Abstract Shearlet Transform

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Abstract

In this paper, the shearlet theory is extended from Euclidean spaces to locally compact groups. More precisely, the abstract shearlet group is defined as a 3-fold semidirect product and the abstract shearlet transform is constructed by means of a quasiregular representation of the semidirect product group. Its properties are investigated and results are illustrated by some examples.

1 Introduction

Various drawbacks of classic wavelet theory in dealing with multidimensional data, including problems concerning image processing, stimulated many mathematicians during the past 20 years to seek for efficient substitutes such as *two dimensional directional wavelets* [1], *wavelets with composite dilation* [14], *curvelets* [5] and *Contourlets* [9]. Curvelets initially emerged as the most efficient tool for image processing as well as for sparse approximation, but they had at least two notable shortcomings: first, they were not generated from a single or even a finite set of functions and second, the curvelet transform did not stem from representation of some locally compact group. In 2005, *shearlets* were introduced to address these shortcomings (cf. [6],[10],[13],[18],[19] and [20]), while preserving all positive features of curvelets. Being motivated by applications in image processing, shearlet theory was initially developed for functions in $L^2(R^2)$. Natural images exhibit edges, that is, discontinuities along curves. Since these discontinuities are

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spatially distributed, in order to obtain efficient representations of images, such representations must contain basis elements with many more shapes and directions than the classical wavelet bases. Shearlet as ultimate systems, are obtained by applying the actions of dilation, shear transformation and translation to a fixed function, and exhibit the geometric and mathematical properties, such as, directionality, elongated shapes and scales.
There are some generalizations of the idea of shearlets. For example, in [7] and [8] Dahlke et al. extended shearlet theory to n-dimensional signals. Also, considering shearlet group as a semidirect product group, it turns out that its standard representation is the quasiregular representation, a fact useful to characterize irreducible subrepresentations of the shearlet group [3]. In addition, we found out that there are groups in both mathematics and physics that can be regarded as 3-fold semidirect product groups. Moreover, there may be useful transforms that act on the phase space as trivariate functions, so if there is no possibility to consider the parameter space as a 3-fold direct product group, one can regard it as a 3-fold semidirect product group and utilize harmonic analysis.
In a forthcoming paper [4], we have considered the shearlet group as a 3-fold semidirect product Lie group and were able to compute its Lie algebra.
In this paper we go beyond the Euclidian space and define shearlet group and shearlet transform relative to locally compact groups.

2 Preliminaries and notations

Fix two locally compact groups H and K, and let $h \mapsto \tau_h$ be a homomorphism of H into the group of automorphisms of K denoted by $Aut(K)$. Also assume that the mapping $(h,k) \mapsto \tau_h(k)$ from $H \times K$ onto K is continuous. Then the set $H \times K$ endowed with the product topology and the operations:

$$(h,k)(h',k') = (hh', k\tau_h(k'))$$

$$(h,k)^{-1} = (h^{-1}, \tau_{h^{-1}}(k^{-1})),$$

is a locally compact group. This group is called the *semidirect product* of H and K, respectively, and is denoted by $H \times_\tau K$.
From [2, Lemma 1.4.3] or (15.29) of [16], we have:

Lemma 2.1. *Let $d\mu_H(h)$ and $d\mu_K(k)$ be the left Haar measures of H and K respectively, and let $G = H \times_\tau K$. Then the left Haar measure of G is $d\mu_G(h,k) = \delta(h)d\mu_H(h)d\mu_K(k)$. Where δ is a positive continuous homomorphism of H and is given by $d\mu_K(k) = \delta(h)d\mu_K(\tau_h(k))$.*

There is a standard unitary representation U of $G = H \times_\tau K$ on the Hilbert space $L^2(K)$ which can be constructed in the following way:

$$U(h,k)f(y) = \delta(h)^{\frac{1}{2}} f(\tau_{h^{-1}}(yk^{-1})),$$

in which, $f \in L^2(K), (h,k) \in G, y \in K$.
This representation is called *quasiregular representation* and in general is not irreducible.

3 Main results

Let H, K and L be three locally compact groups where L is also Abelian. Moreover assume $\tau : H \to Aut(K)$ and $\lambda : H \times_\tau K \to Aut(L)$ are homomorphisms which, $(h,k) \mapsto \tau_h(k)$ from $H \times K$ onto K and $((h,k),l) \mapsto \lambda_{(h,k)}(l)$ from $H \times K \times L$ onto L are continuous.

Definition 3.1. With the notations as above, any locally compact group in the form of $(H \times_\tau K) \times_\lambda L$ denoted by $\mathcal{S}$, is called an *abstract shearlet group* associated to homomorphisms τ and λ, respectively.

By virtue of lemma 2.1, one can compute the left Haar measure of an abstract shearlet group:

Lemma 3.2. *The left Haar measure of $\mathcal{S} = (H \times_\tau K) \times_\lambda L$ is:*

$$d\mu_{\mathcal{S}}(h,k,l) = \delta_\lambda(h,k)\delta_\tau(h)d\mu_H(h)d\mu_K(k)d\mu_L(l),$$

in which, $d\mu_H(h), d\mu_K(k)$ and $d\mu_L(l)$ are, the left Haar measures of the locally compact groups H, K and L, respectively and δ_λ, $\delta_\tau$are given by:

$$d\mu_K(k) = \delta_\tau(h)d\mu_K(\tau_h(k)) \quad d\mu_L(l) = \delta_\lambda(h,k)d\mu_L(\lambda_{(h,k)}(l)).$$

Let U be the quasiregular representation associated with abstract shearlet group $\mathcal{S}$, namely, for $\psi \in L^2(L)$

$$U(h,k,l)\psi(x) = \delta_\lambda(h,k)^{\frac{1}{2}}\psi(\lambda_{(h,k)^{-1}}(xl^{-1})).$$

A function $\psi \in L^2(L)$ is called *admissible* or *abstract shearlet* if, for any $f \in L^2(L)$, the function $(h,k,l) \mapsto \langle f, U(h,k,l)\psi\rangle$ belongs to $L^2(\mathcal{S})$.

Definition 3.3. Let ψ be an abstract shearlet. For $f \in L^2(L)$, the *abstract shearlet transform* of f with respect to ψ is the mapping:

$$\mathcal{SH}_\psi f : \mathcal{S} \longrightarrow \mathbb{C}$$

$$\mathcal{SH}_\psi f(h,k,l) = \langle f, U(h,k,l)\psi\rangle.$$

So, ψ is an abstract shearlet if $\mathcal{SH}_\psi f \in L^2(\mathcal{S})$.

Let $\mathcal{S}$ be an abstract shearlet group associated to homomorphisms τ and λ. Shifting parentheses in the definition of $\mathcal{S}$ yields a new isomorphic locally compact group, say $\widetilde{\mathcal{S}}$, of course associated to new homomorphisms $\widetilde{\tau}$ and $\widetilde{\lambda}$ as denoted in the following theorem:

Theorem 3.4. *The following mapping is an isomorphism of topological groups:*

$$\Omega : H \times_{\widetilde{\tau}} (K \times_{\widetilde{\lambda}} L) \to (H \times_{\tau} K) \times_{\lambda} L$$

$$(h,(k,l)) \mapsto ((h,k),l),$$

in which, $\widetilde{\tau} : H \to Aut(K \times_{\widetilde{\lambda}} L)$ *and* $\widetilde{\lambda} : K \to Aut(L)$ *are homomorphisms defined by*

$$\widetilde{\tau}_h(k,l) = (\tau_h(k), \lambda_{(h,1_K)}(l))$$

$$\widetilde{\lambda}_k(l) = \lambda_{(1_H,k)}(l).$$

Proof. Since τ and λ are homomorphisms, it is easy to check that $\widetilde{\tau}$ and $\widetilde{\lambda}$ are homomorphisms. Moreover, in group $\widetilde{\mathcal{S}} = H \times_{\widetilde{\tau}} (K \times_{\widetilde{\lambda}} L)$ we have

$$\begin{aligned}
(h,(k,l))(h',(k',l')) &= (hh',(k,l)(\widetilde{\tau}_h(k',l')) \\
&= (hh',(k,l)(\tau_h(k'),\lambda_{(h,1_K)}(l'))) \\
&= (hh',(k\tau_h(k'),l\widetilde{\lambda}_k(\lambda_{(h,1_K)}(l')))) \\
&= (hh',(k\tau_h(k'),l\lambda_{(1_H,k)}(\lambda_{(h,1_K)}(l')))) \\
&= (hh',(k\tau_h(k'),l\lambda(h,k)(l'))).
\end{aligned}$$

This implies that Ω is homomorphism. Indeed:

$$\begin{aligned}
\Omega((h,(k,l))(h',(k',l'))) &= ((hh',k\tau_h(k')),l\lambda(h,k)(l')) \\
&= ((h,k),l)((h',k'),l').
\end{aligned}$$

Besides, it is evident that Ω is a continuous bijection. ∎

Remark: One should be noted that not any $\widetilde{\tau}$ and $\widetilde{\lambda}$ are satisfying theorem 3.4. Indeed, if so does, then $\widetilde{\tau}_h(k,l) = (\alpha,\beta)$ and $\widetilde{\lambda}_k = \Gamma$ imply $\tau_h(k) = \alpha$ and $\lambda_{(h,k)}(l) = \Gamma(\beta)$; So, as an example, for the Heisenberg group $\mathbb{H}(G) = G \times_{\widetilde{\tau}} (\widehat{G} \times_{\widetilde{\lambda}} \mathbb{T})$ with $\widetilde{\tau}_g(\omega,z) = (\omega, z\omega(g))$ and $\widetilde{\lambda}_\omega(z) = z$, we can deduce $\tau_g(\omega) = \omega$ and $\lambda_{(g,\omega)}(z) = z\omega(g)$. But it is obvious that λ and $\lambda_{(g,\omega)}$ are not homomorphisms.

The left Haar measure of $\widetilde{\mathcal{S}}$ can be computed as:

Theorem 3.5. *The left Haar measure of* $\widetilde{\mathcal{S}} = H \times_{\widetilde{\tau}} (K \times_{\widetilde{\lambda}} L)$ *is related to that of* $\mathcal{S}$ *as:*

$$d\mu_{\widetilde{\mathcal{S}}}(h,k,l) = \delta_\tau(h^{-1}) d\mu_{\mathcal{S}}(h, \tau_{h^{-1}}(k), l).$$

Proof. we have

$$d\mu_{\mathcal{S}}((h,k),l) = \delta_\lambda(h,k)\delta_\tau(h) d\mu_H(h) d\mu_K(k) d\mu_L(l)$$

$$d\mu_{\widetilde{\mathcal{S}}}(h,(k,l)) = \delta_{\widetilde{\tau}}(h)\delta_{\widetilde{\lambda}}(k) d\mu_H(h) d\mu_K(k) d\mu_L(l),$$

in which,

$$d\mu_K(k) = \delta_\tau(h) d\mu_K(\tau_h(k)) \tag{3.1}$$

$$d\mu_L(l) = \delta_\lambda(h,k) d\mu_L(\lambda_{(h,k)}(k)) \tag{3.2}$$

$$d\mu_{K\times_{\widetilde{\lambda}}L}(k,l) = \delta_{\widetilde{\tau}}(h)d\mu_{K\times_{\widetilde{\lambda}}L}(\widetilde{\tau}_h(k,l)) \tag{3.3}$$

$$d\mu_L(l) = \delta_{\widetilde{\lambda}}(k)d\mu_L(\widetilde{\lambda}_k(l)) = \delta_{\widetilde{\lambda}}(k)d\mu_L(\lambda_{(1_H,k)}(l)). \tag{3.4}$$

Equation (3.4) for $l := \lambda_{(h,1_K)}(l)$ reads:

$$\begin{aligned} d\mu_L(\lambda_{(h,1_K)}(l)) &= \delta_{\widetilde{\lambda}}(k)d\mu_L(\lambda_{(1_H,k)}(\lambda_{(h,1_K)}(l))) && (3,4') \\ &= \delta_{\widetilde{\lambda}}(k)d\mu_L(\lambda_{(h,k)}(l)) \\ &= \delta_{\widetilde{\lambda}}(k)\delta_{\lambda}^{-1}(h,k)d\mu_L(l). && \text{(from(3.2))} \end{aligned}$$

Now from (3.3), we have

$$\begin{aligned} \delta_{\widetilde{\lambda}}(k)d\mu_K(k)d\mu_L(l) &= \delta_{\widetilde{\tau}}(h)d\mu_{K\times_{\widetilde{\lambda}}L}(\tau_h(k),\lambda_{(h,1_K)}(l)) \\ &= \delta_{\widetilde{\tau}}(h)\delta_{\widetilde{\lambda}}(\tau_h(k))d\mu_K(\tau_h(k))d\mu_L(\lambda_{(h,1_K)}(l))) \\ &= \delta_{\widetilde{\tau}}(h)\delta_{\widetilde{\lambda}}(\tau_h(k))\delta_{\tau}^{-1}(h)d\mu_K(k)\delta_{\widetilde{\lambda}}(k)\delta_{\lambda}^{-1}(h,k)d\mu_L(l). \\ & \qquad \text{(from (3.1) and (3.4'))} \end{aligned}$$

Finally we deduce

$$\delta_{\lambda}(h,k)\delta_{\tau}(h) = \delta_{\widetilde{\tau}}(h)\delta_{\widetilde{\lambda}}(\tau_h(k)).$$

In particular for $k := \tau_{h^{-1}}(k)$, we have

$$d\mu_{\widetilde{\mathcal{S}}}(h,k,l) = \delta_{\tau}(h^{-1})d\mu_{\mathcal{S}}(h,\tau_{h^{-1}}(k),l). \qquad \blacksquare$$

Hereafter, we investigate properties of abstract shearlet transform which generalize some classic results. In particular, the abstract shearlet transform acts on translated signals as the following proposition shows.

Proposition 3.6. *Based on the notations as above,*

$$\mathcal{SH}_{\psi}[L_x f](h,k,l) = \mathcal{SH}_{\psi}f(h,k,x^{-1}l) = [L_{(1_H,1_K,x)}o\mathcal{SH}_{\psi}]f(h,k,l).$$

Proof. Since L_x is a unitary operator and $L_{x^{-1}}$ is its adjoint, we have

$$\begin{aligned} \mathcal{SH}_{\psi}[L_x f](h,k,l) &= \langle L_x f, U(h,k,l)\psi\rangle \\ &= \langle f, L_{x^{-1}}U(h,k,l)\psi\rangle \\ &= \int_L f(t)U(h,k,l)\psi(xt)dt \\ &= \int_L f(t)\delta_{\lambda}^{\frac{1}{2}}(h,k)\psi(\lambda_{(h,k)^{-1}}(xtl^{-1}))dt \\ &= \int_L f(t)\delta_{\lambda}^{\frac{1}{2}}(h,k)\psi(\lambda_{(h,k)^{-1}}(t(x^{-1}l)^{-1}))dt \\ &= \langle f, U(h,k,x^{-1}l)\psi\rangle \\ &= \mathcal{SH}_{\psi}f(h,k,x^{-1}l). \end{aligned}$$

But on the other hand,

$$\mathcal{SH}_{\psi}f(h,k,x^{-1}l) = \mathcal{SH}_{\psi}f((1_H,1_K,x)^{-1}(h,k,l)) = [L_{(1_H,1_K,x)}o\mathcal{SH}_{\psi}]f(h,k,l). \qquad \blacksquare$$

For $(h,k) \in H \times_\tau K$, define the (unitary) operator $Q_{(h,k)}$ on $L^2(\widehat{L})$ by

$$Q_{(h,k)}\widehat{\psi}(\xi) = \delta_\lambda^{-\frac{1}{2}}(h,k)\widehat{\psi}(\xi o \lambda_{(h,k)}).$$

By virtue of $Q_{(h,k)}$, we link the abstract shearlet transform to signals living on frequency domain. More precisely,

Proposition 3.7. *With the notations as above, we have*

$$\mathcal{SH}_\psi f(h,k,l) = \langle \widehat{f}, M_l Q_{(h,k)}\widehat{\psi}\rangle,$$

in which, M_l is the modulation operator defined by $[M_l\widehat{\psi}](\xi) = \overline{\xi(l)}\widehat{\psi}(\xi)$.

Proof. For $(h,k) \in H \times K$, assume $P_{(h,k)}$ is the operator on $L^2(L)$ defined by $P_{(h,k)}\psi(x) = \delta_\lambda^{\frac{1}{2}}(h,k)\psi(\lambda_{(h,k)^{-1}}(x))$. Then we can rewrite the abstract shearlet transform as

$$\mathcal{SH}_\psi f(h,k,l) = \langle f, T_l P_{(h,k)}\psi\rangle,$$

in which, $T_l\psi(x) = \psi(l^{-1}x)$. Now by applying the Plancherel theorem, we have

$$\mathcal{SH}_\psi f(h,k,l) = \langle \widehat{f}, M_l \widehat{P_{(h,k)}\psi}\rangle,$$

where, $\widehat{P_{(h,k)}\psi}$ can be computed as

$$\begin{aligned}
\widehat{P_{(h,k)}\psi}(\xi) &= \int_L P_{(h,k)}\psi(l)\overline{\xi(l)}dl \\
&= \delta_\lambda^{\frac{1}{2}}(h,k)\int_L \psi(\lambda_{(h,k)^{-1}}(l))\overline{\xi(l)}dl \\
&= \delta_\lambda^{-\frac{1}{2}}(h,k)\int_L \psi(l)\overline{\xi(\lambda_{(h,k)}(l))}dl \\
&= \delta_\lambda^{-\frac{1}{2}}(h,k)\int_L \psi(l)\overline{\xi o \lambda_{(h,k)}(l)}dl \\
&= \delta_\lambda^{-\frac{1}{2}}(h,k)\widehat{\psi}(\xi o \lambda_{(h,k)}) \\
&= Q_{(h,k)}\widehat{\psi}.
\end{aligned}$$

So we have, $\mathcal{SH}_\psi f(h,k,l) = \langle \widehat{f}, M_l Q_{(h,k)}\widehat{\psi}\rangle$. ∎

Let U be the quasiregular representation of abstract shearlet group $\mathcal{S}$ on the Hilbert space $L^2(L)$ and $F : L^2(L) \to L^2(\widehat{L})$ be the Fourier transform, that is unitary by Plancherel theorem. Then $\widehat{U}$ defined by $\widehat{U} = FUF^{-1}$ is a unitary representation of shearlet group on the Hilbert space $L^2(\widehat{L})$ which is unitarily equivalent to U.
We observe that for any $(h,k,l) \in \mathcal{S}$, $[U(h,k,l)\psi]^\wedge = \widehat{U}(h,k,l)\widehat{\psi}$. So we deduce:

$$\mathcal{SH}_\psi f(h,k,l) = \langle \widehat{f}, \widehat{U}(h,k,l)\widehat{\psi}\rangle.$$

Moreover, by virtue of proposition 3.7 we have, $\widehat{U}(h,k,l) = M_l Q_{(h,k)}$.

We can write abstract shearlet transform as a convolution in the following manner:

Proposition 3.8. *We have:*

$$\mathcal{SH}_\psi f(h,k,l) = [f * U(h,k,1_L)\widetilde{\psi}](l),$$

in which, $\widetilde{\psi}(x) = \overline{\psi}(x^{-1})$.

Proof. Indeed with the notations as above, we have

$$\begin{aligned}
[f * U(h,k,1_L)\widetilde{\psi}](l) &= \int_L f(lt)U(h,k,1_L)\widetilde{\psi}(t^{-1})dt \\
&= \int_L f(lt)\delta_\lambda^{\frac{1}{2}}(h,k)\widetilde{\psi}(\lambda_{(h,k)^{-1}}(t^{-1}))dt \\
&= \int_L f(lt)\delta_\lambda^{\frac{1}{2}}(h,k)\overline{\psi}(\lambda_{(h,k)^{-1}}(t))dt \\
&= \int_L f(t)\delta_\lambda^{\frac{1}{2}}(h,k)\overline{\psi}(\lambda_{(h,k)^{-1}}(l^{-1}t))dt \qquad (t \mapsto l^{-1}t) \\
&= \int_L f(t)\overline{U(h,k,l)\psi}(t)dt \\
&= \langle f, U(h,k,l)\psi\rangle = \mathcal{SH}_\psi f(h,k,l).
\end{aligned}$$

■

Similar to any "good" transform in the setting of applied harmonic analysis, the abstract shearlet transform is an isometry, as the following theorem shows:

Theorem 3.9. $\psi \in L^2(L)$ *is admissible if and only if,*

$$C_\psi = \int_{H\times_\tau K} |\widehat{\psi}(\gamma o \lambda_{(h,k)})|^2 \delta_\tau(h) d\mu_H(h) d\mu_K(k) < \infty,$$

for almost every $\gamma \in \widehat{L}$. *If so, then* $C_\psi^{-\frac{1}{2}}\mathcal{SH}_\psi : L^2(L) \to L^2(\mathcal{S})$ *is an isometry.*

Proof. We have

$$\begin{aligned}
\|\mathcal{SH}_\psi f\|^2_{L^2(\mathcal{S})} &= \int_H\int_K\int_L |\mathcal{SH}_\psi f(h,k,l)|^2\delta_\tau(h)\delta_\lambda(h,k) dl dk dh \\
&= \int_H\int_K\int_L |(f * U(h,k,1_L)\widetilde{\psi})(l)|^2 dl \delta_\tau(h)\delta_\lambda(h,k) dk dh \\
&= \int_H\int_K\int_{\widehat{L}} |\widehat{f}(\xi)|^2 |\widehat{U}(h,k,1_L)\widehat{\widetilde{\psi}}(\xi)|^2 d\xi \delta_\tau(h)\delta_\lambda(h,k) dk dh \\
&= \int_H\int_K\int_{\widehat{L}} |\widehat{f}(\xi)|^2 |\delta_\lambda^{\frac{1}{2}}(h,k) Q_{(h,k)}\widehat{\widetilde{\psi}}(\xi)|^2 d\xi \delta_\tau(h) dk dh \\
&= \int_{\widehat{L}} |\widehat{f}(\xi)|^2 \int_{H\times_\tau K} |\widehat{\widetilde{\psi}}(\gamma o \lambda_{(h,k)})|^2 \delta_\tau(h) dk dh d\xi \\
&= \int_{\widehat{L}} |\widehat{f}(\xi)|^2 \int_{H\times_\tau K} |\widehat{\overline{\psi}}(\gamma o \lambda_{(h,k)})|^2 \delta_\tau(h) dk dh d\xi \\
&= \int_{\widehat{L}} |\widehat{f}(\xi)|^2 \int_{H\times_\tau K} |\widehat{\psi}(\gamma o \lambda_{(h,k)})|^2 \delta_\tau(h) dk dh d\xi \\
&= \int_{\widehat{L}} |\widehat{f}(\xi)|^2 C_\psi d\xi.
\end{aligned}$$

So $C_\psi < \infty$ yields two outcomes: first, $\|\mathcal{SH}_\psi f\|^2_{L^2(\mathcal{S})} < \infty$ which in turn means ψ is admissible and second, $\|\mathcal{SH}_\psi f\| = C_\psi^{\frac{1}{2}}\|f\|$. ■

4 Examples

1. The first and the most important example is the shearlet group introduced in [20]. As mentioned before, we showed that the shearlet group $\mathbb{S} = \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ is a type of abstract shearlet group. More precisely, $\mathbb{S} = (\mathbb{R}^+ \times_\tau \mathbb{R}) \times_\lambda \mathbb{R}^2$, in which, τ and λ are given by $\tau_a(s) = \sqrt{a}s$ and $\lambda_{(a,s)}(t) = S_s A_a t$, where $S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ and $A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$ are respectively, shear and anisotropic dilation matrices. Also in [3, Theorem 3.8] we showed how the admissibility condition for two dimensional shearlets reduces to $\int_{\mathbb{R}^2} \frac{|\widehat{\psi}(\lambda,\eta)|^2}{\lambda^2} d\lambda d\eta < \infty$.
Shearlet group is isomorphic to a group $\widetilde{\mathbb{S}} = \mathbb{R}^+ \times_{\widetilde{\tau}} (\mathbb{R} \times_{\widetilde{\lambda}} \mathbb{R}^2)$, in which, $\widetilde{\tau}$ and $\widetilde{\lambda}$ are given by

$$\widetilde{\tau}_a(s,t) = (\sqrt{a}s, A_a t)$$
$$\widetilde{\lambda}_s(t) = S_s t.$$

The standard representation of shearlet group, namely,

$$\sigma(a,s,t)\psi(x) = \psi_{a,s,t}(x) = a^{-\frac{3}{4}}\psi(A_a^{-1}S_s^{-1}(x-t)),$$

is the quasiregular representation of the semidirect product group $\mathbb{S} = (\mathbb{R}^+ \times_\tau \mathbb{R}) \times_\lambda \mathbb{R}^2$. Indeed,

$$U(a,s,t)\psi(x) = \delta_\lambda^{\frac{1}{2}}(a,s)\psi(\lambda_{(a,s)^{-1}}(x-t)),$$

in which, $\delta_\lambda(a,s)$ is given by:

$$\begin{aligned} d\mu_{\mathbb{R}^2}(t) &= \delta_\lambda(a,s) d\mu_{\mathbb{R}^2}(\lambda_{(a,s)}(t)) \\ &= \delta_\lambda(a,s) d\mu_{\mathbb{R}^2}(S_s A_a t). \end{aligned}$$

By (2.44) of [11] it is known that for any $T \in GL(n,\mathbb{R})$, $d\mu_{\mathbb{R}^n}(Tt) = |detT| d\mu_{\mathbb{R}^n}(t)$. This yields $\delta_\lambda(a,s) = a^{-\frac{3}{2}}$. So

$$\begin{aligned} U(a,s,t)\psi(x) &= a^{-\frac{3}{4}}\psi(\lambda_{(\frac{1}{a},\tau_{\frac{1}{a}}(-s))}(x-t)) \\ &= a^{-\frac{3}{4}}\psi(S_{\tau_{\frac{1}{a}}(-s)} A_{\frac{1}{a}}(x-t)) \\ &= a^{-\frac{3}{4}}\psi(A_a^{-1}S_s^{-1}(x-t)) \\ &= \sigma(a,s,t)\psi(x). \end{aligned}$$

2. Three dimensional shearlets are very useful tools for analyzing signals with three dimensional domain such as seismic waves. We utilize the approach taken in [7] (see also [12] and [15]) and investigate the locally compact group $\mathbb{S}_3 = (\mathbb{R}^* \times_\tau \mathbb{R}^2) \times_\lambda \mathbb{R}^3$ in which, τ and λ are given by $\tau_a(s) = a^{\frac{2}{3}}s$ and $\lambda_{(a,s)}(t) = S_s A_a t$, where for $s = (s_1, s_2)$, S_s and A_a are given by:

$$S_s = \begin{pmatrix} 1 & s_1 & s_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_a = \begin{pmatrix} a & 0 & 0 \\ 0 & \sqrt[3]{a} & 0 \\ 0 & 0 & \sqrt[3]{a} \end{pmatrix}.$$

Since $det(S_s A_a) = a^{\frac{5}{3}}$, so $\delta_\lambda(a,s) = a^{-\frac{5}{3}}$, then, similar to the preceding example, one can compute the quasiregular representation of $\mathbb{S}_3$ as:

$$\begin{aligned} U(a,s,t)\psi(x) &= a^{-\frac{5}{6}}\psi(\lambda_{(\frac{1}{a},\tau_{\frac{1}{a}}(-s))}(x-t)) \\ &= a^{-\frac{5}{6}}\psi(S_{\tau_{\frac{1}{a}}(-s)}A_{\frac{1}{a}}(x-t)) \\ &= a^{-\frac{5}{6}}\psi(A_a^{-1}S_s^{-1}(x-t)). \end{aligned}$$

This is precisely the representation defined in [7].
$\mathbb{S}_3$ is isomorphic to $\widetilde{\mathbb{S}}_3 = \mathbb{R}^* \times_{\widetilde{\tau}} (\mathbb{R}^2 \times_{\widetilde{\lambda}} \mathbb{R}^3)$, in which, $\widetilde{\tau}_a(s,t) = (a^{\frac{2}{3}}s, A_a t)$, and $\widetilde{\lambda}_s(t) = S_s t$.
3. Consider the *Galilei space-time* $\mathbb{R}^4 \cong \mathbb{R}^3 \times \mathbb{R}$ as the space of pairs (q,t) describing *events* in a four dimensional (nonrelativistic) space-time. Here q and t stand, respectively, for the position and (absolute) time of the event.
There are four types of symmetries of this space-time:

$$\begin{aligned} &a)\ \textit{Movements with constant velocity } v: && (q,t) \mapsto (q+vt,t) \\ &b)\ \textit{Rotations by } A \in SO_3(\mathbb{R}): && (q,t) \mapsto (Aq,t) \\ &c)\ \textit{Space translations}: && (q,t) \mapsto (q+p,t) \\ &d)\ \textit{Time translations}: && (q,t) \mapsto (q,t+s) \end{aligned}$$

All these maps are affine maps on $\mathbb{R}^4$. We recall that the subgroup $\Gamma \subset \mathit{Aff}_4(\mathbb{R})$ generated by these four maps is called *proper Galilei group*. Γ is the natural symmetry group of nonrelativistic mechanics. As it is shown in [17, Example 2.2.6], Γ is isomorphic to the abstract shearlet group:

$$\mathcal{G} = (SO_3(\mathbb{R}) \times_\tau \mathbb{R}^3) \times_\lambda \mathbb{R}^4,$$

in which, $\tau_A(v) = Av$, and $\lambda_{(A,v)}(q,t) = (Aq+vt,t)$.
In the matrix form, the action of $\lambda_{(A,v)}$ on (q,t) reads: $\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q \\ t \end{pmatrix}$,
this implies $\delta_\lambda(A,v) = |det(A)|^{-1} = 1$. So the quasiregular representation of this abstract shearlet group reduces to:

$$\begin{aligned} U &: \mathcal{G} \to \mathcal{U}(L^2(\mathbb{R}^4)) \\ U(A,v,(q,t))\psi(p,s) &= \psi(A^{-1}(p-q-(s-t)v), s-t). \end{aligned}$$

4. The similitude group of plane defined by

$$Sim_2(\mathbb{R}) = (\mathbb{R}^+ \times_\tau SO_2(\mathbb{R})) \times_\lambda \mathbb{R}^2,$$

in which, for any $(a,M,t) \in Sim_2(\mathbb{R})$, $\tau_a(M) = M$ and $\lambda_{(a,M)}(t) = aMt$, is a kind of abstract shearlet group. This group is the group theoretical framework of two dimensional continuous wavelet transform. Its quasiregular representation reads

$$U(a,M,t)\psi(y) = \frac{1}{a}\psi(\frac{1}{a}M^{-1}(y-t)) = T_t D_{aM}\psi(y),$$

in which, D and T are respectively, the standard dilation and translation operators on $L^2(\mathbb{R}^2)$. So the abstract shearlet transform on $Sim_2(\mathbb{R})$ is given by

$$\mathcal{SH}_\psi f : Sim_2(\mathbb{R}) \to \mathbb{C}$$

$$\mathcal{SH}_\psi f(a, M, t) = \langle f, T_t D_{aM} \psi \rangle.$$

The group $Sim_2(\mathbb{R})$ is isomorphic to a group given by

$$G = \mathbb{R}^+ \times_{\widetilde{\tau}} (SO_2(\mathbb{R}) \times_{\widetilde{\lambda}} \mathbb{R}^2),$$

in which, $\widetilde{\tau}_a(M, t) = (M, at)$ and $\widetilde{\lambda}_M(t) = Mt$.

5. Consider the abstract shearlet group $\mathcal{S} = (\mathbb{Z}_2 \times_\tau \mathbb{Z}_n) \times_\lambda \mathbb{Z}_n$, in which, τ and λ are given by:

$$\tau_0(r) = r, \quad \tau_1(r) = -r$$

$$\lambda_{(0,r)}(s) = r + s, \quad \lambda_{(1,r)}(s) = r - s.$$

$\mathcal{S}$ is isomorphic to the locally compact group $\widetilde{\mathcal{S}} = \mathbb{Z}_2 \times_{\widetilde{\tau}} (\mathbb{Z}_n \times_{\widetilde{\lambda}} \mathbb{Z}_n)$, in which, $\widetilde{\tau}$ and $\widetilde{\lambda}$ are given by:

$$\widetilde{\tau}_0(r, s) = (r, s), \quad \widetilde{\tau}_1(r, s) = (-r, -s)$$

$$\widetilde{\lambda}_r(s) = r + s.$$

The quasiregular representation of $\mathcal{S}$ can be computed as:

$$U(k, r, s)\psi(x) = \delta_\lambda^{\frac{1}{2}}(k, r)\psi(\lambda_{(k,r)^{-1}}(x - s)),$$

in which, $k \in \mathbb{Z}_2, r, s, x \in \mathbb{Z}_n, \psi \in L^2(\mathbb{Z}_n)$ and $\delta_\lambda(k, r)$ is given by:

$$d\mu_{\mathbb{Z}_n}(s) = \delta_\lambda(k, r) d\mu_{\mathbb{Z}_n}(\lambda_{(k,r)}(s)).$$

But, we have $(0, r)^{-1} = (0, -r)$, $(1, r)^{-1} = (0, r)$. Moreover, for $k = 0, 1$ we have $\delta_\lambda(k, r) = 1$, no matter counting measure or discrete uniform distribution is assumed on $\mathbb{Z}_n$ as Haar measure. So, finally we have:

$$U(0, r, s)\psi(x) = \psi(\lambda_{(0,-r)}(x - s)) = \psi(x - s - r),$$

$$U(1, r, s)\psi(x) = \psi(\lambda_{(1,r)}(x - s)) = \psi(s - x + r).$$

6. Consider these three locally compact groups: $(2^{\mathbb{Z}}, .)$, $(\mathbb{R}, +)$ and *Klein 4-group* $K_4 = \langle a, b | a^2 = b^2 = (ab)^2 = 1 \rangle$. It is well known that K_4 is Abelian and $Aut(K_4) \cong S_3$. Now let $\mathcal{S}$ be the abstract shearlet group $\mathcal{S} = (2^{\mathbb{Z}} \times_\tau \mathbb{R}) \times_\lambda K_4$, in which, for $2^j \in 2^{\mathbb{Z}}$ we have $\tau_{2^j}(x) = 2^j x$ and λ is given by permutations:

$$\lambda_{(1,0)} = \begin{pmatrix} a & b & ab \\ a & b & ab \end{pmatrix}$$

$$\lambda_{(2^j,x)} = \begin{pmatrix} a & b & ab \\ ab & a & b \end{pmatrix}, \quad if \quad either \quad j = 0 \quad or \quad x = 0$$

$$\lambda_{(2^j,x)} = \begin{pmatrix} a & b & ab \\ a & ab & b \end{pmatrix}, \qquad if \qquad j > 0, x > 0$$

$$\lambda_{(2^j,x)} = \begin{pmatrix} a & b & ab \\ ab & b & a \end{pmatrix}, \qquad if \qquad j > 0, x < 0$$

$$\lambda_{(2^j,x)} = \begin{pmatrix} b & a & ab \\ a & b & ab \end{pmatrix}, \qquad if \qquad j < 0, x > 0$$

$$\lambda_{(2^j,x)} = \begin{pmatrix} a & b & ab \\ b & ab & a \end{pmatrix}, \qquad if \qquad j < 0, x < 0.$$

$\mathcal{S}$ is isomorphic to the locally compact group $\widetilde{\mathcal{S}} = 2^{\mathbb{Z}} \times_{\widetilde{\tau}} (\mathbb{R} \times_{\widetilde{\lambda}} K_4)$, in which, $\widetilde{\tau}$ and $\widetilde{\lambda}$ are given by:

$$\widetilde{\tau}_{2^j}(x,a) = (2^j x, ab), \quad \widetilde{\tau}_{2^j}(x,b) = (2^j x, a) \qquad for \quad j \neq 0$$

$$\widetilde{\lambda}_x = \begin{pmatrix} a & b & ab \\ ab & a & b \end{pmatrix}, \qquad for \quad x \neq 0.$$

For $2^j \in 2^{\mathbb{Z}}$, $x \in \mathbb{R}, 1, a, b, ab \in K_4$ and $\psi \in L^2(K_4)$ the quasiregular representation of $\mathcal{S}$ reads as:

$$U(2^j,x,1)\psi(1) = U(2^j,x,a)\psi(a) = U(2^j,x,b)\psi(b) = U(2^j,x,ab)\psi(ab) = \psi(1),$$

$$\begin{aligned} U(2^j,x,1)\psi(a) = U(2^j,x,a)\psi(1) = U(2^j,x,b)\psi(ab) &= U(2^j,x,ab)\psi(b) \\ &= \psi(\lambda_{(2^{-j},-2^{-j}x)}(a)), \end{aligned}$$

$$\begin{aligned} U(2^j,x,1)\psi(b) = U(2^j,x,a)\psi(ab) = U(2^j,x,b)\psi(1) &= U(2^j,x,ab)\psi(a) \\ &= \psi(\lambda_{(2^{-j},-2^{-j}x)}(b)), \end{aligned}$$

$$\begin{aligned} U(2^j,x,1)\psi(ab) = U(2^j,x,a)\psi(b) = U(2^j,x,b)\psi(a) &= U(2^j,x,ab)\psi(1) \\ &= \psi(\lambda_{(2^{-j},-2^{-j}x)}(ab)). \end{aligned}$$

One should note that $\delta_\lambda(2^j,x) = 1$, no matter counting measure or discrete uniform distribution is assumed on K_4 as Haar measure.

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