10}$$

$$
\begin{aligned}
y_3 = 1 - \frac{S_0x^2}{2} + \frac{S_1}{88704}\Big[21S_0^5x^{12} - 308S_0^4x^{10} + 1980S_0^3x^8 - 7392S_0^2x^6 + \\
18480S_0x^4 + 133056S_0x^2 - 44352x^2\Big] + \\
S_2 \times \Bigg[\frac{S_0^5x^{12}}{4224} + \frac{1}{288}S_0^4x^{10} + \frac{5}{224}S_0^3x^8 + \frac{1}{12}S_0^2x^6 + \frac{5S_0x^4}{24} - \frac{35}{36}S_0S_1x^4 + \frac{5S_1x^4}{24} + \\
\frac{9}{2}S_0S_1x^2 - \frac{3S_0x^2}{2} - \frac{3S_1x^2}{2} + \frac{29}{144}S_0S_1x^6 - \frac{37}{60}S_0^2S_1x^6 + \frac{25}{224}S_0^2S_1x^8 - \frac{45}{196}S_0^3S_1x^8 + \\
\frac{505S_0^3S_1x^{10}}{12096} - \frac{119S_0^4S_1x^{10}}{2592} + \frac{1555S_0^4S_1x^{12}}{133056} - \frac{17S_0^8S_1x^{20}}{3852288} + \frac{6185S_0^7S_1x^{18}}{108573696} + \\
\frac{163S_0^6S_1x^{16}}{354816} + \frac{21145S_0^5S_1x^{14}}{8072064} + \frac{89S_0^5S_1x^{12}}{23232} - \frac{1262395S_0^7S_1^4x^{22}}{1529966592} + \frac{26869S_0^6S_1^3x^{20}}{34320384} - \\
\frac{171505S_0^7S_1^3x^{20}}{125841408} + \frac{515S_0^7S_1^3x^{18}}{251328} + \frac{3275S_0^6S_1^3x^{18}}{565488} + \frac{295S_0^7S_1^4x^{18}}{91392} + \frac{63575S_0^8S_1^5x^{24}}{304668672} + \\
\frac{523166665S_0^7S_1^4x^{24}}{592275898368} - \frac{1262395S_0^7S_1^4x^{22}}{1529966592} + \frac{8123S_0^7S_1^4x^{20}}{2996224} + \frac{2145139S_0^8S_1^4x^{26}}{79705866240} - \\
\frac{37445S_0^8S_1^4x^{24}}{171376128} + \ldots + \frac{3755S_0^{23}S_1^5x^{58}}{6365173848985187647488} + \\
\frac{S_0^{25}S_1^5x^{62}}{5085586811194303315968} + \frac{S_0^{24}S_1^5x^{60}}{64911460909191266304}\Bigg]
\end{aligned}
\tag{3.11}
$$

It should be emphasized that y_3 does not represent the third correction term; rather it is the approximate solution after the third iteration. Unknown constants can be determined from the residual

$$R(x, S_0, S_1, S_2) = L\left(y_3(x, S_0, S_1, S_2)\right) + N\left(y_3(x, S_0, S_1, S_2)\right) \tag{3.12}$$

for the third order approximation. Using the Eq. (2.9) with $x = 1, 2, 3$ yields

$$S_0 = 1.53406577, S_1 = 0.98031243, S_2 = -0.10588147 \tag{3.13}$$

Substituting these constants into the Eq.(3.11), the approximate solution of the third order is obtained as:

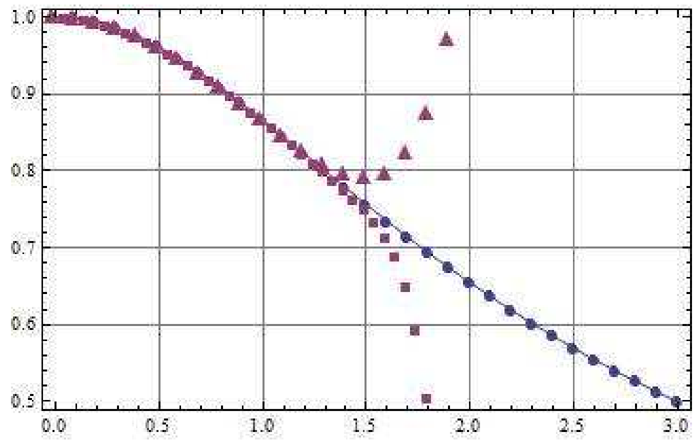


Figure 3.1: Comparison between the fourth order OPIM(•), fifth order HAM-VIM (▲), ninth order HAM-VIM (■) approximate solutions and the numerical results (–) for Example 1.

$$
\begin{aligned}
y_3(x) = {} & 1 - 0.166667x^2 + 0.0416667x^4 - 0.0115741x^6 + 0.00337577x^8 - \\
& 0.00101275x^{10} + 0.000309553x^{12} - 0.0000961702x^{14} + 0.0000305113x^{16} - \\
& 7.757926108 \times 10^{-6}x^{18} - 8.2105715765 \times 10^{-6}x^{20} + 0.0000259462x^{22} \\
& - 0.00002895x^{24} - 3.5887084471241 \times 10^{-7}x^{26} + 0.0000302234x^{28} \\
& - 7.866052351076 \times 10^{-6}x^{30} - 0.00003015x^{32} + 4.4502399674 \times 10^{-6}x^{34} \\
& + 0.00003140x^{36} + 6.5466232556 \times 10^{-6}x^{38} - 0.0000288707x^{40} - 0.000020677x^{42} \\
& + 0.000018640x^{44} + 0.000031354x^{46} + 2.724316366046 \times 10^{-6}x^{48} \\
& - 0.0000345355x^{50} - 0.000011951x^{52} + 0.000034677x^{54} + 0.000017377x^{56} \\
& - 0.0000476036x^{58} + 0.0000265864x^{60} - 4.989939036859478 \times 10^{-6}x^{62} \qquad (3.14)
\end{aligned}
$$

It is obvious that as the number of iterations increase, the approximate solution becomes more and more complicated which requires symbolic computer programs. Mathematica 9.0 is used to deal with the complex calculations in this work. Repeating the same steps by using Mathematica, one can easily get higher order approximate solutions. Dehghan et al.[29] and Singh et al. [30] obtained approximate solutions for this problem using variational iteration method(VIM) and homotopy analysis method(HAM). Figure 3.1 shows that OPIM yields better results than those obtained by VIM and HAM.

Example 3.2. *Consider the Lane-Emden type equation :*

$$y'' + \frac{2}{x}y' + e^y = 0, \quad y(0) = y'(0) = 0 \qquad (3.15)$$

which represents the isothermal gas spheres equation in the case that the temperature stays constant [15, 29, 31].

Reconsider the Eq. (3.15) as:

$$F(y'',y',y,\varepsilon) = y'' + (\frac{2\varepsilon}{x}y' + e^{\varepsilon y}) = L + N = 0 \tag{3.16}$$

where

$$L = y'' \text{ and } N = (\frac{2\varepsilon}{x}y' + e^{\varepsilon y}). \tag{3.17}$$

There is not y'' term in part N. Thus, (2.3) reduces to:

$$N + N_y(y_c)_n\varepsilon + N_{y'}({y_c}')_n\varepsilon + N_\varepsilon\varepsilon = -L. \tag{3.18}$$

With the help of the Eqs. (2.2), (3.17), (3.18) and setting $\varepsilon = 1$, OPIA is constructed as:

$$(y_c)''_n = -(y_n)'' - \frac{2}{x}(y_n)' - y_n - 1, \;\; y(0) = y'(0) = 0. \tag{3.19}$$

By choosing $y_0 = 0$ as a starting guess, (3.19) turns into first order problem:

$$(y_c)''_0 = -(y_0)'' - \frac{2}{x}(y_n)' - y_n - 1, \;\; y(0) = y'(0) = 0. \tag{3.20}$$

Solving (3.20) and using (2.4), one gets the following approximate solutions:

$$y_1 = -\frac{1}{2}S_0x^2 \tag{3.21}$$

$$y_2 = -\frac{1}{2}S_0x^2 + \frac{1}{24}(S_1)\,x^2\left(S_0x^2 + 36S_0 - 12\right) \tag{3.22}$$

$$\begin{aligned} y_3 = -\frac{x^2}{720}\Big[&S_0\Big[S_1\left(S_2\left(x^4+140x^2+3240\right)-\left(x^2+36\right)\right)- \\ &30\;30\left(S_2\left(x^2+36\right)-12\right)\Big]-30\left(S_1\left(S_2\left(x^2+36\right)-12\right)-12S_2\right)\Big] \end{aligned} \tag{3.23}$$

$$\begin{aligned} y_4 = \frac{x^2}{604800}\Big[&-840\left(S_1\left(S_2\left(S_3(x^4+140x^2+3240)-30(x^2+36)\right)\right.\right. - \\ &30\left.\left(S_3(x^2+36)-12\right)\right)-30\left(S_2(S_3(x^2+36)-12)-12S_3\right)\Big) \\ &S_0\left(-840\left(S_2\left(S_3(x^4+140x^2+3240)-30(x^2+36)\right)-30\left(S_3(x^2+36)-12\right)\right)\right. \\ &S_1\Big[-840\left(S_3(x^4+140x^2+3240)-30(x^2+36)\right) \\ &+S_2\left(S_3(15x^6+5096x^4+422800x^2+8164800)-840(x^4+140x^2+3240)\right)\Big] \end{aligned} \tag{3.24}$$

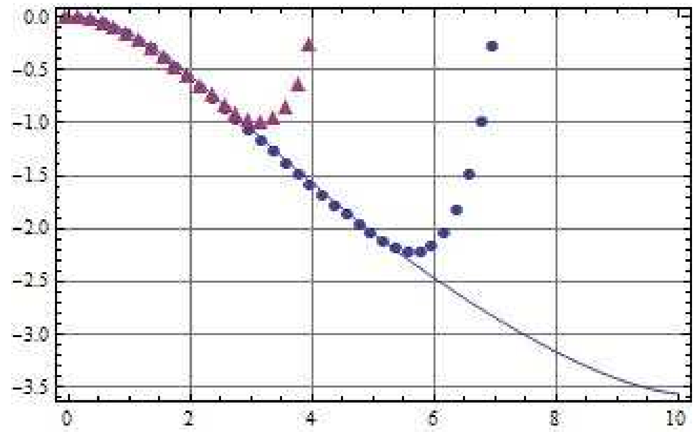


Figure 3.2: Comparison between the fourth order approximate solutions obtained by OPIM(●), ADM-DTM (▲) and the numerical results (–) for Example 2.

Using the Eq. (2.9) with the residual

$$R(x, S_0, S_1, S_2, S_3) = L\left(y_4(x, S_0, S_1, S_2, S_3)\right) + N\left(y_4(x, S_0, S_1, S_2, S_3)\right) \tag{3.25}$$

the unknown constants are obtained as

$$S_0 = 2.0203622551, S_1 = -1.0201147822, S_2 = -0.9963202221, S_3 = 0.020789994 \tag{3.26}$$

for the fourth order approximation. Substituting the Eq.(3.26) into (3.24) yields:

$$y_4(x) = -0.15989962328x^2 + 0.007920058473x^4 + \\ - 0.005608897215631x^6 + 0.000039055217456x^8. \tag{3.27}$$

In [26, 31], the authors have used differential transform method (DTM) and Adomian decomposition method(ADM) to solve the (3.15) and they obtained the series solution:

$$y(x) = -\frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{8}{3 \times 7!}x^6 + \frac{122}{9 \times 9!}x^8 - \frac{5032}{45 \times 11!}x^{10} + \ldots \tag{3.28}$$

Figure 3.2 and Figure 3.3 are sketched for comparison of the higher order approximate solutions of OPIM and ADM-DTM solutions.It is also clear from the figures that OPIM solutions are valid in larger region.

Example 3.3. *Consider the following homogeneous Lane-Emden type equation [32]:*

$$y'' + \frac{2}{x}y' - (4x^2 + 6)y = 0, y(0) = 1, y'(0) = 0, 0 \le x \le 1. \tag{3.29}$$

Exact solution of this problem is given as

$$y(x) = e^{x^2}. \tag{3.30}$$

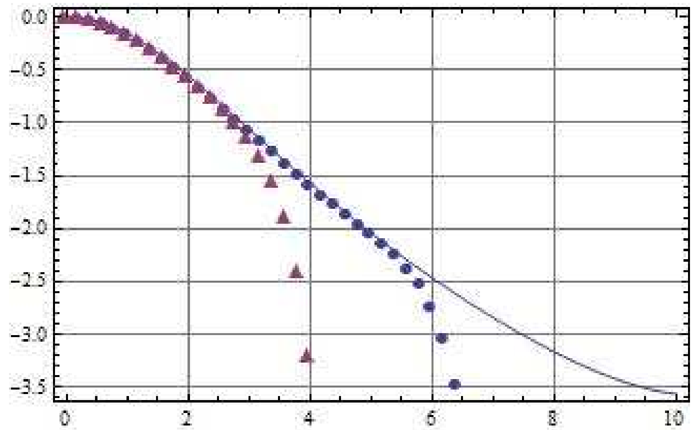


Figure 3.3: Comparison between the fifth order approximate solutions obtained by OPIM(●), ADM-DTM (▲) and the numerical results (–) for Example 2.

Rewrite the Eq. (3.29) as:

$$F(y'',y',y,\varepsilon) = y'' + \varepsilon\left(\frac{2}{x}y' - (4x^2+6)y\right) = L + N = 0 \tag{3.31}$$

where

$$L = y'' \text{ and } N = \varepsilon\left(\frac{2}{x}y' - (4x^2+6)y\right). \tag{3.32}$$

y'' does not appear in the remaining part N. Therefore, Eq. (2.3) becomes:

$$N + N_y(y_c)_n\varepsilon + N_{y'}(y_c{}')_n\varepsilon + N_\varepsilon\varepsilon = -L \tag{3.33}$$

In the light of previous examples, OPIA can be established as

$$(y_c)''_n = -(y_n)'' - \frac{2}{x}(y_n)' + (4x^2+6)y_n. \tag{3.34}$$

by using the Eqs. (2.2), (3.32)and (3.33). Accordingly, first order problem is obtained as:

$$(y_c)''_n = (4x^2+6),\ \ y(0) = y'(0) = 0 \tag{3.35}$$

Starting with the initial condition $y_0 = 1$, iterations can be reached as:

$$y_1 = 1 + \frac{1}{3}S_0\left(x^4+9x^2\right) \tag{3.36}$$

$$y_2 = 1 + \frac{1}{3}S_0\left(x^4+9x^2\right) + \frac{1}{630}S_1x^2\left[15S_0x^6 + 294S_0x^4 + 595S_0x^2 - 5670S_0 + 210x^2 + 1890\right] \tag{3.37}$$

$$y_3 = 1 + \frac{1}{3} S_0 \left(x^4 + 9x^2\right) + \frac{1}{630} S_1 x^2 \left[15 S_0 x^6 + 294 S_0 x^4 + 595 S_0 x^2 - 5670 S_0 + 210 x^2 + 1890\right] + \frac{S_2}{727650} \times \left[2182950 x^2 + 242550 x^4 - 6548850 S_0 x^2 + 687225 S_0 x^4 + 339570 S_0 x^6 + 17325 S_0 x^8 + 687225 S_1 x^4 + 339570 S_1 x^6 + 525 S_0 S_1 x^{12} + 16247 S_0 S_1 x^{10} + 63195 S_0 S_1 x^8 - 1211133 S_0 S_1 x^6 - 4419800 S_0 S_1 x^4 + 19646550 S_0 S_1 x^2 - 6548850 S_1 x^2 + 17325 S_1 x^8\right] \quad (3.38)$$

Unknown constants can be determined from the residual

$$R(x, S_0, S_1, S_2) = L\left(y_3(x, S_0, S_1, S_2)\right) + N\left(y_3(x, S_0, S_1, S_2)\right) \quad (3.39)$$

for the third order approximation. Using the Eq. (2.9) with $x = 0.3, 0.6, 0.9$ yields

$$S_0 = 0.33423439, S_1 = 0.31859877, S_2 = 0.20764389 \quad (3.40)$$

Substituting these constants into the Eq.(3.38), the approximate solution of the third order is obtained as:

$$y_3(x) = 0.000135466 x^{12} + 0.00419221 x^{10} + 0.041724 x^8 + 0.185681 x^6 + 0.483647 x^4 + 1.0041 x^2 + 1. \quad (3.41)$$

Repeating the same steps one can get the following approximations:

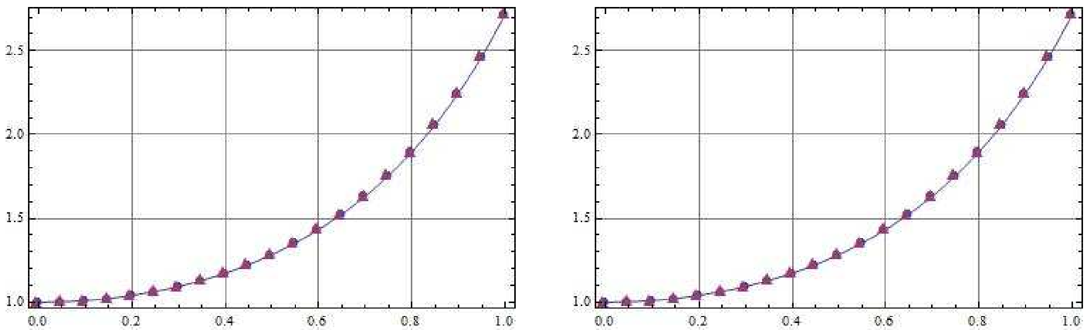
$$y_4(x) = 0.0000407289 x^{16} + 0.000166717 x^{14} + 0.00142143 x^{12} + 0.00831423 x^{10} + 0.0416732 x^8 + 0.166665 x^6 + 0.5 x^4 + x^2 + 1. \quad (3.42)$$

$$y_5(x) = 4.529285401255387 \times 10^{-7} x^{20} + 2.306750146551151 \times 10^{-6} x^{18} + 0.0000254168 x^{16} + 0.000197899 x^{14} + 0.00138916 x^{12} + 0.00833324 x^{10} + 0.0416667 x^8 + 0.166667 x^6 + 0.5 x^4 + x^2 + 1. \quad (3.43)$$

This problem has been also investigated by Ozis et al using variational iteration method(VIM) and homotopy perturbation method (HPM) [32, 33]. Figures 3.4, 3.5 and Table 1 give important information on the convergence and the absolute errors for OPIA and other approximate solutions in literature. It is clear that the results obtained by OPIM are more accurate than those in [32, 33] .

Table 1: Comparison of absolute errors for Example 3 at different orders of approximations.

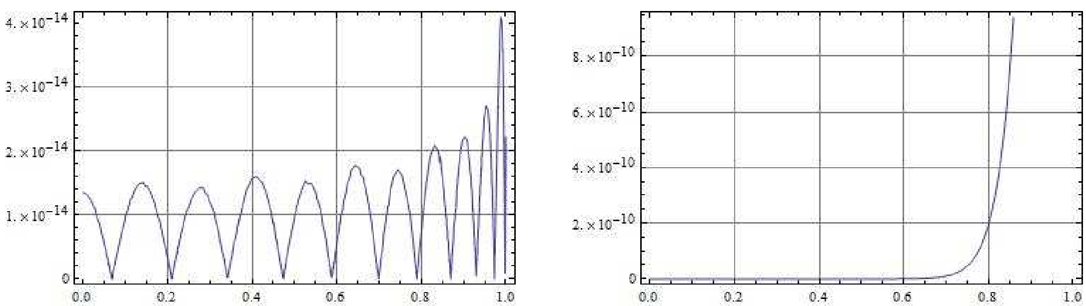
x	Errors for OPIA		Errors for VIM-HPM	
	$\lvert y_{exact} - y_5 \rvert$	$\lvert y_{exact} - y_6 \rvert$	$\lvert y_{exact} - y_5 \rvert$	$\lvert y_{exact} - y_6 \rvert$
0.1	3.08426×10^{-13}	2.22045×10^{-16}	1.11022×10^{-15}	1.00128×10^{-16}
0.2	3.85914×10^{-13}	2.22045×10^{-16}	5.72165×10^{-12}	3.26406×10^{-14}
0.3	5.62883×10^{-13}	1.00025×10^{-17}	7.47710×10^{-10}	9.59788×10^{-12}
0.4	9.64347×10^{-13}	2.44249×10^{-15}	2.38451×10^{-8}	5.43454×10^{-10}
0.5	1.95532×10^{-12}	1.50998×10^{-14}	3.51584×10^{-7}	1.24994×10^{-8}
0.6	4.77649×10^{-12}	1.01037×10^{-13}	3.18608×10^{-6}	1.62772×10^{-7}
0.7	5.45375×10^{-11}	5.02709×10^{-13}	0.0000206568	1.43282×10^{-6}
0.8	2.78031×10^{-11}	2.05902×10^{-12}	0.000104921	9.47740×10^{-6}
0.9	3.08426×10^{-10}	6.38223×10^{-12}	0.000442699	0.000050436
1	2.10201×10^{-9}	2.90235×10^{-12}	0.00161516	0.000226273



(a) Fourth order OPIA(●) and VIM-HPM(▲) solutions.

(b) Fifth order OPIA(●) and VIM-HPM(▲) solutions.

Figure 3.4: Comparison of approximate solutions of OPIM and VIM-HPM with the exact solution(–) for Example 3.



(a) OPIA solution of fifth order.

(b) VIM solution of fifth order.

Figure 3.5: Errors for OPIA and the variational iteration method for Example 3.

4 Conclusion

In this paper, OPIM is applied for the first time to investigate the new approximate solutions for Lane-Emden type differential equations. This new technique provides us to optimally control the convergence of solution series. Also, it gives a very good approximation even in a few terms to these kinds of nonlinear equations. The results obtained in this paper proves that the OPIM is a very effective technique for differential equations. It is worth mentioning that, a symbolic program is necessary for successive calculations after a few iterations. Mathematica 9 has been used to overcome the complicated calculations for this present research.

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