Some remarks on the structure of Lipschitz-free spaces[*]

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Abstract

We give several structural results concerning the Lipschitz-free spaces $\mathcal{F}(M)$, where M is a metric space. We show that $\mathcal{F}(M)$ contains a complemented copy of $\ell_1(\Gamma)$, where $\Gamma = \mathrm{dens}(M)$. If $\mathcal{N}$ is a net in a finite dimensional Banach space X, we show that $\mathcal{F}(\mathcal{N})$ is isomorphic to its square. If X contains a complemented copy of ℓ_p, c_0 then $\mathcal{F}(\mathcal{N})$ is isomorphic to its ℓ_1-sum. Finally, we prove that for all $X \cong C(K)$ spaces, where K is a metrizable compact, $\mathcal{F}(\mathcal{N})$ are mutually isomorphic spaces with a Schauder basis.

1 Introduction

Let (M, d) be a metric space and $0 \in M$ be a distinguished point. The triple $(M, d, 0)$ is called pointed metric space. By $\mathrm{Lip}_0(M)$ we denote the Banach space of all Lipschitz real valued functions $f : M \to \mathbb{R}$, such that $f(0) = 0$. The norm of $f \in \mathrm{Lip}_0(M)$ is defined as the smallest Lipschitz constant $L = \mathrm{Lip}(f)$ of f, i.e.

$$\mathrm{Lip}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)}, \ x, y \in M, \ x \neq y \right\}.$$

The Dirac map $\delta : M \to \mathrm{Lip}_0(M)^*$ defined by $\langle f, \delta(p) \rangle = f(p)$ for $f \in \mathrm{Lip}_0(M)$ and $p \in M$ is an isometric embedding from M into $\mathrm{Lip}_0(M)^*$.

Note that $\delta(0) = 0$. The closed linear span of $\{\delta(p),\ p \in M\}$ is denoted $\mathcal{F}(M)$ and called the Lipschitz-free space over M (or just free space, for short). Clearly,

$$\|m\|_{\mathcal{F}(M)} = \sup\left\{\langle m, f\rangle : f \in \mathrm{Lip}_0(M), \|f\| \le 1\right\}$$

It follows from the compactness of the unit ball of $\mathrm{Lip}_0(M)$ with respect to the topology of pointwise convergence, that $\mathcal{F}(M)$ can be seen as the canonical predual of $\mathrm{Lip}_0(M)$, i.e. $\mathcal{F}(M)^* = \mathrm{Lip}_0(M)$ holds isometrically ([33] Chapter 2 for details).

Lipschitz free spaces have gained importance in the non-linear structural theory of Banach spaces after the appearance of the seminal paper [13] of Godefroy and Kalton, and the subsequent work of these and many other authors e.g. [19], [20], [21], [22], [14], [24], [16], [17], [31], [23], [5], [4], [10], [6], [29] [7], [8], [9]. Free spaces can be used efficiently for constructions of various examples of Lipschitz-isomorphic Banach spaces X, Y which are not linearly isomorphic. To this end, structural properties of their free spaces $\mathcal{F}(X)$, as well as free spaces of their subsets, enter the game. For example, in the separable setting, $\mathcal{F}(X)$ contains a complemented copy of X [13], and it is isomorphic to its ℓ_1-sum. On the other hand, if $\mathcal{N}$ is a net in X then $\mathcal{F}(\mathcal{N})$ is a Schur space [20] and it has the approximation property.

A comprehensive background on free spaces of metric spaces can be found in the book of Weaver [33]. There are several surveys exposing the applications of this notion to the nonlinear structural theory of Banach spaces, in particular [19], [15].

Our first observation in this note is that $\mathcal{F}(M)$ contains a complemented copy of $\ell_1(\Gamma)$, where Γ is the density character of an arbitrary infinite metric space M. Our proof could be adjusted also to the case $\Gamma = \omega_0$, which is one of the main results in [5].

The main purpose of this note is to prove several structural results, focusing mainly on the case when M is a uniformly discrete metric space, in particular a net $\mathcal{N}$ in a Banach space X. Our results run parallel (as we have realized during the preparation of this note) to those of Kaufmann [23], resp. Dutrieux and Ferenczi [10] which are concerned with the bigger (in a sense) space $\mathcal{F}(X)$. However, the space $\mathcal{F}(\mathcal{N})$ is only the linear quotient of $\mathcal{F}(X)$, so the results are certainly not formally transferable. In particular, the discrete setting prohibits the use of the "scaling towards zero" arguments (used e.g. in [23]), which leads to complications in proving that our free spaces are linearly isomorphic to their squares, or even ℓ_1-sums. We are able to show these facts at least for nets in finite dimensional Banach spaces and all classical Banach spaces. Surprisingly, the proofs for the finite dimensional case and the infinite dimensional case are rather different.

Our main technical result is that $\mathcal{F}(\mathcal{N})$ has a Schauder basis for all nets in $C(K)$ spaces, K metrizable compact. The constructive proof is obtained in c_0, and the result is then transferred into the $C(K)$ situation by using the abstract theory developed in the first part of our note.

Let us start with some definitions and preliminary results. Let $N \subset M$ be metric spaces, and assume that the distinguished point $0 \in M$ serves as a distinguished point in N as well. Then the identity mapping leads to the canonical isometric embedding $\mathcal{F}(N) \hookrightarrow \mathcal{F}(M)$ ([33] p.42). In order to study the complementability properties of this subspace, one can rely on the theory of quotients of metric space, as outlined in [33] p.11 or [23]. For our purposes we will outline an alternative (but equivalent) description of the situation.

Definition 1. *Let* $N \subset M$ *be metric spaces,* $0 \in N$. *We denote by*

$$Lip_N(M) = \{f \in Lip_0(M) : f|_N = 0\}.$$

It is clear that $Lip_N(M)$ is a closed linear subspace of $Lip_0(M)$, which is moreover w^*-closed. Indeed, by the general perpendicularity principles ([11] p.56) we obtain

$$Lip_N(M) = \mathcal{F}(N)^\perp, \quad \mathcal{F}(N) = Lip_N(M)_\perp$$

Hence there is a canonical isometric isomorphism

$$Lip_N(M) \cong (\mathcal{F}(M)/\mathcal{F}(N))^*$$

Since the space of all finite linear combinations of Dirac functionals is linearly dense in $\mathcal{F}(M)$, resp. also in $\mathcal{F}(N)$, it is clear that the image of finite linear combinations of Dirac functionals supported outside the set N, under the quotient mapping $\mathcal{F}(M) \to \mathcal{F}(M)/\mathcal{F}(N)$ is linearly dense. Moreover, it is nonzero for nontrivial combinations.

Definition 2. *If* $\mu = \sum_{j=1}^n a_j \delta_{t_j} : t_j \in M \setminus N$ *then we let*

$$\|\mu\|_{\mathcal{F}_N(M)} = \sup \langle \mu, f \rangle, \quad f \in Lip_N(M), \ \|f\| \leq 1.$$

$$\mathcal{F}_N(M) = \overline{\left\{ \mu = \sum_{j=1}^n a_j \delta_{t_j} : t_j \in M \setminus N \right\}}^{\|\cdot\|_{\mathcal{F}_N(M)}}.$$

i.e. we complete the space of finite sums of Dirac functionals with respect to the duality

$$\langle \mathcal{F}_N(M), Lip_N(M) \rangle.$$

Clearly, our definition gives an isometric isomorphism

$$\mathcal{F}_N(M) \cong \mathcal{F}(M)/\mathcal{F}(N)$$

Proposition 1. *Let* $N \subset M$ *be metric spaces. If there exists a Lipschitz retraction* $r : M \to N$ *then*

$$\mathcal{F}(M) \cong \mathcal{F}(N) \oplus \mathcal{F}_N(M).$$

This follows readily from the alternative description using metric quotients (e.g. in [23] Lemma 2.2) using the fact that $\mathcal{F}_N(M) \cong \mathcal{F}(M/N)$.

We say that the metric space (M, d) is δ-uniformly discrete if there exists $\delta > 0$ such that $d(x, y) \geq \delta, x, y \in M$. The metric space is uniformly discrete if it is δ-uniformly discrete for some $\delta > 0$.

If $\alpha, \beta > 0$ we say that a subset $N \subset M$ is a (α, β)-net in M provided it is α-uniformly discrete and $d(x, N) < \beta, x \in M$.

It is easy to see that every maximal δ-separated subset $N \subset M$, which exists due to the Zorn maximal principle, is automatically a $(\delta, \delta + \varepsilon)$-net, for any $\varepsilon > 0$.

Proposition 2. *Let $(M, d, 0)$ be a pointed metric space, $K > 0$, $\{M_\alpha\}_{\alpha \in \Gamma}$ be a system of pairwise disjoint subsets of M, and $0 \in N \subset M \setminus \cup_{\alpha \in \Gamma} M_\alpha$. Suppose that for all $\beta \in \Gamma$ and all $x \in M_\beta$ holds*

$$d(x, \cup_{\alpha \in \Gamma, \alpha \neq \beta} M_\alpha) \geq K d(x, N).$$

Then

$$\mathcal{F}_N(N \cup \cup_{\alpha \in \Gamma} M_\alpha) \cong (\oplus_{\alpha \in \Gamma} \mathcal{F}_N(N \cup M_\alpha))_{\ell_1(\Gamma)}.$$

In particular, if $N = \{0\}$ then

$$\mathcal{F}(\{0\} \cup \cup_{\alpha \in \Gamma} M_\alpha) \cong (\oplus_{\alpha \in \Gamma} \mathcal{F}(\{0\} \cup M_\alpha))_{\ell_1(\Gamma)}.$$

Proof. The result is immediate as any collection of 1-Lipschitz functions $f_\alpha \in \text{Lip}_N(N \cup M_\alpha)$ is the restriction of a $\frac{1}{K}$-Lipschitz function $f \in \text{Lip}_N(N \cup \cup_{\alpha \in \Gamma} M_\alpha)$ ∎

Recall that the density character dens(M), or just density, of a metric space M is the smallest cardinal Γ such that there exists dense subset of M of cardinality Γ.

Let Γ be a cardinal (which is identified with the smallest ordinal of the same cardinality). By the cofinality cof(Γ) we denote the smallest ordinal α (in fact a cardinal) such that $\Gamma = \lim_{\beta < \alpha} \Gamma_\beta$, where Γ_β is an increasing ordinal sequence ([18] p.26).

2 Structural properties

Proposition 3. *Let M be a metric space of density dens$(M) = \Gamma$. Then $\mathcal{F}(M)$ contains a complemented copy of $\ell_1(\Gamma)$.*

Proof. For convenience we may assume that $\Gamma > \omega_0$, because this case has been already proved in [5] (Our proof can be adjusted to this case as well). By ([32] Corollary 1.2) if $c_0(\Gamma) \hookrightarrow X^*$ then $\ell_1(\Gamma)$ is complemented in X. So it suffices to prove that $\text{Lip}_0(M)$ contains a copy of $c_0(\Gamma)$. For every $n \in \mathbb{N}$ let M_n be some maximal $\frac{1}{2^n}$-separated set in M. Denote $\Gamma_n = |M_n|$. It is clear that dens$(M) = \lim_{n \to \infty} \Gamma_n$, in the cardinal sense. In case when the cofinality cof$(\Gamma) > \omega_0$, it is clear that $\Gamma_n = \Gamma$, for some $n \in \mathbb{N}$. In this case, let $\{f_\alpha : \alpha \in \Gamma_n\}$ be a transfinite sequence of 1-Lipschitz functions such that $f_\alpha(x_\alpha) = \frac{1}{2^{n+3}}$ and supp$(f_\alpha) \subset B(x_\alpha, \frac{1}{2^{n+2}})$. Since the supports of f_α are pairwise disjoint it is clear that $\{f_\alpha\}_{\alpha \in \Gamma_n}$ is equivalent to the unit basis of $c_0(\Gamma)$ and the result follows. In the remaining case,

we may assume that $\{\Gamma_{k_n}\}_{n=1}^{\infty}$ is a strictly increasing sequence of cardinals. Denote $M_n = \{x_\alpha^n\}_{\alpha \in \Gamma_{k_n}}$. Let $L_1 = M_1$. By induction we will construct sets $L_n \subset M_n$ as follows. Inductive step towards $n+1$. Consider the sets

$$A_{j,\alpha} = M_{n+1} \cap B(x_\alpha^j, \frac{1}{2^{k_j+1}}), \quad j \leq n, \alpha \in \Gamma_{k_j}$$

If there is some j, α so that $|A_{j,\alpha}| = \Gamma_{k_{n+1}}$ then we let $L_{n+1} = A_{j,\alpha}$. Otherwise we let

$$L_{n+1} = M_{n+1} \setminus \cup_{j \leq n, \alpha \in \Gamma_{k_j}} A_{j,\alpha}$$

In either case we have $|L_{n+1}| = \Gamma_{k_{n+1}}$. By discarding suitable countable subsets of these sets L_n we can assume that

$$\mathrm{dist}(L_n, L_m) \geq \max\left\{\frac{1}{2^{k_n+1}}, \frac{1}{2^{k_m+1}}\right\}$$

To finish, let $\{f_\alpha^n : x_\alpha^n \in L_n, n \in \mathbb{N}\}$ be a transfinite sequence of 1-Lipschitz disjointly supported functions such that $f_\alpha^n(x_\alpha^n) = \frac{1}{2^{k_n+3}}$ and $\mathrm{supp}(f_\alpha^n) \subset B(x_\alpha^n, \frac{1}{2^{k_n+2}})$. This sequence is equivalent to the basis of $c_0(\Gamma)$, which finishes the proof. ∎

Theorem 4. *Let N, M be uniformly discrete infinite sets of the same cardinality such that $N \subset M$ is a net. Then $\mathcal{F}(N) \cong \mathcal{F}(M)$.*

Proof. Let $K > 0$ be such that $\max_{m \in M} \mathrm{dist}(m, N) \leq K$ Let $r : M \to N$ be a retraction such that $d(x, r(x)) \leq K$. As M is uniformly discrete, r is Lipschitz. By Proposition 1

$$\mathcal{F}(M) \cong \mathcal{F}(N) \oplus \mathcal{F}_N(M).$$

It is clear that $\mathcal{F}_N(M) \cong \ell_1(M \setminus N)$. By Proposition 3

$$\mathcal{F}(N) \cong \mathcal{F}(N) \oplus \ell_1(M) \cong \mathcal{F}(M).$$ ∎

Recall that all nets in a given infinite dimensional Banach space are Lipschitz equivalent ([26], or [1] p.239), hence their free spaces are linearly isomorphic. On the other hand, there are examples of non-equivalent nets in $\mathbb{R}^2$ ([28], [3] or [1] p.242), hence the next result is not immediately obvious.

Proposition 5. *Let $\mathcal{N}, \mathcal{M}$ be nets of the same cardinality $\mathrm{dens}(M)$ in a metric space (M, d). Then $\mathcal{F}(\mathcal{N}) \cong \mathcal{F}(\mathcal{M})$.*

Proof. Suppose $\mathcal{N}$ is a (a, b)-net and $\mathcal{M}$ is a (c, d)-net in M, $a \leq c$. Let $K = \mathcal{M} \cup \mathcal{N}$, and let $\mathcal{K} \subset K$ be maximal subset such that from each pair of points $x \in \mathcal{M}, y \in \mathcal{N}$ for which $d(x, y) < \frac{a}{4}$ we choose only one $x \in \mathcal{K}$. It is now clear that both $\mathcal{N}$ and $\mathcal{M}$ are bi-Lipschitz equivalent to a respective subset of $\mathcal{K}$. By Theorem 4, $\mathcal{F}(\mathcal{K}) \cong \mathcal{F}(\mathcal{M}) \cong \mathcal{F}(\mathcal{N})$. ∎

Of course, the above proposition applies to any pair of nets in a given Banach space X, or its subset $S \subset X$ which contains arbitrarily large balls.

Lemma 6. *Let $Y = X \oplus \mathbb{R}$ be Banach spaces, $\mathcal{N}$ be a net in X and $\mathcal{M}$ be the extension of $\mathcal{N}$ into the natural net in Y. Denote $\mathcal{M}^+ = \mathcal{M} \cap X \oplus \mathbb{R}^+$, $\mathcal{M}^- = \mathcal{M} \cap X \oplus \mathbb{R}^-$.*
If $\mathcal{F}(\mathcal{N}) = \mathcal{F}(\mathcal{N}) \oplus \mathcal{F}(\mathcal{N})$ and $\mathcal{F}(\mathcal{M}^+) = \mathcal{F}(\mathcal{M}^+) \oplus \mathcal{F}(\mathcal{M}^+)$ then $\mathcal{F}(\mathcal{M}) = \mathcal{F}(\mathcal{M}^+) = \mathcal{F}(\mathcal{M}) \oplus \mathcal{F}(\mathcal{M})$.

Proof. Thanks to Proposition 5 we are allowed to make additional assumptions on the form of the nets. Let us assume that $\mathcal{M} = \mathcal{N} \times \mathbb{Z}$, which immediately implies that $\mathcal{N} \cup \mathcal{M}^+$ is bi-Lipschitz equivalent with $\mathcal{M}^+$ (and $\mathcal{M}^-$) by translation. Denoting $P : Y \to X$ the canonical projection $P(x, t) = x$, we see that $P : \mathcal{M} \to \mathcal{N}$ is a Lipschitz retraction, so

$$\mathcal{F}(\mathcal{M}^+) \cong \mathcal{F}(\mathcal{N} \cup \mathcal{M}^+) \cong \mathcal{F}(\mathcal{N}) \oplus \mathcal{F}_{\mathcal{N}}(\mathcal{N} \cup \mathcal{M}^+)$$

and using Proposition 2

$$\mathcal{F}(\mathcal{M}) \cong \mathcal{F}(\mathcal{N}) \oplus \mathcal{F}_{\mathcal{N}}(\mathcal{M}) \cong \mathcal{F}(\mathcal{N}) \oplus \mathcal{F}_{\mathcal{N}}(\mathcal{N} \cup \mathcal{M}^+) \oplus \mathcal{F}_{\mathcal{N}}(\mathcal{N} \cup \mathcal{M}^-)$$

Since $\mathcal{F}_{\mathcal{N}}(\mathcal{N} \cup \mathcal{M}^+) \cong \mathcal{F}_{\mathcal{N}}(\mathcal{N} \cup \mathcal{M}^-)$ and $\mathcal{F}(\mathcal{N}) \cong \mathcal{F}(\mathcal{N}) \oplus \mathcal{F}(\mathcal{N})$ the result follows.

∎

Theorem 7. *Let $\mathcal{N}$ be a net in $\mathbb{R}^n$. Then $\mathcal{F}(\mathcal{N}) \cong \mathcal{F}(\mathcal{N}) \oplus \mathcal{F}(\mathcal{N})$.*

Proof. For $n = 1$ it is well known [12] that $\mathcal{F}(\mathcal{N}) \cong \mathcal{F}(\mathcal{N}^+) \cong \ell_1 \cong \mathcal{F}(\mathcal{N}) \oplus \mathcal{F}(\mathcal{N})$.

Inductive step towards $n + 1$. We may assume that $\mathcal{N} = \mathbb{Z}^{n+1}$ is the integer grid. Let us use the following notation (our convention is that $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$, $\mathbb{Z}^- = \{-1, -2, \dots\}$).

$$\mathcal{L} = \mathbb{Z}^{n-1} \times \{0\} \times \{0\}, \ \mathcal{L}_1 = \mathbb{Z}^{n-1} \times \mathbb{Z}^+ \times \{0\},$$
$$\mathcal{L}_2 = \mathbb{Z}^{n-1} \times \{0\} \times \mathbb{Z}^+, \ \mathcal{L}_3 = \mathbb{Z}^{n-1} \times \mathbb{Z}^- \times \{0\}$$
$$\mathcal{M}^+ = \mathbb{Z}^{n-1} \times \mathbb{Z} \times \mathbb{Z}^+, \ \mathcal{M}_1 = \mathbb{Z}^{n-1} \times \mathbb{Z}^+ \times \mathbb{Z}^+, \ \mathcal{M}_2 = \mathbb{Z}^{n-1} \times \mathbb{Z}^- \times \mathbb{Z}^+$$

With this notation, we have the following bi-Lipschitz equivalence

$$\mathcal{L}_1 \cup \mathcal{L} \cup \mathcal{L}_2 \cong \mathcal{L}_1 \cup \mathcal{L} \cup \mathcal{L}_3.$$

By inductive assumption this implies

$$\mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \cong \mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \oplus \mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2). \tag{1}$$

On the other hand, using Proposition 2 in various settings

$$\mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \cong \mathcal{F}(\mathcal{L}) \oplus \mathcal{F}_{\mathcal{L}}(\mathcal{L} \cup \mathcal{L}_1) \oplus \mathcal{F}_{\mathcal{L}}(\mathcal{L} \cup \mathcal{L}_2),$$

$$\mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \cong \mathcal{F}(\mathcal{L} \cup \mathcal{L}_1) \oplus \mathcal{F}_{\mathcal{L} \cup \mathcal{L}_1}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \cong$$
$$\mathcal{F}(\mathcal{L} \cup \mathcal{L}_1) \oplus \mathcal{F}_{\mathcal{L}}(\mathcal{L} \cup \mathcal{L}_2), \tag{2}$$

$$\mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3) \cong \mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_3) \oplus \mathcal{F}_{\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_3}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3) \cong$$
$$\mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_3) \oplus \mathcal{F}_{\mathcal{L}}(\mathcal{L} \cup \mathcal{L}_2). \quad (3)$$

Hence, using the inductive assumption $\mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_3) \cong \mathcal{F}(\mathcal{L} \cup \mathcal{L}_1)$

$$\mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3) \cong \mathcal{F}(\mathcal{L} \cup \mathcal{L}_1) \oplus \mathcal{F}_{\mathcal{L}}(\mathcal{L} \cup \mathcal{L}_2) \quad (4)$$

Comparing (2), (4) and using (1) we obtain

$$\mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3) \cong \mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \cong \mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \oplus \mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \quad (5)$$

By Lemma 6, in order to complete the inductive step, it suffices to prove that $\mathcal{F}(\mathcal{M}^+) = \mathcal{F}(\mathcal{M}^+) \oplus \mathcal{F}(\mathcal{M}^+)$.

Denote $R : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+$ the mapping $R(z) = \frac{z^2}{|z|}$, where z is the complex number represented as $z = x + iy$. It is clear that R is bi-Lipschitz. Indeed, if $z_0 = a + ib$ and $z_1 = x + iy$ are two complex numbers from the first quadrant with $a \le x$, then

$$|R(z_0) - R(z_1)| = |e^{a+2ib} - e^{x+2iy}| \le |e^{a+2ib} - e^{a+2iy}| + |e^{a+2iy} - e^{x+2iy}| =$$
$$= e^a |e^{ib} - e^{iy}| \cdot |e^{ib} + e^{iy}| + |e^a - e^x|$$
$$\le 2|e^{a+ib} - e^{a+iy}| + |e^{a+ib} - e^{x+iy}|$$
$$\le 2|e^{a+ib} - e^{x+iy}| + |e^{a+ib} - e^{x+iy}| = 3|z_0 - z_1|.$$

On the other hand, for any $z_0 = a + ib$ and $z_1 = x + iy$ from the upper half plane with $a \le x$ we have

$$|R^{-1}(z_0) - R^{-1}(z_1)| = |e^{a+\frac{ib}{2}} - e^{x+\frac{iy}{2}}| \le |e^{a+\frac{ib}{2}} - e^{a+\frac{iy}{2}}| + |e^{a+\frac{iy}{2}} - e^{x+\frac{iy}{2}}| =$$
$$= e^a \frac{|e^{ib} - e^{iy}|}{|e^{\frac{ib}{2}} + e^{\frac{iy}{2}}|} + |e^a - e^x|$$
$$\le \frac{\sqrt{2}}{2} |e^{a+ib} - e^{a+iy}| + |e^{a+ib} - e^{x+iy}|$$
$$\le 2|z_0 - z_1|,$$

which we wanted to prove.

The mapping

$$T : \mathcal{M}_1 \to \mathbb{R}^{n+1}, T(u, x, y) = (u, R(x, y))$$

takes the net $\mathcal{M}_1$ from the set $\mathbb{R}^{n-1} \times \mathbb{R}^+ \times \mathbb{R}^+$ in a bi-Lipschitz way to the net $T(\mathcal{M}_1)$ in the set $\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^+$. Hence $\mathcal{F}(\mathcal{M}_1) \cong \mathcal{F}(T(\mathcal{M}_1))$. Since $\mathcal{M}^+ = \mathcal{M}_1 \cup \mathcal{L}_2 \cup \mathcal{M}_2$ is another net in the second set, by Proposition 5 we obtain

$$\mathcal{F}(\mathcal{M}_1) \cong \mathcal{F}(\mathcal{M}^+)$$

Now thanks to the bi-Lipschitz equivalence $\mathcal{M}_1 \cong \mathcal{M}_1 \cup \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2$,

$$\mathcal{F}(\mathcal{M}_1) \cong \mathcal{F}(\mathcal{M}_1 \cup \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \cong$$
$$\mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \oplus \mathcal{F}_{\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2}(\mathcal{M}_1 \cup \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2)$$

Since $\mathcal{M}^+$ is bi-Lipschitz equivalent to $\mathcal{M}^+ \cup \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2$ we get

$$\mathcal{F}(\mathcal{M}^+) \cong \mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3) \oplus \mathcal{F}_{\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2}(\mathcal{M}_1 \cup \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \oplus$$
$$\mathcal{F}_{\mathcal{L} \cup \mathcal{L}_2 \cup \mathcal{L}_3}(\mathcal{M}_2 \cup \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_3) \quad (6)$$

Using (5) and the obvious

$$\mathcal{F}_{\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2}(\mathcal{M}_1 \cup \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \cong \mathcal{F}_{\mathcal{L} \cup \mathcal{L}_2 \cup \mathcal{L}_3}(\mathcal{M}_2 \cup \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_3)$$

we finally obtain

$$\mathcal{F}(\mathcal{M}_1) \oplus \mathcal{F}(\mathcal{M}_1) \cong \mathcal{F}(\mathcal{M}^+) \cong \mathcal{F}(\mathcal{M}_1)$$

which ends the inductive step and the proof.   ∎

Theorem 8. *Let X be a Banach space such that $X \cong Y \oplus X$, where Y is an infinite dimensional Banach space with a Schauder basis. Let $\mathcal{N}$ be a net in X. Then*

$$\mathcal{F}(\mathcal{N}) \cong (\oplus_{j=1}^{\infty} F(\mathcal{N}))_{\ell_1}.$$

Proof. We may assume without loss of generality that the norm of the direct sum $Y \oplus X$ is in fact equal to the maximum norm $Y \oplus_{\infty} X$. Using Proposition 5 it suffices to prove the result for just one particular net $\mathcal{N}$. Let $M_k \subset kS_X$, $k \in \mathbb{N}$ be a $(1,2)$-net. Then $\mathcal{N} = \cup_{k=1}^{\infty} M_k$ is a $(1,3)$-net in X. Let $\{e_k\}$ be a bi-monotone normalized Schauder basis of Y. Set $Z = (\oplus_{j=1}^{\infty} F(\mathcal{N}))_{\ell_1}$. It is clear that
$Z \cong (\oplus_{j=1}^{\infty} Z)_{\ell_1}$

We will use Pelczynski's decomposition technique to prove the theorem. Since $\mathcal{F}(\mathcal{N})$ is complemented in Z it only remains to prove that $\mathcal{F}(\mathcal{N})$ contains a complemented subspace isomorphic to Z. Let

$$V_n = \{ke_n \oplus x : x \in M_k, k \in \mathbb{N}\} \subset Y \oplus X$$
$$M = \cup_{n=1}^{\infty} V_n$$

The sets V_n, as subsets of the pointed metric space $(Y \oplus X, \| \cdot \|, 0)$, satisfy the assumptions of Proposition 2 and so

$$\mathcal{F}(M) = (\oplus_{n=1}^{\infty} \mathcal{F}(V_n))_{\ell_1} \cong (\oplus_{n=1}^{\infty} \mathcal{F}(\mathcal{N}))_{\ell_1} \cong Z.$$

We extend the set M into a $(1,3)$-net $\mathcal{M}$ in $Y \oplus X$. Because $\mathcal{F}(\mathcal{M}) \cong \mathcal{F}(\mathcal{N})$ it suffices to show that $\mathcal{F}(\mathcal{M})$ contains a complemented copy of $\mathcal{F}(M)$. To this end it is enough to find a Lipschitz retraction $R : \mathcal{M} \to M$. Denote by $[a]$ the

integer part of $a \in \mathbb{R}$. First let $r : X \to \mathcal{N}$ be a (non-continuous) retraction such that $[\|x\|] \leq \|r(x)\| \leq \|x\|, \|r(x) - x\| \leq 4$ and $\|r(x)\| = \|x\|$ provided $\|x\| \in \mathbb{N}$. Let $s : Y \to Y$ be a (non-continuous) retraction defined for $x = \sum_{i=1}^{\infty} x_i e_i$ by

$$s\left(\sum_{i=1}^{\infty} x_i e_i\right) = \begin{cases} de_k & \text{if } x_k > \max\{x_i : i \neq k\} \cup \{0\}, d = \min_{i \neq k}[x_k - x_i] \\ \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

It is easy to see that $\|r(x) - r(y)\| \leq 9\|x - y\|$, $\|s(x) - s(y)\| \leq 6\|x - y\|$ provided $\|x - y\| \geq 1$ (i.e. they are Lipschitz for large distances). Indeed,

$$\|r(x) - r(y)\| \leq \|r(x) - x\| + \|r(y) - y\| + \|x - y\| \leq 8 + \|x - y\| \leq 9\|x - y\|$$

Assuming $1 \leq \|x - y\| \leq \lambda$, we get $|x_i - y_i| \leq \lambda, i \in \mathbb{N}$. Suppose that $s(x) = de_k$, $s(y) = te_l$. We claim that $d \leq 3\lambda$. Indeed, assuming by contradiction that $x_k \geq d + \max\{x_i : i \neq k\} \geq 3\lambda + \max\{x_i : i \neq k\}$ we obtain that $y_k \geq \lambda + \max\{y_i : i \neq k\}$. Hence $k = l$ and $|d - t| \leq 2\lambda + 2$. The same argument yields $t \leq 3\lambda$, so finally we obtain $\|s(x) - s(y)\| \leq 6\lambda$.

Let $R : \mathcal{M} \to M$ is now defined as

$$R(y \oplus x) = \begin{cases} s(y) \oplus r\left(\frac{\|s(y)\|}{\|x\|}x\right) & \text{if } \|x\| > \|s(y)\| > 0 \\ \frac{\|r(x)\|}{\|s(y)\|}s(y) \oplus r(x) & \text{if } \|s(y)\| \geq \|x\| > 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

We claim that R is a retraction onto M. If $y \oplus x \in M$ then clearly $s(y) = y, r(x) = x, \|s(y)\| = \|r(x)\|$ and so $R(y \oplus x) = y \oplus x$. Next, observe that $R(y \oplus x) \in M$ holds for every $y \oplus x \in \mathcal{M}$. Indeed, regardless of the case in the definition of R, we see that the first summand of $R(y \oplus x)$ is a non-negative integer multiple of some basis vector e_n in Y. In the first (and third) case it is obvious, in the second case it follows as the norm of $\frac{\|r(x)\|}{\|s(y)\|}s(y)$ is an integer $\|r(x)\|$. The second summand is the result of an application of the retraction r, and its norm equals the norm of the first summand, hence the value of $R(y \oplus x)$ indeed lies in M.

Next, we claim that R is Lipschitz. Recall that $\mathcal{M}$ is a $(1,3)$-net in a Banach space, so it suffices to prove that there exists a $K > 0$ such that $\|R(y_1 \oplus x_1) - R(y \oplus x)\| \leq K$ whenever $\|y_1 \oplus x_1 - y \oplus x\| \leq D$, for say $D = 8$. This is well-known and easy to see, as every pair of distinct points $p, q \in \mathcal{M}$ can be connected by a straight segment of length $\|p - q\|$, and a sequence of $[\|p - q\|] + 1$ points on this segment of distance (of consecutive elements) at most one. Each of these points has an approximant from $\mathcal{M}$ of distance at most 3, so it clear that there exists a sequence of $[\|p - q\|] + 1$ points in $\mathcal{M}$ of (consecutive) distance at most $D - 1 = 7$, "connecting" the points p, q, and the result follows by a simple summation of the increments of R along the mentioned sequence.

Let us start the proof of Lipschitzness of R by partitioning $\mathcal{M}$ into three disjoint subsets

$$D_1 = \{y \oplus x : \|x\| > \|s(y)\| \geq 20D\},$$

$$D_2 = \{y \oplus x : \|s(y)\| \geq \|x\| \geq 20D\},$$

$$D_3 = \{y \oplus x : \min\{\|s(y)\|, \|x\|\} < 20D\}.$$

The set D_1 (resp. D_2) corresponds to the case 1 (resp. 2) in the definition of R.

Observe that $\|R(y \oplus x)\| \leq \min\{\|y\|, \|x\|\}$ so it suffices to prove the Lipschitzness of R on the set $D_1 \cup D_2$. Moreover, the sets D_1 and D_2 have in a sense a common "boundary" (in the intuitive sense, which is not contained in D_1) consisting of those elements for which $\|x\| = \|s(y)\|$. It is easy to see that for such elements the first two cases in definition of R may be applied with the same result (although formally we are forced to apply the second case). Suppose now that $p \in D_1, q \in D_2$. A similar argument as above using the straight segment connecting p, q (and a short finite sequence from $\mathcal{M}$ which approximates this segment) we see that the segment essentially has to "cross the boundary" between D_1, D_2, and so the proof of the Lipschitzness of R will follow provided we can do it for each of the sets D_1, D_2 separately.

Suppose $y_1 = y + \tilde{y}, x_1 = x + \tilde{x}$ are such that $\|\tilde{y}\|, \|\tilde{x}\| \leq D$.

Case 1. We consider first the case $y_1 \oplus x_1, y \oplus x \in D_1$. Then

$$\frac{\|s(y_1)\|}{\|x_1\|}x_1 - \frac{\|s(y)\|}{\|x\|}x = \frac{\|s(y+\tilde{y})\|}{\|x+\tilde{x}\|}(x+\tilde{x}) - \frac{\|s(y)\|}{\|x\|}x =$$

$$\left(\frac{\|s(y+\tilde{y})\|}{\|x+\tilde{x}\|} - \frac{\|s(y)\|}{\|x\|}\right)x + \frac{\|s(y+\tilde{y})\|}{\|x+\tilde{x}\|}\tilde{x}$$

Now

$$\left|\frac{\|s(y+\tilde{y})\|}{\|x+\tilde{x}\|} - \frac{\|s(y)\|}{\|x\|}\right| \leq \max\left\{\frac{\|s(y)\|+9D}{\|x\|-D} - \frac{\|s(y)\|}{\|x\|}, \frac{\|s(y)\|}{\|x\|} - \frac{\|s(y)\|-9D}{\|x\|+D}\right\}$$

$$\frac{\|s(y)\|+9D}{\|x\|-D} - \frac{\|s(y)\|}{\|x\|} = \frac{(\|s(y)\|+9D)\|x\| - \|s(y)\|(\|x\|-D)}{(\|x\|-D)\|x\|}$$

$$= \frac{9D\|x\| + D\|s(y)\|}{(\|x\|-D)\|x\|} \leq \frac{10D\|x\|}{(\|x\|-D)\|x\|} \leq \frac{10D}{\frac{9}{10}\|x\|} \leq \frac{100D}{9\|x\|} \leq \frac{12D}{\|x\|}$$

Similarly, we obtain

$$\frac{\|s(y)\|}{\|x\|} - \frac{\|s(y)\|-9D}{\|x\|+D} \leq \frac{10D}{\|x\|}.$$

Hence we obtain

$$\left|\frac{\|s(y+\tilde{y})\|}{\|x+\tilde{x}\|} - \frac{\|s(y)\|}{\|x\|}\right| \leq \frac{12D}{\|x\|}.$$

The last term is also estimated similarly:

$$\frac{\|s(y+\tilde{y})\|}{\|x+\tilde{x}\|}\|\tilde{x}\| \leq \frac{\|s(y)\|+9D}{\|x\|-D}D \leq \frac{\|x\|+9D}{\|x\|-D}D \leq 3D$$

So combining the above computations we get

$$\left\| \frac{\|s(y_1)\|}{\|x_1\|} x_1 - \frac{\|s(y)\|}{\|x\|} x \right\| \leq 15D$$

So the mapping $y \oplus x \to \frac{\|s(y)\|}{\|x\|} x$ from D_1 to M takes vectors of distance at most D to vectors of distance at most $15D$. It is now clear that R is Lipschitz on D_1.

Case 2. We consider now $y_1 \oplus x_1, y \oplus x \in D_2$, and denote $z = s(y + \tilde{y}) - s(y)$ (recall that $\|z\| \leq 9D$):

$$\frac{\|r(x_1)\|}{\|s(y_1)\|} s(y_1) - \frac{\|r(x)\|}{\|s(y)\|} s(y) = \frac{\|r(x + \tilde{x})\|}{\|s(y + \tilde{y})\|} s(y + \tilde{y}) - \frac{\|r(x)\|}{\|s(y)\|} s(y)$$

Therefore

$$\left\| \frac{\|r(x + \tilde{x})\|}{\|s(y + \tilde{y})\|} s(y + \tilde{y}) - \frac{\|r(x)\|}{\|s(y)\|} s(y) \right\|$$

$$\leq \max \left\{ \left\| \frac{\|r(x)\| + 9D}{\|s(y)\| - 9D} (s(y) + z) - \frac{\|r(x)\|}{\|s(y)\|} s(y) \right\|, \right.$$

$$\left. \left\| \frac{\|r(x)\| - 9D}{\|s(y)\| + 9D} (s(y) + z) - \frac{\|r(x)\|}{\|s(y)\|} s(y) \right\| \right\}$$

The first term could be rewritten and estimated as follows:

$$\left\| \frac{(\|r(x)\| + 9D)\|s(y)\|}{(\|s(y)\| - 9D)\|s(y)\|} (s(y) + z) - \frac{(\|s(y)\| - 9D)\|r(x)\|}{(\|s(y)\| - 9D)\|s(y)\|} s(y) \right\|$$

$$\leq \left\| \left(\frac{(\|r(x)\| + 9D)\|s(y)\|}{(\|s(y)\| - 9D)\|s(y)\|} - \frac{(\|s(y)\| - 9D)\|r(x)\|}{(\|s(y)\| - 9D)\|s(y)\|} \right) s(y) \right\| + \frac{\|r(x)\| + 9D}{\|s(y)\| - 9D} 9D$$

$$\leq \left\| \left(\frac{(\|r(x)\| + 9D)\|s(y)\| - (\|s(y)\| - 9D)\|r(x)\|}{(\|s(y)\| - 9D)\|s(y)\|} \right) s(y) \right\| + 27D$$

$$\leq \left| \frac{(\|r(x)\| + 9D)\|s(y)\| - (\|s(y)\| - 9D)\|r(x)\|}{\|s(y)\| - 9D} \right| + 27D$$

$$\leq \left| \frac{9D\|s(y)\| + 9D\|r(x)\|}{\|s(y)\| - 9D} \right| + 27D \leq \frac{18D\|s(y)\|}{\|s(y)\| - 9D} + 27D \leq 63D.$$

The second term we estimate analogously

$$\left\| \frac{\|r(x)\| - 9D}{\|s(y)\| + 9D} (s(y) + z) - \frac{\|r(x)\|}{\|s(y)\|} s(y) \right\|$$

$$\leq \left\| \left(\frac{(\|r(x)\| - 9D)\|s(y)\|}{(\|s(y)\| + 9D)\|s(y)\|} - \frac{(\|s(y)\| + 9D)\|r(x)\|}{(\|s(y)\| + 9D)\|s(y)\|} \right) s(y) \right\| + \frac{\|r(x)\| - 9D}{\|s(y)\| + 9D} 9D$$

$$\leq \left| \frac{9D\|s(y)\| + 9D\|r(x)\|}{\|s(y)\| + 9D} \right| + 9D \leq \frac{18D\|s(y)\|}{\|s(y)\|} + 9D \leq 27D.$$

We conclude that R is Lipschitz on the whole domain $\mathcal{M}$. Hence $\mathcal{F}(M)$ is isomorphic to a complemented subspace of $\mathcal{F}(\mathcal{M}) \cong \mathcal{F}(\mathcal{N})$. ∎

A simple situation which fits the above assumptions is when X contains a complemented subspace with a symmetric basis (e.g. ℓ_p, c_0 or an Orlicz sequence space). By the standard structural theorems for classical Banach spaces ([27]) we obtain.

Corollary 9. *Let X be a Banach space isomorphic to any of the (classical) spaces ℓ_p, L_p, $1 \le p < \infty$, $C(K)$, or an Orlicz space h_M, $\mathcal{N}$ be a net in X. Then*

$$\mathcal{F}(\mathcal{N}) \cong (\oplus_{j=1}^{\infty} F(\mathcal{N}))_{\ell_1}.$$

Recall that a metric space M is an absolute Lipschitz retract if, for some $K > 0$, M is a K-Lipschitz retract of every metric superspace $M \subset N$ ([1] p.13). We are going to use the discretized form of this condition. This concept is almost explicit in the work of Kalton [21], where it would probably be called absolute coarse retract.

Definition 3. *Let M be a δ-uniformly discrete space, $\delta > 0$. We say that M is an absolute uniformly discrete Lipschitz retract if, for some $K > 0$, the space M is a K-Lipschitz retract of every δ-uniformly discrete superspace $M \subset N$.*

Lemma 10. *Let X be Banach space which is an absolute Lipschitz retract, $\mathcal{N}$ be a net in X. Then $\mathcal{N}$ is absolute uniformly discrete Lipschitz retract. Conversely, if $\mathcal{N}$ is absolute uniformly discrete Lipschitz retract and X is a Lipschitz retract of X^{**} then X is an absolute Lipschitz retract.*

Proof. The first implication is obvious. To prove the second one, suppose that $X \subset \ell_\infty(\Gamma) = Y$ is a linear embedding. Since $\ell_\infty(\Gamma)$ is an injective space, it suffices to prove that there is a Lipschitz retraction from $\ell_\infty(\Gamma)$ onto X. Since X is a Lipschitz retract of X^{**}, it suffices to follow verbatim the proof of Theorem 1 in [25]. Indeed, consider a net $\mathcal{N}$ in X with extension into a net $\mathcal{M}$ in Y. By assumption, there exists a Lipschitz retraction $r : \mathcal{M} \to \mathcal{N}$. This retraction r can be easily extended to a coarsely continuous retraction R from Y onto X (using the terminology of [21]), which is of course Lipschitz for large distances. It is this condition on R that is used in the proof of Theorem 1 in [25]. ∎

Remark. It is an open problem if the retraction from X^{**} to X exists for every separable Banach space (see [21]).

Important examples of absolute uniformly discrete Lipschitz retract are the nets in $C(K)$ spaces, K metrizable compact, [1] p.15.

Corollary 11. *Let $\mathcal{M}$ be a countable absolute uniformly discrete Lipschitz retract which contains a bi-Lipschitz copy of the net $\mathcal{N}$ in c_0. Then $\mathcal{F}(\mathcal{M}) \cong \mathcal{F}(\mathcal{N})$.*

Proof. There is a Lipschitz retraction from $\mathcal{M}$ onto $\mathcal{N}$, and on the other hand using Aharoni's theorem ([11] p. 546) $\mathcal{M}$ is bi-Lipschitz embedded into $\mathcal{N}$ (and hence also a retract). Thus $\mathcal{F}(\mathcal{M})$ is complemented in $\mathcal{F}(\mathcal{N})$ and vice versa. To finish, apply Theorem 8 for c_0 together with the Pelczynski decomposition principle. ∎

To give concrete applications of the above corollary, we obtain the following result. The case of c_0^+ follows from the Pelant c_0^+-version of Aharoni's result [30].

Theorem 12. *Let $\mathcal{N}$ be a net in c_0 and $\mathcal{M}$ be a net in any of the following metric spaces: $C(K)$, K infinite metrizable compact, or c_0^+ (the subset of c_0 consisting of elements with non-negative coordinates). Then $\mathcal{F}(\mathcal{M}) \cong \mathcal{F}(\mathcal{N})$.*

3 Schauder basis

Theorem 13. *Let X be a metric space. Suppose there exist a set $M \subseteq X$ and a sequence of distinct points $\{\mu_n\}_{n=1}^\infty \subseteq M$, together with a sequence of retractions $\{\varphi_n\}_{n=1}^\infty$, $\varphi_n : M \to M$, $n \in \mathbb{N}$, which satisfy the following conditions:*

(i) $\varphi_n(M) = M_n := \bigcup_{j=1}^n \{\mu_j\}$ for every $n \in \mathbb{N}$,

(ii) $\overline{\bigcup_{j=1}^\infty \{\mu_j\}}^X = M$,

(iii) There exists $K > 0$ such that φ_n is K-Lipschitz for every $n \in \mathbb{N}$,

(iv) $\varphi_m \varphi_n = \varphi_n \varphi_m = \varphi_n$ for every $m, n \in \mathbb{N}$, $n \le m$.

Then the space $\mathcal{F}(M)$ has a Schauder basis with the basis constant at most K.

Proof. It is a well-known fact that every Lipschitz mapping $L : A \to B$ between pointed metric spaces A, B, such that $L(0) = 0$ extends uniquely to a linear mapping
$\widehat{L} : \mathcal{F}(A) \to \mathcal{F}(B)$ in a way that that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{F}(A) & \xrightarrow{\widehat{L}} & \mathcal{F}(B) \\
\delta_A \uparrow & & \uparrow \delta_B \\
A & \xrightarrow{\;\;L\;\;} & B
\end{array}
$$

Moreover, the norm of $\widehat{L}$ is at most $\mathrm{Lip}(L)$. Therefore for every $n \in \mathbb{N}$ there is a linear mapping $P_n = \widehat{\varphi_n} : \mathcal{F}(M) \to \mathcal{F}(M)$ extending $\varphi_n : M \to M$ with $\|P_n\| \le K$. We want to prove that $\{P_n\}$ is a sequence of canonical projections associated with some Schauder basis of $\mathcal{F}(M)$, namely that

a) $\dim P_n(\mathcal{F}(M)) = n - 1$ for every $n \in \mathbb{N}$,

b) $P_n P_m = P_m P_n = P_m$ for all $m, n \in \mathbb{N}$, $m \le n$,

c) $\lim_n P_n(x) = x$ for all $x \in \mathcal{F}(M)$.

The first condition is easy: as $\varphi_n(M) = M_n = \{\mu_i\}_{i=1}^n$ we have $P_n(\mathcal{F}(M)) = \mathcal{F}(M_n)$, which is a $(n-1)$-dimensional space. Let us check the commutativity. Note first that for $m, n \in \mathbb{N}$ the diagram

$$
\begin{array}{ccccc}
\mathcal{F}(M) & \xrightarrow{P_m} & \mathcal{F}(M) & \xrightarrow{P_n} & \mathcal{F}(M) \\
\delta_M \uparrow & & \uparrow \delta_M & & \uparrow \delta_M \\
M & \xrightarrow{\varphi_m} & M & \xrightarrow{\varphi_n} & M
\end{array}
$$

commutes, which means that $\widehat{\varphi_n \circ \varphi_m} = P_n P_m$. But then from the condition iv follows $P_n P_m = P_m P_n = P_m$ for $m \le n$.

The validity of the limit equation is proved easily. Note that elements of the form $\sum_{i=1}^m \alpha_i \delta_{x_i}$, where $m \in \mathbb{N}$, $x_i \in \{\mu_n\}_{n=1}^\infty$, $\alpha_i \in \mathbb{R}$ for all $i \in \{1, \cdots, m\}$, are norm dense in $\mathcal{F}(M)$. Indeed, it is a well-known fact that elements $\mu \in \mathcal{F}(M)$ of the same form $\sum_{i=1}^m \alpha_i \delta_{x_i}$ with $x_i \in M$ are norm dense in $\mathcal{F}(M)$ and the condition ii gives the more general result. By uniform boundedness of the family $\{P_n\}_{n=1}^\infty$, it suffices to check the limit for elements mentioned above. Thus pick a measure $\mu = \sum_{i=1}^m \alpha_i \delta_{x_i}$, $m \in \mathbb{N}$, $a_i \in \mathbb{R}$, $x_i \in \{\mu_j\}_{j=1}^\infty$ for all $i \in \{1, ..., m\}$. Find $k \in \mathbb{N}$ such that $\{x_1, \cdots, x_m\} \subseteq M_k$. Then for all $n \ge k$ we have

$$
\begin{aligned}
\|P_n \mu - \mu\| &= \sup_{\|f\| \le 1} \left| \left\langle f, \sum_{i=1}^m \alpha_i (\delta_{\varphi_n(x_i)} - \delta_{x_i}) \right\rangle \right| \\
&= \sup_{\|f\| \le 1} \left| \sum_{i=1}^m (\alpha_i f(\varphi_n(x_i)) - \alpha_i f(x_i)) \right| \\
&= \sup_{\|f\| \le 1} \left| \sum_{i=1}^m (\alpha_i f(x_i) - \alpha_i f(x_i)) \right| = 0.
\end{aligned}
$$

This was to prove. ∎

Definition 4. *Let X be a Banach space with a Schauder basis $E = \{e_i\}_{i=1}^\infty$. The set $M(E) = \{x \in X| \ x = \sum_{i=1}^\infty x_i e_i, \ x_i \in \mathbb{Z}, i \in \mathbb{N}\}$ we call the integer-grid to the basis E. If it is clear what basis we are working with, we will denote the set M and speak simply about a grid.*

It is not difficult to see that if a basis E is normalized, then the grid $M(E)$ is a $\frac{1}{2bc(E)}$-separated set, where $bc(E)$ denotes the basis constant of E. For E an unconditional basis we will denote $uc(E)$ the unconditional constant of E. We will now show that for a normalized, unconditional basis E the space $\mathcal{F}(M)$ has a Schauder basis.

Lemma 14. *Let X be a Banach space with a normalized, unconditional Schauder basis $E = \{e_i\}_{i \in \mathbb{N}}$ and a grid $M(E) = M$. Then there exists a sequence of retractions $\varphi_n : M \to M$ together with a sequence of distinct points $\mu_n \in M$, $n \in \mathbb{N}$ satisfying the conditions from the Theorem 13 with the constant at most $K = uc(E) + 2bc(E)$.*

Proof. Before we define the retractions $\{\varphi_n\}_{n=1}^\infty$ and the points $\{\mu_n\}_{n=1}^\infty$ rigorously, let us give the reader some geometric idea of how will the retractions look like. We will add points from M so that first the set $C_1^1 = \{x_1 e_1| \ |x_1| \le 1\}$ is created, then the set $C_1^2 = \{x_1 e_1 + x_2 e_2| \ |x_i| \le 1, i = 1, 2\}$, then the set $C_2^2 = \{x_1 e_1 + x_2 e_2| \ |x_i| \le 2, i = 1, 2\}$, then $C_2^3 = \left\{ \sum_{i=1}^3 x_i e_i| \ |x_i| \le 2, i = 1, 2, 3 \right\}$ and so on. Note that coordinates of each $\mu \in C_i^j$ are entire numbers.

The retractions will cut coordinates of the argument so that if $x = \sum_{i=1}^\infty x_i e_i \in M$ and $\{\mu_i\}_{i=1}^n = M_n = \varphi_n(M)$, $n \in \mathbb{N}$, then $\varphi_n(x)$ is obtained by following algorithm: Choose all $\mu_i \in M_n$ minimizing the value $|x_1 - (\mu_i)_1|$, out of them choose those μ_{i_j} minimizing $|x_2 - (\mu_{i_j})_2|$ and so on. Note the process will stop eventually

because $x = \sum_{i=1}^{k} x_i e_i$ for some $k \in \mathbb{N}$ as $x \in M$ and the basis E is normalized. It will be a matter of choosing (ordering) the points $\{\mu_i\}_{i=1}^{\infty}$ so that the process ends with only one point $\mu_i = \varphi_n(x)$.

We are now going to describe the construction of the sequence φ_n in the following way. We will build the sequence of points μ_n and to each $n \in \mathbb{N}$, we associate the sets $\varphi_n^{-1}(\mu_i), i \in \{1, \cdots, n\}$. As we want the image $\varphi_n(M) = M_n = \bigcup_{i=1}^{n} \{\mu_i\}$, the only things needed for the mapping φ_n to be well-defined is to check $\bigcup_{i=1}^{n} \{\varphi_n^{-1}(\mu_i)\} = M$ and $\varphi_n^{-1}(\mu_i) \cap \varphi_n^{-1}(\mu_j) = \varnothing$ for $i \neq j$. For simplicity, we denote the set-valued mapping $\varphi_n^{-1} = F_n$ and we will define the mappings $\varphi_n, n \in \mathbb{N}$ through defining $F_n : M_n \rightarrow 2^M$. Note that if for every $i \in \{1, ..., n\}$ holds $\mu_i \in F_n(\mu_i)$, then the mapping φ_n is a retraction.

In the sequel, by the n-tuple $(a_1, a_2, ...a_n)$, $a_i \in \mathbb{R}$ we will mean the linear combination $\sum_{i=1}^{n} a_i e_i$ and for a point $x \in X$, $x = \sum_{i=1}^{\infty} x_i e_i$ we will always identify x with $(x_1, x_2, x_3, ...)$.

Set

$$\mu_1 = 0 \qquad\qquad F_1(\mu_1) = M,$$

$$\mu_2 = (1,0) \qquad\quad F_2(\mu_2) = \{x \in M | \ x_1 \geq 1\}$$
$$\qquad\qquad\qquad\qquad F_2(\mu_1) = M \setminus F_2(\mu_2),$$

$$\mu_3 = (-1,0) \qquad F_3(\mu_3) = \{x \in M | \ x_1 \leq -1\}$$
$$\qquad\qquad\qquad\qquad F_3(\mu_1) = F_2(\mu_1) \setminus F_3(\mu_3)$$
$$\qquad\qquad\qquad\qquad F_3(\mu_2) = F_2(\mu_2).$$

It is not difficult to see $\varphi_1, \varphi_2, \varphi_3$ are retractions satisfying the conditions i,iii,iv from the Theorem 13 with Lipschitz constant which equals to $\mathrm{uc}(E) \leq K$. Indeed, for φ_1 it is clear as its image is only $\{0\}$. For φ_2, $x, y \in M$ and $i \in \mathbb{N}$ we have

$$|\varphi_2(x)_i - \varphi_2(y)_i| = \begin{cases} 0 & i > 1 \vee (x_1 \geq 1 \vee y_1 \geq 1) \vee (x_1 \leq 0 \vee y_1 \leq 0), \\ 1 & i = 1 \wedge ((x_1 \geq 1 \wedge y_1 \leq 0) \vee (y_1 \geq 1 \wedge x_1 \leq 0)), \end{cases} \qquad (9)$$

and similarily for $n = 3$, $x \in M$ and $i \in \mathbb{N}$ we have

$$\varphi_3(x)_i = \begin{cases} 0 & i > 1 \vee x_1 = 0, \\ 1 & i = 1 \wedge x_1 \geq 1, \\ -1 & i = 1 \wedge x_1 \leq -1 \end{cases}$$

and therefore for $x, y \in M$

$$|\varphi_3(x)_i - \varphi_3(y)_i| = \begin{cases} 0 & i > 1 \vee x_1 y_1 \geq 1 \vee x_1 = y_1 = 0, \\ 1 & i = 1 \wedge ((|x_1| \geq 1 \wedge y_1 = 0) \vee (|y_1| \geq 1 \wedge x_1 = 0)), \\ 2 & i = 1 \wedge x_1 y_1 \leq -1. \end{cases} \qquad (10)$$

Due to the unconditionality of E, it is true that for every $x \in X$ and $z \in \mathbb{R}$, $|z| \leq x_1$ holds $\|(z, x_2, x_3, x_4, ...)\| \leq \mathrm{uc}(E)\|x\|$. But for every $i \in \mathbb{N}$ the expression in (10) is

less or equal to $|x_i - y_i|$, which gives us Lipschitz condition on φ_n with constant $\mathrm{uc}(E)$.

Moreover, the last retraction φ_3 maps M onto the set $C_1^1 \subseteq M$ containing all points $x \in M$ with $x = (x_1)$ and $|x_1| \leq 1$. Let us denote $C_r^d = \{x \in M \mid x = (x_1, x_2, ..., x_d), |x_i| \leq r, \ i \leq d\}$. From now on, we will proceed inductively. Suppose we have a sequence of retractions $\{\varphi_i\}_{i=1}^m$ together with the points μ_i, such that $\varphi_m(M) = C_r^r$ and that $\{\varphi_i\}_{i=1}^m$ satisfy the conditions i,iii,iv from the Theorem 13. Note that $m = (2r+1)^r$.

We proceed by induction which we divide into two steps. First we find points $\mu_{m+1}, ..., \mu_s$ together with retractions $\varphi_{m+1}, ..., \varphi_s$, where $s = (2r+1)^{r+1}$, such that $M_s = C_r^{r+1}$ and such that $\{\varphi_i\}_{i=1}^s$ satisfy the conditions i,iii,iv from theorem 13. Then we find points $\mu_{s+1}, ..., \mu_t$ and retractions $\varphi_{s+1}, ..., \varphi_t$, where $t = (2r+3)^{r+1}$, $\varphi_t : M \to C_{r+1}^{r+1}$ which satisfy i,iii,iv. As $\bigcup_{r=1}^\infty C_r^r = M$, the condition ii from theorem 13 is obtained as well, which will conclude the proof.

On the bounded set C_r^r we define an ordering by the formula

$$(x_1, x_2, ..., x_r) \prec (y_1, y_2, ..., y_r) \Leftrightarrow (x_1 > y_1) \vee \atop \exists i \in \{1, ..., r-1\} \, \forall j \in \{1, ..., i\} : (x_j = y_j) \wedge (x_{i+1} > y_{i+1}). \tag{11}$$

There exists a bijection $w : \{1, ..., (2r+1)^r\} \to C_r^r$, which preserves order.

Let us shorten the notation by introducing indexing functions a, b. If $j \in \{1, ..., r\}$ and $i \in \{1, ..., (2r+1)^r\}$, let $a(j, i) = j(2r+1)^r + i$ and $b(j, i) = (r+j)(2r+1)^r + i$. We set $\mu_{a(j,i)} = (w(i), j) = w(i) + je_{r+1}$ and $\mu_{b(j,i)} = (w(i), -j) = w(i) - je_{r+1}$. Moreover, we formally put $\mu_{a(0,i)} = \mu_{b(0,i)} = w(i)$. Then we define sets

$$F_{a(j,i)}(\mu_{a(j,i)}) = \left\{ x \in F_{a(j,i)-1}(\mu_{a(j-1,i)}), x_{r+1} \geq j \right\},$$
$$F_{a(j,i)}(\mu_{a(j-1,i)}) = F_{a(j,i)-1}(\mu_{a(j-1,i)}) \setminus F_{a(j,i)}(\mu_{a(j,i)}),$$
$$F_{a(j,i)}(\mu_q) = F_{a(j,i)-1}(\mu_q), \ q \in \{1, ..., a(j,i)-1\}, \mu_q \neq \mu_{a(j-1,i)},$$

and

$$F_{b(j,i)}(\mu_{b(j,i)}) = \left\{ x \in F_{b(j,i)-1}(\mu_{b(j-1,i)}), x_{r+1} \leq -j \right\},$$
$$F_{b(j,i)}(\mu_{b(j-1,i)}) = F_{b(j,i)-1}(\mu_{b(j-1,i)}) \setminus F_{b(j,i)}(\mu_{b(j,i)}),$$
$$F_{b(j,i)}(\mu_q) = F_{b(j,i)-1}(\mu_q), \ q \in \{1, ..., b(j,i)-1\}, \mu_q \neq \mu_{b(j-1,i)}.$$

It is easy to see that the formulae above define mappings $\varphi_{a(j,i)}$ and $\varphi_{b(j,i)}$. Supposed it holds for the mappings $\{\varphi_i\}_{i=1}^m$ it is clear that $F_n(\mu_p) \cap F_n(\mu_q) = \varnothing$ for $p \neq q$ and all $n \in \{1, ..., s\}$, and that $\mu_n \in F_n(\mu_n)$ and $\bigcup_{i=1}^n F_n(\mu_i) = M$, which means each mapping φ_n is well-defined and is a retraction onto M_n.

Let us check the uniform Lipschitz boundedness. Fix $n \in \{m+1, ..., s\}$. Note first that

$$\forall x = \sum_{i=1}^\infty x_i e_i \in X, \forall z \in \ell_\infty :$$

$$\forall i \in \mathbb{N} : 0 \leq |z_i| \leq |x_i| \Rightarrow \left\| \sum_{i=1}^\infty z_i e_i \right\| \leq \|x\| \cdot \mathrm{uc}(E)$$

From this we deduce the Lipschitz boundedness.

If $x, y \in M$, then for $i > r + 1$ we have $|\varphi_n(x)_i - \varphi_n(y)_i| = |0 - 0| = 0 \leq |x_i - y_i|$. If $i < r + 1$ then we distinguish three cases:

a) $|x_i|, |y_i| \leq r$. Then $\varphi_n(x)_i = x_i$, $\varphi_n(y)_i = y_i$ and therefore $|\varphi_n(x)_i - \varphi_n(y)_i| = |x_i - y_i|$.

b) $|x_i| \leq r$, $|y_i| > r$. Then $\varphi_n(x)_i = x_i$ and $\varphi_n(y)_i = r\,\mathrm{sgn}(y_i)$. Therefore $|\varphi_n(x)_i - \varphi_n(y)_i| = |x_i - r\,\mathrm{sgn}(y_i)| \leq |x_i - y_i|$.

c) $|x_i|, |y_i| > r$. Then $\varphi_n(x)_i = r\,\mathrm{sgn}(x_i)$, $\varphi_n(y)_i = r\,\mathrm{sgn}(y_i)$ and therefore

$$|\varphi_n(x)_i - \varphi_n(y)_i| = |r\,\mathrm{sgn}(x_i) - r\,\mathrm{sgn}(y_i)| = \begin{cases} 0 \leq |x_i - y_i|, & x_i y_i > 0, \\ 2r \leq |x_i - y_i|, & x_i y_i < 0. \end{cases}$$

Finally, let $i = r + 1$. If now $x_i y_i < 0$, then either $0 \leq \varphi_n(x)_i \leq x_i$ and $y_i \leq \varphi_n(y)_i \leq 0$ or vice versa. Both options give $|\varphi_n(x)_i - \varphi_n(y)_i| \leq |x_i - y_i|$, which is what we need.

Let $x_i, y_i \geq 0$. Suppose $n = a(j, k)$ for eligible j, k. Then $\varphi_n(x)_i = j$ or $\varphi_n(x)_i = j - 1$ or $\varphi_n(x)_i = x_i$, which occurs whenever $0 \leq x_i < j - 1$. Of course the same holds for y. From this we have either $|\varphi_n(x)_i - \varphi(y)_i| \leq |x_i - y_i|$ or $|\varphi_n(x)_i - \varphi(y)_i| \leq 1$. If $n = b(j, k)$ for some j, k, then $\varphi_n(x)_i = r$ whenever $x_i \geq r$ and $\varphi_n(x)_i = x_i$ whenever $x_i < r$, the same for y. It is clear that $|\varphi_n(x)_i - \varphi(y)_i| \leq |x_i - y_i|$.

Let $x_i, y_i \leq 0$. If $n = a(j, k)$ for some j, k, then $|\varphi_n(x)_i - \varphi(y)_i| = |0 - 0| = 0 \leq |x_i - y_i|$. If $n = b(j, k)$ for some j, k, then $\varphi_n(x)_i = -j$ or $\varphi_n(x)_i = -j + 1$ or $\varphi_n(x)_i = x_i$, which holds whenever $0 \geq x_i > -j + 1$. Again, we get either $|\varphi_n(x)_i - \varphi(y)_i| \leq |x_i - y_i|$ or $|\varphi_n(x)_i - \varphi(y)_i| \leq 1$.

To sum up all cases, if $x, y \in M$, then either $x_{r+1} = y_{r+1}$ or not. In the first case we have

$$\|\varphi_n(x) - \varphi_n(y)\| = \left\| \sum_{i=1}^{r+1} (\varphi_n(x)_i - \varphi_n(y)_i)\, e_i \right\| \leq \left\| \sum_{i=1}^{r} (\varphi_n(x)_i - \varphi_n(y)_i)\, e_i \right\| + 1$$
$$\leq \mathrm{uc}(E)\|x - y\| + 2\mathrm{bc}(E)\|x - y\| =$$
$$= \|x - y\|(\mathrm{uc}(E) + 2\mathrm{bc}(E)), \tag{12}$$

as M is a $\frac{1}{2\mathrm{bc}(E)}$-separated set, while in the $x_{r+1} \neq y_{r+1}$ case we have

$$\|\varphi_n(x) - \varphi_n(y)\| = \left\| \sum_{i=1}^{r+1} (\varphi_n(x)_i - \varphi_n(y)_i)\, e_i \right\| \leq \mathrm{uc}(E)\|x - y\|. \tag{13}$$

Considering both cases we get the mapping φ_n is Lipschitz with constant $K = \mathrm{uc}(E) + 2\mathrm{bc}(E)$.

It remains to prove that the mappings $\{\varphi_n\}_{n=1}^{s}$ satisfy the commutativity condition iv, provided the mappings $\{\varphi_n\}_{n=1}^{m}$ do. Note that for any $m, n \in \mathbb{N}$, $m \leq n$ holds

$$F_n(\mu_n) \cap F_m(\mu_m) \in \{\emptyset, F_n(\mu_n)\}. \tag{14}$$

Out of this fact the commutativity follows easily: Consider $i < j \in \{1,...,s\}$. First, because φ_i is a retraction onto M_i and the same holds for φ_j and M_j, from $M_i \subseteq M_j$ follows $\varphi_j\varphi_i = \varphi_i$. It remains to prove $\varphi_i\varphi_j(x) = \varphi_i(x)$ for every $x \in M$.

Take $x \in M$. There exists a maximal finite sequence of indices $1 = k_0 < ... < k_l \leq s$ such that

$$x \in F_{k_l}(\mu_{k_l}) \subseteq \cdots \subseteq F_{k_0}(\mu_{k_0}).$$

Clearly if $c(i)$ is the biggest index such that $k_{c(i)} \leq i$, then $\varphi_i(F_{k_d}(\mu_{k_d})) = \mu_{c(i)}$ for all d, $c(i) \leq d \leq l$. This applies analogously for φ_j with $c(j)$. From the fact that both $x, \mu_{k_{c(j)}} \in F_{k_{c(j)}}(\mu_{k_{c(j)}}) \subseteq F_{k_{c(i)}}(\mu_{k_{c(i)}})$ we get simply

$$\varphi_i\varphi_j(x) = \varphi_i(\mu_{k_{c(j)}}) = \mu_{k_{c(i)}} = \varphi_i(x),$$

which finishes the proof of commutativity.

To finish the proof, it remains to show the construction of retractions $\varphi_{s+1}, ..., \varphi_t$, where $t = (2r+3)^{r+1}$, $\varphi_t : M \to C_{r+1}^{r+1}$ which satisfy i,iii,iv.

For $i \in \mathbb{N}$ let us define an *i-predecessor* function $p_i : M \to M$ by

$$p_i\left(\sum_{n=1}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} x_n e_n - \text{sgn}(x_i)e_i.$$

Now for every $j \in \{1,...,r+1\}$ we introduce sets

$$A_{j,1} = \big\{(x_1,...,x_{j-1}, r+1, x_{j+1},...,x_{r+1}) :$$
$$x_i \in \mathbb{Z} \wedge |x_i| \leq r+1 \text{ for } i < j \wedge |x_i| \leq r \text{ for } i > j\big\},$$
$$A_{j,-1} = \big\{(x_1,...,x_{j-1}, -r-1, x_{j+1},...,x_{r+1}) :$$
$$x_i \in \mathbb{Z} \wedge |x_i| \leq r+1 \text{ for } i < j \wedge |x_i| \leq r \text{ for } i > j\big\}.$$

Clearly, $A_{j,-1}, A_{j,1} \subseteq C_{r+1}^{r+1}$ and $|A_{j,-1}| = |A_{j,1}| = (2r+1)^{r+1-j}(2r+3)^{j-1}$. Moreover,

$$\bigcup_{\substack{j \in \{1,...,r+1\} \\ i \in \{-1,1\}}} A_{j,i} = C_{r+1}^{r+1} \setminus C_r^{r+1}$$

and it is a disjoint union. For each j, choose any bijection $w_j : \{1,...,|A_{j,1}|\} \to A_{j,1}$ and fix it. Define $\overline{w}_j : \{1,...,|A_{j,1}|\} \to A_{j,-1}$, by $\overline{w}_j(i) = (w_j(i)_1, w_j(i)_2, ..., -w_j(i)_j, ..., w_j(i)_{r+1})$. For simplicity, for $j \in \{1,...,r+1\}, i \in \{1,...,|A_{j,1}|\}$ put

$$\alpha(j,i) = s + 2\sum_{k=1}^{j-1} |A_{k,1}| + i, \quad \beta(j,i) = s + 2\sum_{k=1}^{j-1} |A_{k,1}| + |A_{j,1}| + i.$$

Then we finally set $\mu_{\alpha(j,i)} = w_j(i)$, $\mu_{\beta(j,i)} = \overline{w}_j(i)$. Now we define mappings $\{F_n\}_{n=s+1}^{t}$ via

$$F_{\alpha(j,i)}(\mu_{\alpha(j,i)}) = \Big\{x \in F_{\alpha(j,i)-1}(p_j(\mu_{\alpha(j,i)})), x_j \geq r+1\Big\},$$
$$F_{\alpha(j,i)}(p_j(\mu_{\alpha(j,i)})) = F_{\alpha(j,i)-1}(p_j(\mu_{\alpha(j,i)})) \setminus F_{\alpha(j,i)}(\mu_{\alpha(j,i)}),$$
$$F_{\alpha(j,i)}(\mu_q) = F_{\alpha(j,i)-1}(\mu_q), \quad q \in \{1,...,\alpha(j,i)-1\}, \mu_q \neq p_j(\mu_{\alpha(j,i)}),$$

and

$$F_{\beta(j,i)}(\mu_{\beta(j,i)}) = \left\{ x \in F_{\beta(j,i)-1}(p_j(\mu_{\beta(j,i)})), x_j \leq -r-1 \right\},$$
$$F_{\beta(j,i)}(p_j(\mu_{\beta(j,i)})) = F_{\beta(j,i)-1}(p_j(\mu_{\beta(j,i)})) \setminus F_{\beta(j,i)}(\mu_{\beta(j,i)}),$$
$$F_{\beta(j,i)}(\mu_q) = F_{\beta(j,i)-1}(\mu_q), \quad q \in \{1, ..., \beta(j,i)-1\}, \mu_q \neq p_j(\mu_{\beta(j,i)}).$$

Obviously, the upper equations define mappings $\varphi_{\alpha(j,i)}$ and $\varphi_{\beta(j,i)}$ for all $j \in \{1, ..., r+1\}$ and $i \in \{1, ..., |A_{j,1}|\}$, hence the mappings $\{\varphi_n\}_{n=s+1}^t$ are well-defined and it is an easy check that each such φ_n is a retraction onto the set M_n.

Note that the sets $\{F_n(\mu_n)\}_{n=1}^t$ still satisfy the condition (14) so the commutativity condition iv from theorem 13 is obtained similarly as it was done for retractions $\{\varphi_n\}_{n=1}^s$.

It remains to show the mappings are Lipschitz-bounded. Let us for simplicity denote $\beta_k = \beta(k-1, |A_{k-1,1}|)$ for $1 < k \leq r+1$ and $\beta_1 = s$, the index of first retraction φ_{β_k} such that $A_{k-1,-1} \subseteq M_{\beta_k}$. Fix $n \in \{s+1, ..., t\}$. We will prove that there exists at most one $j = j(n) \in \mathbb{N}$ such that for all $l \in \mathbb{N}, l \neq j$ and all $x, y \in M$ we have $|\varphi_n(x)_l - \varphi_n(y)_l| \leq |x_l - y_l|$ out of which the Lipschitz boundedness of φ_n follows. If $n = \alpha(j,i)$ for some eligible j, i, then for every $x \in M$ holds

$$\varphi_n(x)_l = \begin{cases} 0 & l > r+1, \\ x_l & (l \leq r+1, |x_l| \leq r) \vee (l < j, |x_l| = r+1), \\ r\,\mathrm{sgn}(x_l) & (j < l \leq r+1, |x_l| > r) \vee (j = l, x_l < -r) \vee \\ & \left(j = l, x_l > r, \forall \mu \in M_n : \varphi_{\beta_j}(x)_j \neq p_j(\mu) \right), \\ (r+1)\,\mathrm{sgn}(x_l) & (l < j, |x_l| > r+1) \vee \\ & \left(l = j, x_l \geq r+1, \exists \mu \in M_n : \varphi_{\beta_j}(x)_j = p_j(\mu) \right), \end{cases}$$

while if $n = \beta(j,i)$ for some j, i, then for every $x \in M$ we have

$$\varphi_n(x)_l = \begin{cases} 0 & l > r+1, \\ x_l & (l \leq r+1, |x_l| \leq r) \vee (l < j, |x_l| = r+1) \vee \\ & (l = j, x_l = r+1), \\ r\,\mathrm{sgn}(x_l) & (j < l \leq r+1, |x_l| > r) \vee \\ & \left(j = l, x_l < -r, \forall \mu \in M_n : \varphi_{\beta_j}(x)_j \neq p_j(\mu) \right), \\ (r+1)\,\mathrm{sgn}(x_l) & (l < j, |x_l| > r+1) \vee (l = j, x_l > r+1) \vee \\ & \left(l = j, x_l \leq -r-1, \exists \mu \in M_n : \varphi_{\beta_j}(x)_j = p_j(\mu) \right). \end{cases}$$

If $x, y \in M$, it is not difficult to see that if $|\varphi_n(x)_l - \varphi_n(y)_l| > |x_l - y_l|$, then $l = j$ and $\varphi_n(x)_l = (r+1)\,\mathrm{sgn}(x_l)$, $\varphi_n(y)_l = r\,\mathrm{sgn}(y_l)$ or vice versa and $x_l y_l > 0$. Particularly $|\varphi_n(x)_l - \varphi_n(y)_l| = 1$ and $|x_l - y_l| = 0$. For all other l, i.e. $l \neq j, l \in \mathbb{N}$ holds $|\varphi_n(x)_l - \varphi_n(y)_l| \leq |x_l - y_l|$, which is what we need.

Therefore we get by computation similar to those done in (12) and (13) that φ_n is a Lipschitz mapping with constant $K = \mathrm{uc}(E) + 2\mathrm{bc}(E)$, which concludes the induction.

As $\bigcup_{r=1}^\infty C_r^r = M$ the condition ii from theorem 13 is also satisfied and hence our proof is finished. ∎

Remark. In (11) it was not necessary for our construction to choose exactly this order. In fact, any bijection $w : \{1, ..., (2r + 1)^r\} \to C_r^r$ would suit our purpose. We chose the order (11) for simplicity. In this case we have $\mu_{a(j-1,i)} = p_j(\mu_{a(j,i)})$ and $\mu_{b(j-1,i)} = p_j(\mu_{b(j,i)})$ for p_j the j-predecessor function and $i \in \{1, ..., (2r + 1)^r\}$, $j \in \{1, ..., r\}$.

Corollary 15. *If $E = \{e_i\}_{i=1}^{\infty}$ denotes the canonical basis in c_0 and $M = M(E) \subseteq c_0$ the integer grid, then the Free-space $\mathcal{F}(M)$ has a monotone Schauder basis.*

Proof. applying the construction of the retractions from the lemma 14 to (c_0, E), we get Lipschitz constant $K = 1$, (see estimates (12) and (13)). Therefore, $\mathcal{F}(M)$ has a monotone Schauder basis. ∎

Corollary 16. *Let $\mathcal{N} \subseteq c_0$ be a net. Then the Free-space $\mathcal{F}(\mathcal{N})$ has a Schauder basis.*

Proof. If we use the notation from previous corollary, M is a $(1, 1)$-net in c_0. But as all nets in an infinite-dimensional space are Lipschitz equivalent ([1], p.239, Proposition 10.22), $\mathcal{N}$ is Lipschitz equivalent to the grid M and therefore $\mathcal{F}(\mathcal{N})$ is isomorphic to $\mathcal{F}(M)$, which concludes the proof. ∎

Corollary 17. *Let $\mathcal{N}$ be a net in any of the following metric spaces: $C(K)$, K metrizable compact, or c_0^+ (the subset of c_0 consisting of elements with non-negative coordinates). Then $\mathcal{F}(\mathcal{N})$ has a Schauder basis.*

Proof. Follows immediately from Theorem 12. ∎

Corollary 18. *Let $\mathcal{N} \subseteq \mathbb{R}^n$ be a net. Then $\mathcal{F}(\mathcal{N})$ has a Schauder basis.*

Proof. It follows from the proof of lemma 14 that $\mathcal{F}(\mathbb{Z}^n)$ has a Schauder basis and $\mathcal{F}(\mathbb{Z}^n) \cong \mathcal{F}(\mathcal{N})$ by Proposition 5, which gives the result. ∎

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