Asymptotic stability of solutions to quasi-geostrophic equation

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Abstract

We show that sufficiently small mild solutions of the initial value problem to the quasi-geostrophic equation in $\mathbb{R}^2$ are asymptotically stable under arbitrary large initial L^2-perturbations. We obtain also the decay rate.

1 Introduction

The Cauchy problem of the two-dimensional dissipative quasi-geostrophic model given by Constantin [2] has the form

$$\theta_t + u_\theta \cdot \nabla\theta + \kappa(-\Delta)^{\frac{\alpha}{2}}\theta = 0, \qquad \mathbb{R}^2 \times (0, +\infty), \tag{1.1}$$

$$\theta(x, 0) = \theta_0(x), \tag{1.2}$$

where the pseudo-differential operator $\Lambda^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ with $0 < \alpha < 2$ is defined by the Fourier transformation:

$$\widehat{\Lambda^\alpha w}(\xi) = (2\pi|\xi|)^\alpha \hat{w}(\xi). \tag{1.3}$$

and $\kappa > 0$ is a dissipative coefficient. Here, a scalar function $\theta = \theta(x,t)$ representing potential temperature and the velocity field $u_\theta = u_\theta(x,t)$ are unknown. Moreover, the velocity field $\theta = \theta(x,t)$ is determined by the scalar stream function ψ through

$$u_\theta = (u_\theta^1, u_\theta^2) = \left(-\frac{\mathrm{d}}{\mathrm{d}x_2}\psi, -\frac{\mathrm{d}}{\mathrm{d}x_1}\psi\right),$$

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where ψ satisfies the equation $\Lambda\psi = -\theta$.

The quasi-geostrophic fluid is an important model in geophysical fluid dynamics, they are special cases of the general quasi-geostrophic approximations for atmospheric and oceanic fluid flow with the small local Rossby number which ensures the validity of the geostrophic balance between the pressure gradient and the Coriolis force. Furthermore, this quasi-geostrophic fluid motion equation shares many features with fundamental fluid motion equations. When $\kappa = 0$, this equation is comparable to the vorticity formulation of the Euler equations, and (1.1) with $\alpha = 0$ is similar to a non-viscous wind driven circulation equation. What is more, equation (1.1) with $\alpha = 1$ shares similar features with the three-dimensional Navier-Stokes equations. Thus, $\alpha = 1$ is therefore referred as the critical case, while the cases $0 < \alpha < 1$ and $1 < \alpha \leqslant 2$ are supercritical and subcritical, respectively.

Due to its mathematical importance and its potential for applications in meteorology and oceanography, this equation has been recently intensively investigated by many authors. They were interested in two issues: the existence of solutions and their asymptotic behavior for large times. For the subcritical case $1 < \alpha \leqslant 2$, Constantin and Wu [3] proved the existence of global in time smooth solutions for every sufficiently smooth initial data. For $\alpha = 1$, Kiselev et al. [8] proved the existence of the global smooth solution for any C^∞ periodic initial data. The issue of global existence and uniqueness of smooth solution is more difficult, when $0 < \alpha < 1$, and has still unanswered aspects although many good results on global solutions under small initial data and regularity criteria of weak solutions have been examined by many authors (see e.g. [4, 12]).

On the other hand, it is desirable to understand the asymptotic behavior of the quasi-geostrophic equation, especially for the asymptotic stability of solutions. The asymptotic stability for zero solution is well understand by the different methods (see [3, 10]). Chae and Lee studied the asymptotic stability of solutions to the critical quasi-geostrophic (1.1) in suitable Besov space, while Dong and Chen recently obtained the global stability of the critical and supercritical problem in the Serrin-type class under the large initial and external perturbations. Recently, Ren and Ma [11] described asymptotic stability of weak solutions in L^2-norm.

The aim of this paper is to obtain new, more subtle result concerning asymptotic behavior of solutions to the equation (1.1), especially solutions global-in-time. First, we recall that Carrillo and Ferreira [1] considered self-similar solutions for equation (1.1). Below, in Theorem 2.1 and Theorem 2.2 we show that those self-similar solutions are globally asymptotically stable under arbitrary large L^2-perturbation. We use ideas introduced in [6] and [7].

2 Notation and preliminaries

Let us recall Lorentz spaces, denoted as usual by $L^{p,q} = L^{p,q}(\mathbb{R})$. The decreasing rearrangement of f is the function f^* defined on $[0, \infty)$ by

$$f^*(t) = \inf\{s > 0 : d_f(s) \leqslant t\},$$

where

$$d_f(s) = |\{x \in \mathbb{R} : |f(x)| > s\}|.$$

It is easy to check that d_f and f^* are nonnegative and nonincreasing functions. Moreover, if d_f is strictly decreasing and continuous, then f^* is the inverse function of d_f and both f^* and f have the same distribution function d_f. Hence, we deduce that

$$\left(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} = \left(\int_0^\infty [f^*(t)]^p \,\mathrm{d}t\right)^{\frac{1}{p}},$$

which can be written in the form

$$\|f\|_p = \left(\frac{p}{p}\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}}.$$

The Lorentz space $L^{p,q}$ is defined as the set of all functions f such that $\|f\|^*_{p,q} < \infty$ with

$$\|f\|^*_{p,q} = \begin{cases} \left(\frac{q}{p}\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}, 0 < p < \infty, 0 < q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t), 0 < p \leqslant \infty, q = \infty. \end{cases}$$

Notice that $L^{p,\infty}$ are called the Marcinkiewicz spaces or the weak Lebesgue spaces. The quantity $\|f\|^*_{p,q}$ gives a natural topology for the Lorentz space as a topological vector space. However, the triangle inequality is not true for $\|f\|^*_{p,q}$. A natural way of metrizing the space $L^{p,q}$ is to define

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s \qquad \text{for} \quad t > 0$$

which can be computed as

$$f^{**}(t) = \sup_{|E| \geqslant t} \left\{\frac{1}{|E|}\int_E |f(x)|\,\mathrm{d}x\right\}$$

for every set E with finite measure. Hence, we define the norm $\|f\|_{p,q}$ in the following way

$$\|f\|_{p,q} = \begin{cases} \left(\frac{q}{p}\int_0^\infty [t^{\frac{1}{p}} f^{**}(t)]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}, & \text{if} \quad 1 < p < \infty, 1 \leqslant q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t), & \text{if} \quad 1 < p \leqslant \infty, q = \infty. \end{cases}$$

The spaces $L^{p,q}$ endowed with the norm are Banach spaces and

$$\|f\|^*_{p,q} \leqslant \|f\|_{p,q} \leqslant \frac{p}{p-1}\|f\|^*_{p,q}$$

holds. An alternative definition of the norm in the Marcinkiewicz spaces is

$$\|f\|_{p,\infty} = \sup\left\{|E|^{-1+\frac{1}{p}}\int_E |f(x)|\,\mathrm{d}x : E \in \mathcal{B}\right\}$$

where $\mathcal{B}$ is the collection of all Borel sets with finite, positive measure. The Lorentz spaces have the same scaling property as the Lebesgue spaces, namely for all $\delta > 0$ we have

$$\|f(\delta\cdot)\|_{p,q} = \delta^{-\frac{n}{p}}\|f\|_{p,q},$$

where $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$.

Moreover, the following inequalities hold true: *the weak Hölder inequality*:

$$\|fg\|_{r,\infty} \leqslant \|f\|_{p,\infty}\|g\|_{q,\infty} \tag{2.1}$$

for every $1 < p \leqslant \infty$ (here $L^{\infty,\infty} = L^\infty$), $0 < q < \infty$ and $0 < r < \infty$ satisfying $\frac{1}{r} = \frac{1}{q} + \frac{1}{p}$, and *the weak Young inequality*

$$\|f * g\|_{r,\infty} \leqslant C\|f\|_{p,\infty}\|g\|_{q,\infty} \tag{2.2}$$

for every $0 < p < \infty$, $0 < q < \infty$ and $0 < r < \infty$ satisfying $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

The crucial role in our reasoning below play some useful inequalities. We recall *the Stroock-Varopoulos inequality* [9]

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{2}} f(x)|f(x)|^{p-1}\,\mathrm{d}x \geqslant \frac{4(p-1)}{p^2}\int_{\mathbb{R}^n}\left((-\Delta)^{\frac{\alpha}{4}}|f(x)|^{\frac{p}{2}}\right)^2\mathrm{d}x, \tag{2.3}$$

which holds true for each $f \in L^p(\mathbb{R}^n)$ such that $(-\Delta)^{\frac{\alpha}{2}} \in L^p(\mathbb{R}^n)$. Notice that for $\alpha = 2$, namely for the usual Laplacian, we have equality in (2.3). Important for us is also *the fractional Gagliardo-Nirenberg inequality* under Lorentz spaces [5]

$$\|f\|_{p,q} \leqslant B\|(-\Delta)^{\frac{\alpha}{4}} f\|_{p_1,q_1}^{\frac{\beta}{q}}\|f\|_{p_2,q_2}^{\frac{q-\beta}{q}} \tag{2.4}$$

with

$$\frac{\beta}{q_1} + \frac{q-\beta}{q_2} = 1,$$
$$\beta\left(\frac{1}{p_1} - \frac{\alpha}{2n}\right) + \frac{q-\beta}{p_2} = \frac{q}{p}$$

and $1 \leqslant p < \infty$, $1 \leqslant p_2, q, q_1, q_2 < \infty$, $0 < \beta < q$, $0 < \alpha < 2n$, $1 < q < \frac{2n}{\alpha}$ and $1 < p_1 < \frac{2n}{\alpha}$. Moreover, we use *the fractional Sobolev inequality*

$$\|f\|_{\frac{2pn}{2n-p\alpha}} \leqslant C\|(-\Delta)^{\frac{\alpha}{4}} f\|_p \tag{2.5}$$

for $1 < p < p < \infty$ and $0 < \alpha < \frac{2n}{p}$.

It is known [1] that there exists $\varepsilon > 0$ such that for every $v_0 \in L^{\frac{2}{\alpha-1},\infty}(\mathbb{R}^2)$ homogeneous of degree $-\alpha+1$ and $\|v_0\|_{\frac{2}{\alpha-1},\infty}(\mathbb{R}^2) < \varepsilon$ the initial value problem (1.1)–(1.2) has a global-in-time solution which is self-similar, namely

$$v(x,t) = t^{\frac{1}{\alpha}-1} V\left(\frac{x}{t^{\frac{1}{\alpha}}}\right). \tag{2.6}$$

Moreover, according to [1, Thm. 3.4] and the interpolation theorem for any $r \in (\frac{2}{\alpha-1}, \infty)$ there exists $0 < \varepsilon_r < \varepsilon$ such that $V \in L^r(\mathbb{R}^n)$ with $\|V\|_r < \varepsilon_r$. Thus, we have

$$\|v(\cdot,t)\|_r = t^{\frac{2}{\alpha}(\frac{1}{r}+\frac{1}{2})}\|V\|_r \tag{2.7}$$

for $r > \frac{2}{\alpha-1}$. Furthermore, by inspection of the proof of [1, Prop. 3.15] we conclude that $\|\nabla v\|_{\frac{2}{\alpha},\infty} < \varepsilon$.

We now state the main results of this work: a type of asymptotic stability for global-in-time solutions $v = v(x,t)$ under arbitrary large $L^2_\sigma(\mathbb{R}^2)$-perturbations.

Theorem 2.1 (Existence of weak solutions). *Let $v = v(x,t)$ be a global-in-time self-similar solution to the initial value problem* (1.1)–(1.2) *in* $C_w\left([0,\infty), L^{\frac{2}{\alpha-1},\infty}(\mathbb{R}^2)\right)$ *satisfying properties* (2.6)–(2.7). *Denote $v_0 = v(\cdot,0)$ and let $w_0 \in L^2_\sigma(\mathbb{R}^2)$ be arbitrary. Then, the Cauchy problem* (1.1)–(1.2) *with the initial condition $u_0 = v_0 + w_0$ has a global-in-time solution $\theta = \theta(x,t)$ of the form $\theta(x,t) = v(x,t) + w(x,t)$, where $w = w(x,t)$ is a weak solution of the corresponding perturbed problem* (*see* (2.9)–(2.10) *below*) *satisfying*

$$w \in X_T = L^\infty\big([0,T], L^2_\sigma(\mathbb{R}^2)\big) \cap L^2\big([0,T], \dot{H}^\alpha(\mathbb{R}^2)\big) \qquad \textit{for each } \; T > 0. \tag{2.8}$$

Theorem 2.2 (Asymptotic behavior of weak solutions). *A solution $\theta = \theta(x,t)$ of problem* (1.1)–(1.2) *considered in Theorem 2.1 can be constructed in a such way to satisfy $\|w(t)\|_2 = \|\theta(t) - v(t)\|_2 \to 0$ as $t \to \infty$.*

For the proofs of Theorems 2.1 and 2.2, denote by $\theta = \theta(x,t)$ a solution of the dissipative quasi-geostrophic equation (1.1) and initial data $\theta_0 = v_0 + w_0$, where $w_0 \in L^2_\sigma(\mathbb{R}^2)$. Then, the functions $w(x,t) = \theta(x,t) - v(x,t)$ satisfy the perturbed initial value problem

$$w_t + u_w \cdot \nabla v + u_v \cdot \nabla w + \Lambda^\alpha w + u_w \cdot \nabla w = 0, \qquad \mathbb{R}^2 \times (0,+\infty), \tag{2.9}$$

$$w(x,0) = w_0(x). \tag{2.10}$$

Thus, our main goal is to construct a weak solution w of problem (2.9)–(2.10) and show its L^2-decay to zero as $t \to \infty$.

3 Existence of solutions

First, we recall a standard definition.

Definition 3.1. *A vector field $w = w(x,t)$ is called a weak solution to problem* (2.9)–(2.10) *if it belongs to the classical energy space X_T defined in* (2.8) *and if*

$$\begin{aligned}\big\langle w(t),\varphi(t)\big\rangle + \int_0^t \Big[&\big\langle(-\Delta)^{\frac{\alpha}{4}}u(\tau),(-\Delta)^{\frac{\alpha}{4}}\varphi(\tau)\big\rangle + \big\langle u_w\cdot\nabla w,\varphi\big\rangle - \big\langle(u_w\cdot\nabla)\varphi, v\big\rangle \\ &+ \big\langle(u_v\cdot\nabla)w,\varphi\big\rangle\Big]\,\mathrm{d}\tau \\ &= \big\langle w_0,\varphi(0)\big\rangle + \int_0^t \big\langle w,\varphi_\tau\big\rangle\,\mathrm{d}\tau\end{aligned} \tag{3.1}$$

for all $0 \geqslant t$ and all $\varphi \in C\big([0,\infty), H^1_\sigma(\mathbb{R}^2)\big) \cap C^1\big([0,\infty), L^2_\sigma(\mathbb{R}^2)\big)$, where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2_\sigma(\mathbb{R}^2)$.

Theorem 3.2. *Let $w_0 \in L^2_\sigma(\mathbb{R}^2)$ be arbitrary and $\nabla v \in L^{\frac{2}{\alpha},\infty}$ satisfies $\|\nabla v\|_{\frac{2}{\alpha},\infty} < \varepsilon$, where ε is sufficiently small. Then, the Cauchy problem* (2.9)–(2.10) *with the initial condition $z_0 = v_0 + w_0$ has a solution $w = w(x,t)$ such that $w \in X_T$ for each $T > 0$ and satisfies*

$$\|w(t)\|_2^2 + 2(1 - C\varepsilon) \int_0^t \|(-\Delta)^{\frac{\alpha}{4}} w(\tau)\|_2^2 \,\mathrm{d}\tau \leqslant 0 \tag{3.2}$$

for all $t \geqslant 0$.

Proof. We multiply equation (2.9) by w and integrate over $\mathbb{R}^2$ to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|w(s)\|_2^2 + \int_{\mathbb{R}^2} u_w \cdot \nabla v w \,\mathrm{d}x + \|(-\Delta)^{\frac{\alpha}{4}} w\|_2^2 = 0 \tag{3.3}$$

using the fact that $\int_{\mathbb{R}^2} u_v \cdot \nabla w w \,\mathrm{d}x = 0$ and $\int_{\mathbb{R}^2} w \cdot \nabla w w \,\mathrm{d}x = 0$ since $\operatorname{div} w = 0$. Now, applying Hölder inequality in Lorentz spaces (2.1) we estimate

$$\left| \int_{\mathbb{R}^2} u_w \cdot \nabla v w \,\mathrm{d}x \right| \leqslant C\|u_w \cdot \nabla v\|_{\frac{4}{2+\alpha},2} \|w\|_{\frac{4}{2-\alpha},2} \leqslant C\|\nabla v\|_{\frac{2}{\alpha},\infty} \|w\|^2_{\frac{4}{2-\alpha},2}$$

since $\|u_w\|_{p,q} \leqslant C\|w\|_{p,q}$ for suitable p and q. Next, we use the fractional Gagliardo-Nirenberg inequality under Lorentz spaces (2.4) to obtain

$$\left| \int_{\mathbb{R}^2} u_w \cdot \nabla v w \,\mathrm{d}x \right| \leqslant C\|\nabla v\|_{\frac{2}{\alpha},\infty} \|(-\Delta)^{\frac{\alpha}{4}} w\|_2^2. \tag{3.4}$$

Combining (3.3) and (3.4) we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\|w(s)\|_2^2 + 2\Big(1 - C\|\nabla v\|_{\frac{2}{\alpha},\infty}\Big) \|(-\Delta)^{\frac{\alpha}{4}} w\|_2^2 \leqslant 0.$$

Integrating from 0 to t we obtain the energy inequality

$$\|w(t)\|_2^2 + 2(1 - C\varepsilon) \int_0^t \|(-\Delta)^{\frac{\alpha}{4}} w(\tau)\|_2^2 \,\mathrm{d}\tau \leqslant \|w_0\|_2^2$$

for sufficiently small $\|\nabla v\|_{\frac{2}{\alpha},\infty} < \varepsilon$. ∎

4 Energy inequality in $L^p(\mathbb{R}^2)$ - space

Theorem 4.1. *Let $w_0 \in L^2_\sigma(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ for some $p > 2$ be arbitrary. Then there exists a constant $C = C(p)$ such that the solution obtained in Theorem 3.2 satisfies the following energy inequality*

$$\|w(t)\|_p^p + p(1 - C\|\nabla v(t)\|_{\frac{2}{\alpha},\infty}) \int_0^t \|(-\Delta)^{\frac{\alpha}{4}} |w|^{\frac{p}{2}}(\tau)\|_2^2 \,\mathrm{d}\tau \leqslant 0. \tag{4.1}$$

Proof. We multiply equation (2.9) by $|w|^{p-1}$, integrate over $\mathbb{R}^2$ and applying Stroock-Varopoulos inequality (2.3) to get

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}s}\|w(s)\|_p^p + \int_{\mathbb{R}^2} u_w \cdot \nabla v|w|^{p-1}\,\mathrm{d}x + \|(-\Delta)^{\frac{\alpha}{4}}|w|^{\frac{p}{2}}\|_2^2 \leqslant 0 \tag{4.2}$$

using the fact that $\int_{\mathbb{R}^2} u_v \cdot \nabla w|w|^{p-1}\,\mathrm{d}x = 0$ and $\int_{\mathbb{R}^2} w \cdot \nabla w|w|^{p-1}\,\mathrm{d}x = 0$ since $\operatorname{div} w = 0$. Hence, it is enough to estimate the term $\int_{\mathbb{R}^2} u_w \cdot \nabla v|w|^{p-1}\,\mathrm{d}x$. Applying Hölder inequality in Lorentz spaces (2.1), we get

$$\begin{aligned}\left|\int_{\mathbb{R}^2} u_w \cdot \nabla v|w|^{\frac{p}{2}-1}\,\mathrm{d}x\right| &\leqslant C\|u_w \cdot \nabla v|w|^{p-1}\|_{\frac{4}{2+\alpha},2}\|w^{\frac{p}{2}}\|_{\frac{4}{2-\alpha},2}\\
&\leqslant C\|\nabla v\|_{\frac{2}{\alpha},\infty}\|u_w|w|^{\frac{p}{2}-1}\|_{\frac{4}{2-\alpha},2}\|w^{\frac{p}{2}}\|_{\frac{4}{2-\alpha},2}\\
&\leqslant C\|\nabla v\|_{\frac{2}{\alpha},\infty}\|u_w^{\frac{p}{2}}\|^{\frac{2}{p}}_{\frac{4}{2-\alpha},2}\|w^{\frac{p}{2}}\|^{1-\frac{2}{p}}_{\frac{4}{2-\alpha},2}\|w^{\frac{p}{2}}\|_{\frac{4}{2-\alpha},2}.\end{aligned}$$

Finally, using the fractional Sobolev inequality (2.5), we arrive at

$$\left|\int_{\mathbb{R}^2} u_w \cdot \nabla v|w|^{\frac{p}{2}-1}\,\mathrm{d}x\right| \leqslant C\|\nabla v\|_{\frac{2}{\alpha},\infty}\|(-\Delta)^{\frac{\alpha}{4}}|w|^{\frac{p}{2}}\|_2^2,$$

which completes the proof of Theorem 4.1. ∎

5 Decay of solution

Theorem 5.1. *For $w_0 \in L^2_\sigma(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ and $p > 2$ the solution obtained in Theorem 3.2 satisfies*

$$\|w(t)\|_p \leqslant Ct^{-\frac{2}{\alpha}\left(\frac{1}{2}-\frac{1}{p}\right)}\|w_0\|_2. \tag{5.1}$$

Proof. First, observe that using the fractional Gagliardo-Nirenberg inequality (2.4), we have the following estimate

$$\begin{aligned}\|w(t)\|_p^p &\leqslant C\left(\|(-\Delta)^{\frac{\alpha}{4}}|w|^{\frac{p}{2}}\|_2^2\right)^{\frac{s}{2}}\|w\|_2^{\frac{p(2-s)}{2}}\\
&\leqslant C\left(\|(-\Delta)^{\frac{\alpha}{4}}|w|^{\frac{p}{2}}\|_2^2\right)^{\frac{s}{2}}\|w_0\|_2^{\frac{p(2-s)}{2}},\end{aligned}$$

since the L^2- norm of the solution is bounded, where $s = \frac{2(2-p)}{2-\alpha-p}$. Hence, we get

$$\|(-\Delta)^{\frac{\alpha}{4}}|w|^{\frac{p}{2}}\|_2^2 \geqslant C\|w_0\|_2^{\frac{2}{s}(2-s)}\|w(t)\|^{p\frac{s}{2}}.$$

The above inequality together with energy inequality (4.1) leads to the differential inequality of the form

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}f(t) \leqslant -C\|w_0\|_2^{\frac{p}{s}(s-2)}f^{\frac{2}{s}}(t)$$

for the function $f(t) = \|u(t)\|_2^2$, which immediately gives the algebraic decay of the L^2-norm

$$\|w(t)\|_p \leqslant Ct^{-\frac{2}{\alpha}\left(\frac{1}{2}-\frac{1}{p}\right)}\|w_0\|_2. \qquad \blacksquare$$

Theorem 5.2. *For $w_0 \in L^2_\sigma(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ the solution obtained in Theorem 3.2 satisfies*

$$\lim_{t\to\infty} \|w(t)\|_2 = 0. \tag{5.2}$$

Proof. We define

$$A = \limsup_{t\to\infty} \|w(t)\|_2 \quad \text{and} \quad B = \limsup_{t\to\infty} \frac{1}{t}\int_0^t \|w(s)\|_2 \,\mathrm{d}s. \tag{5.3}$$

Let us notice that if $A = 0$ if and only if $B = 0$ which is a straightforward consequence of the fact that the norm $\|w(t)\|_2$ is nonincreasing, the inequality $A \leqslant B$ and the Lebesgue dominated convergence theorem.

Using the Duhamel principle, we have

$$\begin{aligned}\|w(s)\|_2 \leqslant\ & \|e^{-s\Lambda^\alpha} * w_0\|_2 + \|\int_0^s \nabla e^{-(s-\tau)\Lambda^\alpha} * u_w(\tau)w(\tau)\,\mathrm{d}\tau\|_2 \\ & + \|\int_0^s \nabla e^{-(s-\tau)\Lambda^\alpha} * u_w(\tau)v(\tau)\,\mathrm{d}\tau\|_2 + \|\int_0^s \nabla e^{-(s-\tau)\Lambda^\alpha} * u_v(\tau)w(\tau)\,\mathrm{d}\tau\|_2 \\ =\ & I_1(s) + I_2(s) + I_3(s) + I_4(s),\end{aligned} \tag{5.4}$$

where $e^{-s\Lambda^\alpha}$ is a fundamental solution of the equation $u_s = -\Lambda^\alpha u$.

First, we prove that $\lim_{s\to\infty} I_1(s) = 0$, hence we get that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t I_1(s)\,\mathrm{d}s = 0. \tag{5.5}$$

It is know that $\|e^{-s\Lambda^\alpha} * w_0\|_2 \leqslant \|w_0\|_2$ and moreover, $\|e^{-s\Lambda^\alpha} * \varphi\|_2 \leqslant Cs^{-\frac{1}{2\alpha}}\|\varphi\|_1$ for $\varphi \in C_0^\infty(\mathbb{R}^2)$. Let $\varepsilon > 0$. We choose $\varphi \in C_0^\infty(\mathbb{R}^2)$ such that $\|w_0 - \varphi\|_2 < \varepsilon$. Then we have

$$\|e^{-s\Lambda^\alpha} * w_0\|_2 \leqslant \|e^{-s\Lambda^\alpha} * (w_0-\varphi)\|_2 + \|e^{-s\Lambda^\alpha} * \varphi\|_2 \leqslant Cs^{-\frac{1}{2\alpha}} + \varepsilon,$$

which proves (5.5).

Next, we deal with the second term in (5.4) applying the Young inequality (2.2), the Hölder inequality (2.1) and the fractional Gagliardo-Nirenberg inequality (2.4)

$$\begin{aligned}I_2(s) &\leqslant C\int_0^s (s-\tau)^{-\frac{1}{\alpha}}\|u_w(\tau)w(\tau)\|_2\,\mathrm{d}\tau \leqslant C\int_0^s (s-\tau)^{-\frac{1}{\alpha}}\|w(\tau)\|_4^2\,\mathrm{d}\tau \\ &\leqslant C\int_0^s (s-\tau)^{-\frac{1}{\alpha}}\|(-\Delta)^{\frac{\alpha}{4}}w(\tau)\|_2^{\frac{2}{\alpha}}\|w(\tau)\|_2^{2\frac{\alpha-1}{\alpha}}\,\mathrm{d}\tau.\end{aligned}$$

Integrating $I_2(s)$ from 0 to t and multiply by $\frac{1}{t}$, we get

$$\frac{1}{t}\int_0^t I_2(s)\,\mathrm{d}s \leqslant C\frac{1}{t}\int_0^t |\cdot|^{-\frac{1}{\alpha}} * \left(\|(-\Delta)^{\frac{\alpha}{4}}w(\tau)\|_2^2\right)^{\frac{1}{\alpha}}\,\mathrm{d}s,$$

since from energy inequality (3.2) the norm $\|w(t)\|_2^2$ is bounded for $t > 0$. Now, using the Young inequality in the Marcinkiewicz space (2.2), we obtain

$$\frac{1}{t}\int_0^t I_2(s)\,\mathrm{d}s \leqslant Ct^{-\frac{2-\alpha}{\alpha}}\int_0^t \|(-\Delta)^{\frac{\alpha}{4}}w(\tau)\|_2^2\,\mathrm{d}\tau \leqslant Ct^{-\frac{2-\alpha}{\alpha}}, \tag{5.6}$$

because $\int_0^t \|(-\Delta)^{\frac{\alpha}{4}}w(\tau)\|_2^2\,\mathrm{d}\tau$ is bounded according to energy inequality (3.2).

The last two terms, namely integrals $I_3(s)$ and $I_4(s)$, we treat in the same manner. We apply the Young inequality with $\frac{1}{2}+1 = \frac{1}{q}+\frac{1}{2}+\frac{1}{r}$, where $r > \frac{2}{\alpha-1}$, properties of the fundamental solution $e^{t\Lambda^\alpha}$ and properties of the self-similar solution $v = v(x,t)$ to get

$$\begin{aligned} I_3(s) &\leqslant C\int_0^s (s-\tau)^{-\frac{2}{\alpha}(\frac{1}{r}+\frac{1}{2})}\|v(\tau)\|_r\|w(\tau)\|_2\,\mathrm{d}\tau \\ &\leqslant C\|V\|_r\|\int_0^s (s-\tau)^{-\frac{2}{\alpha}(\frac{1}{r}+\frac{1}{2})}\tau^{\frac{2}{\alpha}(\frac{1}{2}+\frac{1}{r})-1}\|w(\tau)\|_2\,\mathrm{d}\tau. \end{aligned}$$

Taking the limit superior as $s \to \infty$ of both sides and using notation (5.3) and substituting $\tau = sz$, we have

$$\limsup_{s\to\infty} I_3(s) \leqslant C\|V\|_r A\int_0^1 (1-z)^{-\frac{2}{\alpha}(\frac{1}{r}+\frac{1}{2})}z^{\frac{2}{\alpha}(\frac{1}{2}+\frac{1}{r})-1}\,\mathrm{d}z \leqslant CB\|V\|_r, \tag{5.7}$$

since the function $f(z) = (1-z)^{-\frac{1}{\alpha}}z^{-1+\alpha}$ is integrable for $\alpha \in (1,2)$.

Analogously, we arrive at

$$\limsup_{s\to\infty} I_4(s) \leqslant CB\|V\|_r. \tag{5.8}$$

Now, we go back to (5.4), integrate from 0 to t and multiply by $\frac{1}{t}$ to get

$$\begin{aligned} \frac{1}{t}\int_0^t \|w(s)\|_2\,\mathrm{d}s \leqslant{}& \frac{1}{t}\int_0^t \|e^{-s\Lambda^\alpha} * w_0\|_2\,\mathrm{d}s \\ &+\frac{1}{t}\int_0^t \|\int_0^s \nabla e^{-(s-\tau)\Lambda^\alpha} * u_w(\tau)w(\tau)\,\mathrm{d}\tau\|_2\,\mathrm{d}s \\ &+\frac{1}{t}\int_0^t \|\int_0^s \nabla e^{-(s-\tau)\Lambda^\alpha} * u_w(\tau)v(\tau)\,\mathrm{d}\tau\|_2\,\mathrm{d}s+ \\ &\frac{1}{t}\int_0^t \|\int_0^s \nabla * e^{-(s-\tau)\Lambda^\alpha}u_v(\tau)w(\tau)\,\mathrm{d}\tau\|_2\,\mathrm{d}s. \end{aligned}$$

Taking the limit superior as $t \to \infty$ of both sides, using notation (5.3) and combining (5.5), (5.6), (5.7) and (5.8), we have

$$B \leqslant 2CB\|V\|_r,$$

which is equivalent to

$$(1-2C\|V\|_r)B \leqslant 0.$$

Hence, if the norm $\|V\|_r$ is sufficiently small, $B = 0$ and, in the consequence, $A = 0$, which completes the proof of Theorem 5.2. ∎

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