Existence of Solution for an Elliptic Problem with a Sublinear Term

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Abstract

In this work we prove the existence of a classical positive solution for an elliptic equation with a sublinear term. We use Galerkin approximations to show existence of such solution on bounded domains in $\mathbb{R}^N$.

1 Introduction

In this paper, we study the existence of solution for the problem

$$\begin{cases} -\Delta v = \lambda v^q + f(v), & \text{in} \quad \Omega, \\ v > 0 & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary, $\lambda > 0$ is a parameter, $0 < q < 1$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$0 \leq f(s)s \leq C|s|^{p+1}, \tag{2}$$

where $1 < p \leq \frac{N+2}{N-2}$ if $N \geq 3$ or $1 < p$ if $N = 2$.

Our main result in this paper is the following:

Theorem 1.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (2). Then, there exists $\lambda^* > 0$ such that, for every $\lambda \in (0, \lambda^*)$, the problem (1) has a positive solution $u \in C^{2,\gamma}(\overline{\Omega})$, for some $\gamma \in (0, 1)$.*

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Elliptic problems of the type

$$\begin{cases} -\Delta v = g(x,v) & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

where the nonlinearity $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous, have been extensively studied; see for example [2, 3] for a survey. Furthermore, we also refer [4], where the authors considered problem (3) with nonlinearity combined effects of concave and convex; namely, they considered $g(x,u) = \lambda u^q + u^p$ with $0 < q < 1 < p$.

We say that g has sublinear growth at infinite if

$$\lim_{|s| \to +\infty} \frac{g(x,s)}{s} = 0 \text{ uniformly in } x.$$

We say that g has superlinear growth at infinite if

$$\lim_{|s| \to +\infty} \frac{g(x,s)}{s} = +\infty \text{ uniformly in } x.$$

We would like to highlight that the only assumptions which we assume are that $0 < q < 1$ and that f is continuous and satisfies the growth condition (2). This way, the nonlinearity $g(x,s) = \lambda s^q + f(s)$ of problem (1) can have sublinear or superlinear growth at infinite.

Most papers treat problem (3) by means of variational methods, then it is usually assumed that g has sublinear or superlinear growth and, sometimes, $sg(x,s) \geq c|s|^p$, where $c > 0$ is a constant and $p > 2$; see for example [11]. Another common assumption on g is the so-called Ambrosetti-Rabinowitz condition that means the following:

$$\exists R > 0 \text{ and } \theta > 2 \text{ such that } 0 < \theta G(x,s) \leq sg(x,s) \ \forall |s| \geq R \text{ and } x \in \Omega,$$

where $G(x,s) = \int_0^s g(x,\tau)d\tau$. Even when the Ambrosetti-Rabinowitz condition can be dropped, it must be assumed some condition to give compactness of Palais-Smale sequences or Cerami sequences. See for instance [6], where they assume

$$g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \text{ is continuous and } g(x,0) = 0;$$

$$\exists t_0 > 0 \text{ and } M > 0 \text{ such that } 0 < G(x,s) \leq Mg(x,s) \ \forall |s| \geq t_0 \text{ and } x \in \Omega;$$

$$0 < 2G(x,s) \leq sg(x,s) \ \forall |s| \geq 0 \text{ and } x \in \Omega.$$

See also [9].

We are able to solve (1) under weaker assumptions by using the Galerkin method. For that matter we approximate f by Lipschitz functions in Section 2. In Section 3 we solve approximate problems. In Section 4 we prove a regularity result to approximate problems. Section 5 is devoted to prove Theorem 1.1; in doing so we show that solutions v_n of approximate problems are bounded away from zero and converge to a positive solution of (1).

At last in this introduction, we would like to emphasize that a similar approach was already used in [1], but different to that, we do not assume that the nonlinearity f is Lipschitz continuous.

2 Approximating functions

In order to prove Theorem 1.1, we make use of the following approximation result by Lipschitz functions, proved by Strauss in [10].

Lemma 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $sf(s) \geq 0$ for all $s \in \mathbb{R}$. Then, there exists a sequence $f_k : \mathbb{R} \to \mathbb{R}$ of continuous functions satisfying $sf_k(s) \geq 0$ and*

(i) $\forall\, k \in \mathbb{N}$, $\exists c_k > 0$ *such that* $|f_k(\xi) - f_k(\eta)| \leq c_k|\xi - \eta|$, *for all* $\xi, \eta \in \mathbb{R}$.

(ii) (f_k) *converges uniformly to f in bounded subsets of* $\mathbb{R}$.

The proof consists in considering the following family of approximation functions $f_k : \mathbb{R} \to \mathbb{R}$ defined by

$$f_k(s) = \begin{cases} -k[G(-k - \frac{1}{k}) - G(-k)], & \text{if} \quad s \leq -k, \\ -k[G(s - \frac{1}{k}) - G(s)], & \text{if} \quad -k \leq s \leq -\frac{1}{k}, \\ k^2 s[G(-\frac{2}{k}) - G(-\frac{1}{k})], & \text{if} \quad -\frac{1}{k} \leq s \leq 0, \\ k^2 s[G(\frac{2}{k}) - G(\frac{1}{k})], & \text{if} \quad 0 \leq s \leq \frac{1}{k}, \\ k[G(s + \frac{1}{k}) - G(s)], & \text{if} \quad \frac{1}{k} \leq s \leq k, \\ k[G(k + \frac{1}{k}) - G(k)], & \text{if} \quad s \geq k. \end{cases} \tag{4}$$

where $G(s) = \int_0^s f(\tau)d\tau$.

The sequence (f_k) of the previous lemma has some additional properties.

Lemma 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $sf(s) \geq 0$ for all $s \in \mathbb{R}$. Let us suppose that there exist constants $C > 0$ and $1 < p \leq \frac{N+2}{N-2}$ such that*

$$sf(s) \leq C|s|^{p+1} \ , \ \forall s \in \mathbb{R}. \tag{5}$$

Then, the sequence $(f_k)_{k\in\mathbb{N}}$ from Lemma 2.1 satisfies

(i) $0 \leq sf_k(s) \leq C_1|s|^{p+1}$ *for all* $|s| \geq \frac{1}{k}$,

(ii) $0 \leq sf_k(s) \leq C_2|s|^2$ *for all* $|s| \leq \frac{1}{k}$,

where C_1 and C_2 do not depend on k.

Proof: Everywhere in this proof, the constant C is the one given by (2).

First step: Suppose $-k \leq s \leq -\frac{1}{k}$.

By the mean value theorem, there exists $\eta \in (s - \frac{1}{k}, s)$ such that

$$f_k(s) = -k[G(s - \frac{1}{k}) - G(s)] = -kG'(\eta)(s - \frac{1}{k} - s) = f(\eta)$$

and

$$sf_k(s) = sf(\eta).$$

As $s - \frac{1}{k} < \eta < s < 0$ and $f(\eta) < 0$, we have $sf(\eta) \leq \eta f(\eta)$. Therefore,

$$sf_k(s) \leq \eta f(\eta) \leq C|\eta|^{p+1} \leq C|s - \frac{1}{k}|^{p+1} \leq C(|s| + \frac{1}{k})^{p+1} \leq C2^{p+1}|s|^{p+1}.$$

Second step: Suppose $\frac{1}{k} \leq s \leq k$.
By the mean value theorem, there exists $\eta \in (s, s+\frac{1}{k})$ such that

$$f_k(s) = k[G(s+\frac{1}{k}) - G(s)] = kG'(\eta)(s+\frac{1}{k} - s) = f(\eta)$$

and

$$sf_k(s) = sf(\eta).$$

As $0 < s < \eta < s+\frac{1}{k}$ and $f(\eta) > 0$, we have $sf(\eta) \leq \eta f(\eta)$. Therefore,

$$sf_k(s) \leq \eta f(\eta) \leq C|\eta|^{p+1} \leq C|s+\frac{1}{k}|^{p+1} = C(|s|+\frac{1}{k})^{p+1} \leq C2^{p+1}|s|^{p+1}.$$

Third step: Suppose $|s| \geq k$.
Define

$$f_k(s) = \begin{cases} -k[G(-k-\frac{1}{k}) - G(-k)], & \text{if } \ s \leq -k, \\ k[G(k+\frac{1}{k}) - G(k)], & \text{if } \ s \geq k. \end{cases}$$

If $s \leq -k$, by the mean value theorem, there exists $\eta \in (-k-\frac{1}{k}, -k)$ such that

$$f_k(s) = k[G(-k-\frac{1}{k}) - G(-k)] = -kG'(\eta)(-k-\frac{1}{k} - (-k)) = f(\eta)$$

and

$$sf_k(s) = sf(\eta).$$

As $-k-\frac{1}{k} < \eta < -k < 0$ and $k < |\eta| < k+\frac{1}{k}$, we have $sf(\eta) = \frac{s}{\eta}\eta f(\eta)$. Therefore,

$$sf_k(s) = \frac{s}{\eta}\eta f(\eta) \leq \frac{|s|}{|\eta|}C|\eta|^{p+1} =$$

$$= C|s||\eta|^p \leq C|s|(k+\frac{1}{k})^p \leq C|s|(|s|+\frac{1}{k})^p \leq C2^p|s|^{p+1}.$$

If $s \geq k$, by the mean value theorem, there exists $\eta \in (k, k+\frac{1}{k})$ such that

$$f_k(s) = k[G(k+\frac{1}{k}) - G(k)] = kG'(\eta)(k+\frac{1}{k} - k) = f(\eta)$$

and

$$sf_k(s) = sf(\eta) = \frac{s}{\eta}\eta f(\eta) \leq \frac{|s|}{|\eta|}C|\eta|^{p+1} =$$

$$= C|s||\eta|^p \leq C|s|(k+\frac{1}{k})^p \leq C|s|(|s|+\frac{1}{k})^p \leq C2^p|s|^{p+1}.$$

Fourth step: Suppose $-\frac{1}{k} \leq s \leq \frac{1}{k}$.
Define

$$f_k(s) = \begin{cases} k^2 s[G(-\frac{2}{k}) - G(-\frac{1}{k})], & \text{if } \ -\frac{1}{k} \leq s \leq 0, \\ k^2 s[G(\frac{2}{k}) - G(\frac{1}{k})], & \text{if } \ 0 \leq s \leq \frac{1}{k}. \end{cases}$$

If $-\frac{1}{k} \leq s \leq 0$, by the mean value theorem, there exists $\eta \in (-\frac{2}{k}, -\frac{1}{k})$ such that

$$f_k(s) = k^2 s[G(-\frac{2}{k}) - G(-\frac{1}{k})] = k^2 s G'(\eta)(-\frac{2}{k} - (-\frac{1}{k})) = -ksf(\eta).$$

Therefore,

$$\begin{aligned} sf_k(s) &= -ks^2 f(\eta) = -k\frac{s^2}{\eta}\eta f(\eta) \leq k\frac{s^2}{|\eta|}\eta f(\eta) \\ &\leq Ck|s|^2|\eta|^p \leq Ck|s|^2(\frac{2}{k})^p \leq C2^p|s|^2. \end{aligned}$$

If $0 \leq s \leq \frac{1}{k}$, by the mean value theorem, there exists $\eta \in (\frac{1}{k}, \frac{2}{k})$ such that

$$f_k(s) = k^2 s[G(\frac{2}{k}) - G(\frac{1}{k})] = k^2 s G'(\eta)(\frac{2}{k} - \frac{1}{k}) = ksf(\eta).$$

Therefore,

$$\begin{aligned} sf_k(s) &= ks^2 f(\eta) = k\frac{s^2}{|\eta|}\eta f(\eta) \leq \\ &\leq Ck|s|^2|\eta|^p \leq Ck|s|^2(\frac{2}{k})^p \leq C2^p|s|^2. \end{aligned}$$

The proof of the lemma follows by taking $C_1 = C2^{p+1}$ and $C_2 = C2^p$, where C is like in (5).

3 Approximate problem

In order to prove Theorem 1.1, we first study the auxiliary problem

$$\begin{cases} -\Delta v = \lambda v^q + f_n(v) + \frac{1}{n} & \text{in} \quad \Omega, \\ v > 0 & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{6}$$

where $0 < q < 1$, $\lambda > 0$ is a parameter and $f_n : \mathbb{R} \to \mathbb{R}$ is a function of the sequence given by Lemma 2.1 and Lemma 2.2.

We will use the Galerkin method together with the following fixed point theorem, see [10] and [8, Theorem 5.2.5]. A similar approach was already used in [1].

In the following proposition, $\langle ., . \rangle$ denotes the Euclidean inner product of $\mathbb{R}^d$.

Proposition 3.1. *Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function such that $\langle F(\xi), \xi \rangle \geq 0$ for every $\xi \in \mathbb{R}^d$ with $|\xi| = r$ for some $r > 0$. Then, there exists z_0 in the closed ball $\overline{B}_r(0)$ such that $F(z_0) = 0$.*

The main result in this section is the following theorem.

Theorem 3.2. *There exists $\lambda^* > 0$ and $n^* \in \mathbb{N}$ such that (6) has a weak positive solution for all $\lambda \in (0, \lambda^*)$ and $n \geq n^*$.*

Proof: Fix $\mathcal{B} = \{w_1, w_2, \dots, w_m, \dots\}$ a orthonormal basis of $H_0^1(\Omega)$ and define

$$W_m = [w_1, w_2, \dots, w_m]$$

to be the space generated by $\{w_1, w_2, \dots, w_m\}$. Given $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$, let $v = \sum_{i=1}^m \xi_i w_i \in W_m$ and consider the function $F : \mathbb{R}^m \to \mathbb{R}^m$ such that $F(\xi) = (F_1(\xi), F_2(\xi), \dots, F_m(\xi))$, where

$$F_j(\xi) = \int_\Omega \nabla v \nabla w_j - \lambda \int_\Omega (v_+)^q w_j - \int_\Omega f_n(v_+) w_j - \frac{1}{n}\int_\Omega w_j, \;\; j = 1, 2, \dots, m$$

. Therefore,

$$\langle F(\xi), \xi \rangle = \int_\Omega |\nabla v|^2 - \lambda \int_\Omega (v_+)^{q+1} - \int_\Omega f_n(v_+) v_+ - \frac{1}{n}\int_\Omega v. \tag{7}$$

Given $v \in W_m$, we define

$$\Omega_n^+ = \{x \in \Omega : |v(x)| \geq \frac{1}{n}\}$$

and

$$\Omega_n^- = \{x \in \Omega : |v(x)| < \frac{1}{n}\}.$$

Thus, we rewrite (7) as

$$\langle F(\xi), \xi \rangle = \langle F(\xi), \xi \rangle_P + \langle F(\xi), \xi \rangle_N,$$

where

$$\langle F(\xi), \xi \rangle_P = \int_{\Omega_n^+} |\nabla v|^2 - \lambda \int_{\Omega_n^+} (v_+)^{q+1} - \int_{\Omega_n^+} f_n(v_+) v_+ - \frac{1}{n}\int_{\Omega_n^+} v$$

and

$$\langle F(\xi), \xi \rangle_N = \int_{\Omega_n^-} |\nabla v|^2 - \lambda \int_{\Omega_n^-} (v_+)^{q+1} - \int_{\Omega_n^-} f_n(v_+) v_+ - \frac{1}{n}\int_{\Omega_n^-} v.$$

Step 1. Since $0 < q < 1$, then

$$\int_{\Omega_n^+} (v_+)^{q+1} \leq \int_\Omega |v|^{q+1} = \|v\|^{q+1}_{L^{q+1}(\Omega)} \leq C_1 \|v\|^{q+1}_{H_0^1(\Omega)}. \tag{8}$$

By virtue of (i) Lemma 2.2, we get

$$\int_{\Omega_n^+} f_n(v_+) v_+ \;\; \leq \;\; C \int_\Omega |v_+|^{p+1} dx \leq C_2 \|v\|^{p+1}_{H_0^1(\Omega)}. \tag{9}$$

It follows from (8) and (9) that

$$\begin{aligned} \langle F(\xi), \xi \rangle_P \;\; &\geq \;\; \int_{\Omega_n^+} |\nabla v|^2 - \lambda C_1 \|v\|^{q+1}_{H_0^1(\Omega)} \\ &- \;\; C_2 \|v\|^{p+1}_{H_0^1(\Omega)} - \frac{C_3}{n} \|v\|_{H_0^1(\Omega)}, \end{aligned} \tag{10}$$

where C_1, C_2 and C_3 depends on C and $|\Omega|$.

Step 2. Since $0 < q < 1$, then

$$\int_{\Omega_n^-} (v_+)^{q+1} \leq \int_{\Omega_n^-} |v|^{q+1} \leq |\Omega| \frac{1}{n^{q+1}}. \tag{11}$$

By virtue of (*ii*) Lemma 2.2, we get

$$\int_{\Omega_n^-} f_n(v_+)v_+ \leq C \int_{\Omega_n^-} |v_+|^2 dx \leq C|\Omega| \frac{1}{n^2}. \tag{12}$$

It follows from (11) and (12) that

$$\langle F(\xi), \xi \rangle_N \geq \int_{\Omega_n^-} |\nabla v|^2 - \lambda |\Omega| \frac{1}{n^{q+1}} - C|\Omega| \frac{1}{n^2} - |\Omega| \frac{1}{n^2}. \tag{13}$$

It follows from (10) and (13) that

$$\begin{aligned} \langle F(\xi), \xi \rangle &\geq \|v\|^2_{H_0^1(\Omega)} - \lambda C_1 \|v\|^{q+1}_{H_0^1(\Omega)} - C_2 \|v\|^{p+1}_{H_0^1(\Omega)} \\ &- \frac{C_3}{n} \|v\|_{H_0^1(\Omega)} - \lambda |\Omega| \frac{1}{n^{q+1}} - C|\Omega| \frac{1}{n^2} - |\Omega| \frac{1}{n^2}. \end{aligned}$$

Assume now that $\|v\|_{H_0^1(\Omega)} = r$ for some $r > 0$ which will be fixed later. We have

$$\langle F(\xi), \xi \rangle \geq r^2 - \lambda C_1 r^{q+1} - C_2 r^{p+1} - \frac{C_3}{n} r - \lambda |\Omega| \frac{1}{n^{q+1}} - C|\Omega| \frac{1}{n^2} - |\Omega| \frac{1}{n^2}.$$

We want to choose r such that

$$r^2 - C_2 r^{p+1} \geq \frac{r^2}{2},$$

that is,

$$r \leq \frac{1}{(2C_2)^{\frac{1}{p-1}}}.$$

Then, choosing $r = \frac{1}{2(2C_2)^{\frac{1}{p-1}}}$, we obtain

$$\langle F(\xi), \xi \rangle \geq \frac{r^2}{2} - \lambda C_1 r^{q+1} - \frac{C_3}{n} r - \lambda |\Omega| \frac{1}{n^{q+1}} - C|\Omega| \frac{1}{n^2} - |\Omega| \frac{1}{n^2}.$$

Now, defining $\rho = \frac{r^2}{2} - \lambda C_1 r^{q+1}$, we choose $\lambda^* > 0$ such that $\rho > 0$ for $\lambda < \lambda^*$. Therefore, we choose $\lambda^* = \frac{r^{1-q}}{4C_1}$. Now, we choose $n^* \in \mathbb{N}$ such that

$$\frac{C_3}{n} r + \lambda |\Omega| \frac{1}{n^{q+1}} + C|\Omega| \frac{1}{n^2} + |\Omega| \frac{1}{n^2} < \frac{\rho}{2},$$

for every $n \geq n^*$. Let $\xi \in \mathbb{R}^m$ such that $|\xi| = r$. Then, for $\lambda < \lambda^*$ and $n \geq n^*$, we obtain

$$\langle F(\xi), \xi \rangle \geq \frac{\rho}{2} > 0.$$

Since f_n is a Lipschitz continuous function for every n, by standard arguments it is shown that F is continuous, that is, given (x_k) in $\mathbb{R}^m$ and $x \in \mathbb{R}^m$ such that $x_k \to x$ we obtain $F(x_k) \to F(x)$. Therefore, by Proposition 3.1, for all $m \in \mathbb{N}$, there exists $y \in \mathbb{R}^m$ satisfying $|y| \leq r$ and $F(y) = 0$, that is, there exists $v_m \in W_m$ verifying $\|v_m\|_{H_0^1(\Omega)} \leq r$, for every $m \in \mathbb{N}$, and

$$\int_\Omega \nabla v_m \nabla w = \lambda \int_\Omega (v_{m+})^q w + \int_\Omega f_n(v_{m+}) w + \frac{1}{n} \int_\Omega w, \ \forall\, w \in W_m.$$

Since $W_m \subset H_0^1(\Omega)$ for all $m \in \mathbb{N}$ and r does not depend on m, we have that (v_m) is a bounded sequence of $H_0^1(\Omega)$. Then, for some subsequence, there exists $v = v_n \in H_0^1(\Omega)$ such that

$$v_m \rightharpoonup v \ \text{ weakly in } \ H_0^1(\Omega) \tag{14}$$

and

$$v_m \to v \ \text{ in } \ L^2(\Omega) \ \text{ and a.e. in } \ \Omega. \tag{15}$$

Fixing $k \in \mathbb{N}$ such that $m \geq k$ we obtain

$$\int_\Omega \nabla v_m \nabla w_k = \lambda \int_\Omega (v_{m+})^q w_k + \int_\Omega f_n(v_{m+}) w_k + \frac{1}{n} \int_\Omega w_k, \ \forall\, w_k \in W_k. \tag{16}$$

Now, considering $g : H_0^1(\Omega) \to \mathbb{R}$ defined by $g(u) = \int_\Omega \nabla u \nabla w_k$, for every $u \in H_0^1(\Omega)$, we have that g is a continuous linear functional and it follows from (14) that

$$\int_\Omega \nabla v_m \nabla w_k \to \int_\Omega \nabla v \nabla w_k \ \text{ as } \ m \to \infty. \tag{17}$$

On the other hand, note that, from (15), we obtain

$$\int_\Omega f_n(v_{m+}) w_k \to \int_\Omega f_n(v_+) w_k \ \text{ as } \ m \to \infty. \tag{18}$$

Indeed, by Lemma 2.1 (*ii*) it follows that $|f_n(v_{m+}) - f_n(v_+)| \leq c_n |v_{m+} - v_+|$; hence

$$\left| \int_\Omega f_n(v_{m+}) w_k - \int_\Omega f_n(v_+) w_k \right| \leq c_n \|w_k\|_{L^2(\Omega)} \|v_m - v\|_{L^2(\Omega)} \ \text{ as } \ m \to \infty,$$

and then, (15) implies (18). By (14), (18) and Sobolev compact embedding, letting $m \to \infty$, we obtain

$$\lambda \int_\Omega (v_{m+})^q w_k + \int_\Omega f_n(v_{m+}) w_k + \frac{1}{n} \int_\Omega w_k \to \lambda \int_\Omega (v_+)^q w_k + \int_\Omega f_n(v_+) w_k + \frac{1}{n} \int_\Omega w_k. \tag{19}$$

By (16), (17), (19) and by the uniqueness of the limit, we obtain

$$\int_\Omega \nabla v \nabla w_k = \lambda \int_\Omega (v_+)^q w_k + \int_\Omega f_n(v_+) w_k + \frac{1}{n} \int_\Omega w_k, \ \forall\, w_k \in W_k.$$

For density of $[W_k]_{k \in \mathbb{N}}$ in $H_0^1(\Omega)$ and by linearity, we conclude that

$$\int_\Omega \nabla v \nabla w = \lambda \int_\Omega (v_+)^q w + \int_\Omega f_n(v_+) w + \frac{1}{n} \int_\Omega w, \ \forall\, w \in H_0^1(\Omega). \tag{20}$$

Furthermore, $v \geq 0$ a.e. in Ω. In fact, as $v_- \in H_0^1(\Omega)$, we obtain from (20) that

$$\int_\Omega \nabla v \nabla v_- = \lambda \int_\Omega (v_+)^q v_- + \int_\Omega f_n(v_+) v_- + \frac{1}{n} \int_\Omega v_-.$$

Hence, we have from Lemma 2.1 that

$$0 \geq -\|v_-\|^2_{H_0^1(\Omega)} = \int_\Omega \nabla v \nabla v_- = \int_\Omega f_n(v_+) v_- + \frac{1}{n} \int_\Omega v_- \geq 0,$$

that is, $\|v_-\|_{H_0^1(\Omega)} = 0$ and consequently, $v_-(x) = 0$ a.e. in Ω. Therefore, $v(x) = v_+(x) \geq 0$ a.e. in Ω and we conclude the proof of the theorem.

4 Regularity of Solution of the Approximate Problem

In this section, we show that all weak solutions of the problem (6) are regular. Let $v \in H_0^1(\Omega)$ be a weak solution of the problem (6) and define

$$g(x) := \lambda v^q(x) + f_n(v(x)) + \frac{1}{n}.$$

We have that

$$|g| \leq \lambda |v|^q + |f_n(v)| + \frac{1}{n}. \tag{21}$$

Notice that

$$|v|^q \leq 1 + |v|^{t-1}, \tag{22}$$

where $2 \leq t \leq 2^*$. Here, 2^* is the critical Sobolev exponent, that is,

$$2^* = \frac{2N}{N-2}.$$

Furthermore, since $f_n : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function and $f_n(0) = 0$, we have for each $n \in \mathbb{N}$ that

$$|f_n(v)| \leq C_n |v|,$$

and consequently,

$$|f_n(v)| \leq C_n(1 + |v|^{t-1}), \tag{23}$$

where $2 \leq t \leq 2^*$. This way, by combining (21), (22) and (23), we obtain

$$|g| \leq C_1 + C_2 |v|^{t-1}, \tag{24}$$

where

$$C_1 := \lambda + C_n + \frac{1}{n}$$

and

$$C_2 := \lambda + C_n.$$

Then, using (24) and well-known Bootstrap arguments, similar to those found in [7], we conclude that $v \in C^{2,\gamma}(\overline{\Omega})$, for some $\gamma \in (0, 1)$.

5 Proof of the Theorem 1.1

In this section, we demonstrate Theorem 1.1. The following lemma of [10, Theorem 1.1] is used to show that v_n converges to a solution v of (1).

Lemma 5.1. *Let Ω be a bounded open set in $\mathbb{R}^N$, $u_k : \Omega \to \mathbb{R}$ be a sequence of functions and $g_k : \mathbb{R} \to \mathbb{R}$ be a sequence of functions such that $g_k(u_k)$ are measurable in Ω for every $k \in \mathbb{N}$. Assume that $g_k(u_k) \to v$ a.e. in Ω and $\int_\Omega |g_k(u_k)u_k|dx < C$ for a constant C independent of k. Suppose that for every bounded set $B \subset \mathbb{R}$ there is a constant C_B depending only on B such that $|g_k(x)| \leq C_B$, for all $x \in B$ and $k \in \mathbb{N}$. Then, $v \in L^1(\Omega)$ and $g_k(u_k) \to v$ in $L^1(\Omega)$.*

Since $v \in C^{2,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$, satisfies $v \geq 0$ and

$$-\Delta v = \lambda v^q + f_n(v) + \frac{1}{n},$$

it follows by assumptions on f_n that

$$-\Delta v \geq 0.$$

Then, by Maximum Principle, we have $v > 0$ in Ω, that is, v is a solution of the problem (6). For each $n \in \mathbb{N}$, let us denote by v_n the solution of (6). It follows from (14) that

$$v_m^{(n)} \rightharpoonup v_n \text{ weakly in } H_0^1(\Omega) \text{ as } m \to \infty,$$

where, for each $n \in \mathbb{N}$, $(v_m^{(n)})_{m\in\mathbb{N}}$ is a sequence in $H_0^1(\Omega)$ satisfying

$$||v_m^{(n)}|| \leq r, \ \forall m \in \mathbb{N}.$$

Then,

$$\|v_n\| \leq \liminf_{m\to\infty} \|v_m^{(n)}\| \leq r, \ \forall\, n \in \mathbb{N}.$$

Since r does not depend on n, there exists $v \in H_0^1(\Omega)$ such that

$$v_n \rightharpoonup v \text{ weakly in } H_0^1(\Omega).$$

By compact embedding, up to a subsequence, we have

$$v_n \to v \text{ in } L^s(\Omega), \text{ for } 1 \leq s < 2^* \text{ if } N \geq 3 \text{ or for } 1 \leq s < +\infty \text{ if } N = 2,$$

and then, up to a subsequence,

i) $v_n(x) \to v(x)$ a.e. in Ω;

ii) $|v_n(x)| \leq h(x)$, $\forall\, n \in \mathbb{N}$ a.e. in Ω, for some $h \in L^s(\Omega)$.

Notice that the following inequality holds:

$$\begin{cases} -\Delta v_n \geq \lambda v_n^q, & \text{in} \quad \Omega, \\ v_n > 0 & \text{in} \quad \Omega, \\ v_n = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

This way, considering $w_n = \lambda^{\frac{1}{q-1}} v_n$, we obtain

$$-\Delta \left(\frac{w_n}{\lambda^{\frac{1}{q-1}}} \right) \geq \lambda \left(\frac{w_n}{\lambda^{\frac{1}{q-1}}} \right)^q,$$

and consequently,

$$-\Delta\, w_n \geq w_n^q.$$

Let us denote by $\widetilde{w}$ the unique solution of the problem

$$\begin{cases} -\Delta \widetilde{w} = \widetilde{w}^q, & \text{in} \quad \Omega, \\ \widetilde{w} > 0 & \text{in} \quad \Omega, \\ \widetilde{w} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

The existence and uniqueness of such solution is proved in [5]. By Lemma 3.3 of [4], it follows that $w_n \geq \widetilde{w}, \forall\, n \in \mathbb{N}$, that is,

$$v_n(x) \geq \lambda^{\frac{1}{1-q}} \widetilde{w}(x), \text{ a.e. in } \Omega, \forall\, n \in \mathbb{N}. \tag{25}$$

Taking the limit as $n \to +\infty$ in (25), we obtain

$$v(x) \geq \lambda^{\frac{1}{1-q}} \widetilde{w}(x), \text{ a.e. in } \Omega$$

and hence, $v > 0$ a.e. in Ω.

Recall that, from (20),

$$\int_\Omega \nabla v_n \nabla w = \lambda \int_\Omega (v_n)^q w + \int_\Omega f_n(v_n) w + \frac{1}{n} \int_\Omega w, \quad \forall\, w \in H_0^1(\Omega),$$

and using that v_n is a classical solution, we have

$$-\Delta\, v_n = \lambda (v_n)^q + f_n(v_n) + \frac{1}{n} \text{ in } L^2(\Omega). \tag{26}$$

Since

$$v_n \to v \text{ a.e. in } \Omega,$$

we have

$$f_n(v_n(x)) \to f(v(x)) \text{ a.e. in } \Omega \tag{27}$$

by the uniform convergence of Lemma 2.1 (*ii*).

Multiplying the equation (26) by $w = v_n$ and since v_n is bounded in $H_0^1(\Omega)$, we obtain

$$\int_\Omega f_n(v_n) v_n dx \leq C, \tag{28}$$

for every $n \in \mathbb{N}$, where $C > 0$ is a constant independent of n. By (27), (28) and by the expression of f_n defined in (4), the assumptions of Lemma 5.1 are satisfied implying

$$f_n(v_n) \to f(v) \text{ strongly in } L^1(\Omega).$$

Multiplying (26) by $w \in \mathcal{D}(\Omega)$, integrating on Ω and using the previous convergences, we have

$$-\Delta v = \lambda v^q + f(v) \text{ in } \mathcal{D}'(\Omega). \tag{29}$$

Since $f(v) \in L^{\frac{p+1}{p}}(\Omega)$ and $\lambda v^q \in L^{\frac{p+1}{p}}(\Omega)$, we conclude from (29) that $v \in H_0^1(\Omega) \cap W^{2,\frac{p+1}{p}}(\Omega)$ and

$$-\Delta v = \lambda v^q + f(v)$$

in the strong sense. Notice that the assumption (2) implies that

$$|f(s)| \leq C|s|^{t-1},$$

where $2 \leq t \leq 2^*$. Thus, using well-known Bootstrap arguments, we conclude that $v \in C^{2,\gamma}(\overline{\Omega})$, for some $\gamma \in (0,1)$, and it is a classical positive solution of problem (1).

References

[1] C. O. Alves and D. G. de Figueiredo, *Nonvariational Elliptic Systems via Galerkin Methods*, D. Haroske, T. Runst and H. J. Schmeisser (eds.) Function Spaces, Differential Operators and Nonlinear Analysis. The Hans Triebel Anniversary Volume, 2003

[2] A. Ambrosetti, *Critical Points and Nonlinear Variational Problems*, Bull. Soc. Math. France 120, Memoire No. 49 (1992).

[3] A. Ambrosetti and M. Badiale, *The Dual Variational Principle and Elliptic Problems with Discontinuous Nonlinearities*, J. Math. Anal. Appl. 140 (1989), 363-373.

[4] A. Ambrosetti, H. Brezis, G. Cerami, *Combined Effects of Concave and Convex Nonlinearities in Some Elliptic Problems*, Journal of Functional Analysis , **122**, (1994), 519–543.

[5] H. Brezis and L. Oswald, *Remarks on sublinear elliptic equations*. Nonlinear Analysis TMA. **10** 55–64 (1986)

[6] N. Lam and G. Lu, *Elliptic equations and systems with subcritical and critical exponential growth without the Ambrosetti-Rabinowitz condition*. J. Geom. Anal. **24** 118–143 (2014)

[7] O. Kavian, *Introduction à la théorie de Points Critiques*, vol. 13, Springer-Verlag, 1993

[8] S. Kesavan, Topics in functional analysis and applications, John Wiley & Sons (1989).

[9] O. H. Miyagaki and M. A. S. Souto, *Superlinear problems without Ambrosetti and Rabinowitz growth condition*, J. Differential Equations 245 (2008) 3628-3638.

[10] W. A. Strauss, *On weak solutions of semilinear hyperbolic equations*, An. Acad. Brasil. Ciênc. **42** 645–651 (1970)

[11] M. Willem and W. Zou, *On a Schrödinger equation with periodic potential and spectrum point zero*, Indiana Univ. Math. J. 52 (1), (2003), 109-132.

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