Functional differential inequalities with partial derivatives

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Abstract

Initial problems for Hamilton–Jacobi functional differential equations and initial boundary value problems of the Dirichlet type for parabolic equations are considered. It is proved that classical solutions of functional differential equations can be estimated by solutions of initial problems for ordinary functional differential equations. Theorems on the uniqueness of solutions are obtained as consequences of comparison results. A method of differential inequalities is used. Here, the involved operators do not satisfy the Volterra condition.

1 Introduction

Two types of results on partial differential inequalities are taken into considerations in the literature. The first type allows to estimate a function of several variables be means of another function of several variables, while the second one, the so called comparison theorems, give estimates of functions of several variables by means of functions of one variable. Functions of one variable are solutions of initial problems for ordinary differential equations.

In this paper we deal with comparison theorems corresponding to functional differential equations of Hamilton–Jacobi type and for parabolic functional differential problems.

The papers [6] and [23] initiated the theory of first order partial differential inequalities. The main result, known as the Haar–Ważewski inequality, shows

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that a function of several variables which is of class C^1 on the Haar pyramid and satisfies a linear differential inequality can be estimated by a solution of a suitable initial value problem for an ordinary differential equation. Uniqueness results for initial problems on the Haar pyramid and estimates of solutions are obtained as consequences of the Haar–Ważewski theorem.

There are many generalizations and extensions of the above classical result. We list some of them.

The main differential inequality may be nonlinear with respect to an unknown function and consequently, the comparison problem may be nonlinear. The assumption on the regularity of an unknown function which satisfied a differential inequality may be weakened. It is assumed that solutions of differential inequalities admit first order partial derivatives on the Haar pyramid and they are totally differentiable on a subset of the boundary of the Haar pyramid ([14] Chapter VII). For the Cauchy problem on an unbounded domain the assumption of total differentiability can be omitted. Comparison theorems for solutions which are global with respect to spatial variables are presented in [3]. An interesting result on the global uniqueness of the Cauchy problem when the right hand sides of equations satisfy the Hölder condition can be found in [2].

Vector-valued functions satisfying nonlinear systems of partial differential inequalities can be estimated by solutions of initial problems for ordinary differential systems. The above results are presented in the monographs [11] (Chapter IX) and [14], (Chapter VII).

The Haar–Ważewski theorem has been extended to semi classical solutions of nonlinear functional differential inequalities, see the papers [5], [20], [21] and the monograph [22]. Solutions in the Cinquini–Cibrario sense of nonlinear functional differential inequalities have been investigated in [9]. Comparison theorems for generalized entropy solutions of nonlinear differential or functional differential problems are discussed in [8], [10]. Comparison results for initial boundary value problems related to functional differential equations are given in [4]. Differential and functional differential inequalities with Kamke type comparison problems can be found in [1]. Viscosity solutions of functional differential inequalities are developed in [17]. Functional differential versions of the Haar–Ważewski theorem for initial and initial boundary value problems can be found [7] (Chapters I and V).

The classical theory of parabolic differential inequalities was developed in the monographs [11], [14], [19]. Functional differential inequalities of parabolic type and uniqueness results for initial boundary value problems were treated in [12], [13] and [15], [16]. Finite systems of weakly coupled functional differential equations are considered in these paper. Every equation in the system contains the vector of unknown functions as a functional variable and the derivatives of only one function. Partial derivatives appear in a classical sense.

The results on functional differential equations and inequalities presented in the above papers have the following property: it is assumed that given operators satisfy the Volterra condition with respect to time variable.

The aim of this paper is to make further contributions to comparison results for evolution functional differential equations. We consider initial problems for functional differential equations of the Hamilton–Jacobi type and initial boundary value problems of the Dirichlet type for parabolic functional differential equations. We do not assume that given operators satisfy the Volterra condition. We give estimates of solutions and prove theorems on the uniqueness of classical solutions.

The paper is organized as follows. In Section 2 we prove a theorem on ordinary functional differential inequalities. In the next section we give estimates of functions of several variables satisfying functional differential inequalities by means of solutions of suitable initial problems for ordinary differential equations. As a consequence we obtain a general theorem for the uniqueness of a classical solution of the original problem.

In Section 4 we give a result on estimates of solutions of parabolic problems and a theorem on uniqueness of solutions of initial boundary value problems of the Dirichlet type.

Differential equations with deviated variables and integral differential equations are special cases of problems considered here.

2 Ordinary functional differential inequalities

We prove a theorem on ordinary functional differential inequalities. We give estimates of functions of several variables satisfying functional differential inequalities by means of solutions of initial problems for ordinary functional differential equations.

For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions defined on X and taking values in Y. Suppose that $a > 0$ and $b_0 \in \mathbb{R}_+$, $\mathbb{R}_+ = [0,+\infty)$, are fixed and $I = [-b_0 - a, 0]$. For a function $\omega : I \cup [0,a] \to \mathbb{R}$ and for a point $t \in [0,a]$. we define $\omega_t : I \to \mathbb{R}$ as follows: $\omega_t(\tau) = \omega(t+\tau)$, $\tau \in I$. Suppose that $\sigma : [0,a] \times C(I,\mathbb{R}_+) \to \mathbb{R}_+$ and $L : [0,a] \to \mathbb{R}_+$ are given functions. We will say that σ satisfies condition (V_0) if for each $t \in [0,a]$ and for ξ, $\tilde{\xi} \in C(I,\mathbb{R}_+)$ such that $\xi(\tau) = \tilde{\xi}(\tau)$ for $\tau \in [-b_0 - t, 0]$ we have $\sigma(t,\xi) = \sigma(t,\tilde{\xi})$. Condition (V_0) means that the value of σ at (t,ξ) depends on t and on the restriction of ξ to the set $[-b_0 - t, 0]$ only.

Given $\eta \in C([-b_0,0],\mathbb{R}_+)$, we consider the Cauchy problem

$$\omega'(t) = \sigma(t,\omega_t) + L(t)\,\omega(a), \quad \omega(t) = \eta(t) \ \text{ for } \ t \in [-b_0, 0]. \tag{1}$$

We assume that σ satisfies condition (V_0) and we consider classical solutions of (1) and solutions of functional differential inequalities corresponding to (1).

Remark 2.1. *Suppose that $H : [0,a] \times C(I \cup [0,a],\mathbb{R}) \to \mathbb{R}$ and $\mu : I \to \mathbb{R}$ are given functions. Let us consider the Cauchy problem*

$$\omega'(t) = H(t,\omega), \quad \omega(t) = \mu(t) \text{ for } t \in I, \tag{2}$$

where ω is the functional variable. It is clear that (1) *is a special case of* (2). *We will say that H satisfies the Volterra condition if for each $t \in [0,a]$ and for ξ, $\tilde{\xi} \in C(I \cup [0,a],\mathbb{R})$ such that $\xi(\tau) = \tilde{\xi}(\tau)$ for $\tau \in [-b_0 - a, t]$ we have $H(t,\xi) = H(t,\tilde{\xi})$. Note that the Volterra condition is not satisfied for $H(t,\omega) = \sigma(t,\omega_t) + L(t)\omega(a)$.*

The theory of functional differential equations and inequalities presented in [11] (*vol.* II) *concerns equations satisfying the Volterra condition. Until now there have not been any results on functional differential inequalities which do not satisfy the Volterra condition.*

In the present paper we use comparison functional differential equations which do not satisfy the Volterra condition.

For $\xi \in C(I, \mathbb{R})$ and $t \in [0, a]$ we put

$$\|\xi\|_t = \max\{|\xi(\tau)| : \tau \in [-b_0 - t, 0]\}.$$

Lemma 2.2. *If* ζ, $\vartheta \in C([0, a], \mathbb{R})$ *and*

$$D_-\zeta(t) \le \vartheta(t) \text{ for } t \in (0, a],$$

where D_- *is the left hand lower Dini derivative then*

$$\zeta(t) - \zeta(0) \le \int_0^t \vartheta(\tau)\, d\tau, \quad t \in [0, a]. \tag{3}$$

Proof. Write

$$\Delta(t) = \zeta(t) - \int_0^t \vartheta(\tau)\, d\tau, \quad t \in [0, a].$$

Then $D_-\Delta(t) \le 0$ for $t \in (0, a]$. It follows from the Dini theorem ([14], Theorem 2.1) that Δ is nonincreasing on $[0, a]$. This gives $\Delta(t) \le \Delta(0)$ for $t \in [0, a]$ and assertion (3) follows.

Assumption $H[\sigma, L]$. The functions $\sigma : [0, a] \times C(I, \mathbb{R}_+) \to \mathbb{R}_+$, $L : [0, a] \to \mathbb{R}_+$ satisfy the conditions:

(i) σ is continuous, it is nondecreasing with respect to the second variable, and satisfies condition (V_0).

(ii) There is $K \in C([0, a], \mathbb{R}_+)$ such that

$$\sigma(t, \xi) - \sigma(t, \tilde{\xi}) \le K(t)\|\xi - \tilde{\xi}\|_t$$

where $t \in [0, a]$, ξ, $\tilde{\xi} \in C(I, \mathbb{R}_+)$ and $\xi(\tau) \ge \tilde{\xi}$ for $\tau \in I$.

(iii) $L \in C([0, a], \mathbb{R}_+)$ and

$$\int_0^a L(s) \exp\Big\{\int_s^a K(\tau)\, d\tau\Big\}\, ds < 1. \tag{4}$$

The following theorem on functional differential inequalities is important in our considerations.

Theorem 2.3. *Suppose that Assumption* $H[\sigma, L]$ *is satisfied and*

(i) $\varphi \in C([-b_0, a], \mathbb{R}_+)$, φ *is nondecreasing on* $[-b_0, a]$, *and*

$$D_-\varphi(t) \le \sigma(t, \varphi_t) + L(t)\varphi(a) \text{ for } t \in (0, a].$$

(ii) $\eta \in C([-b_0, 0], \mathbb{R}_+)$ *and* $\varphi(t) \le \eta(t)$ *for* $t \in [-b_0, 0]$.

Then there exists exactly one solution $\omega(\,\cdot\,, \eta)$ *of the Cauchy problem* (1). *The solution* $\omega(\,\cdot\,, \eta)$ *is defined on* $[-b_0, a]$ *and*

$$\varphi(t) \le \omega(t, \eta) \text{ for } t \in [0, a]. \tag{5}$$

Proof. Consider the sequence of functions $\{y^{(k)}\}$ defined in the following way. The function $y^{(0)}$ is given by the relations:
(i) $y^{(0)}(t) = \eta(t)$ for $t \in [-b_0, 0]$,
(ii) $y^{(0)}$ is a solution of the Cauchy problem

$$\omega'(t) = K(t)\omega(t) + L(t)\omega(a) + \sigma(t, \theta), \;\; \omega(0) = \eta(0), \tag{6}$$

where $\theta : I \to \mathbb{R}_+$ is given by $\theta(\tau) = 0$ for $\tau \in I$.
If $y^{(k)} : [-b_0, a] \to \mathbb{R}_+$ is a known function then

$$y^{(k+1)}(t) = \eta(t) \text{ for } t \in [-b_0, 0]$$

and

$$y^{(k+1)}(t) = \eta(0) + \int_0^t \sigma(\tau, y^{(k)}_\tau)\, d\tau + y^{(k)}(a) \int_0^t L(s)\, ds \text{ for } t \in [0, a].$$

From (4) we conclude that there exists exactly one $y^{(0)}$ and it is given by

$$y^{(0)}(t) = \eta(0)\, \exp\{\int_0^t K(\tau)\, d\tau\}$$

$$+A \int_0^t L(s)\, \exp\{\int_s^t K(\tau)\, d\tau\}\, ds + \int_0^t \sigma(s, \theta)\, \exp\{\int_s^t K(\tau)\, d\tau\}\, ds$$

where

$$A = \tilde{A}\left[\eta(0)\, \exp\{\int_0^a K(\tau) d\tau\} + \int_0^a \sigma(s, \theta)\, \exp\{\int_s^t K(\tau) d\tau\}\, ds\right],$$

$$\tilde{A} = \left[1 - \int_0^a L(s)\, \exp\{\int_s^a K(\tau)\, d\tau\} ds\right]^{-1}. \tag{7}$$

It follows that the sequence $\{y^{(k)}\}$ is defined on $[-b_0, a]$. It is easily seen that

$$0 \le y^{(k+1)}(t) \le y^{(k)}(t) \text{ for } t \in [0, a] \text{ and } k \ge 0. \tag{8}$$

Now we prove that

$$\varphi(t) \le y^{(k)}(t) \text{ for } t \in [0, a] \text{ and } k \ge 0. \tag{9}$$

For $k = 0$ we have

$$D_- \varphi(t) \le K(t)\varphi(t) + L(t)\varphi(a) + \sigma(t, \theta), \; t \in (0, a],$$

and consequently

$$D_{-}\Big[\varphi(t)\ \exp\{-\int_0^t K(\tau)d\tau\}\Big] \le \big[L(t)\varphi(a)+\sigma(t,\theta)\big]\ \exp\{-\int_0^t K(\tau)d\tau\}.$$

It follows from Lemma 2.2 that

$$\varphi(t) \le \eta(0)\ \exp\{\int_0^t K(\tau)d\tau\} + \varphi(a)\int_0^t L(s)\ \exp\{\int_s^t K(\tau)\}\,ds$$

$$+\int_0^t \sigma(s,\theta)\ \exp\{\int_s^t K(\tau)\}\,ds,\ t\in(0,a].$$

The above inequality and (4) imply the estimate $\varphi(t) \le y^{(0)}(t)$ for $t\in[0,a]$.

Suppose that $k\ge 0$ is fixed and $\varphi(t)\le y^{(k)}(t)$ for $t\in[0,a]$. Then we have

$$D_{-}\varphi(t) \le \sigma(t,y_t^{(k)}) + y^{(k)}(a)L(t),\ t\in(0,a],$$

and consequently

$$\varphi(t) \le \eta(0) + \int_0^t \sigma(\tau,y_\tau^{(k)}) + y^{(k)}(a)\int_0^t L(\tau)d\tau = y^{(k+1)}(t),$$

where $t\in(0,a]$. Then we obtain (9) by induction.

It follows from (8) that there is

$$\lim_{k\to\infty} y^{(k)}(t) = \omega(t,\eta),\ t\in[0,a],$$

and $\omega(\,\cdot\,,\eta)$ satisfies (1). From (9) we obtain (5). Uniqueness of a solution of (1) follows from Assumption $H[\sigma,L]$. This completes the proof.

Remark 2.4. *It is important in classical theorems on functional differential inequalities that given operators satisfy the Volterra condition. Let us consider the following example. The function*

$$\tilde\omega(t) = e^t + \frac{e^a}{2-e^a}\big(e^t-1\big)$$

satisfies the functional differential inequality

$$\omega'(t) \le \omega(t)+\omega(a),\ t\in[0,a] \tag{10}$$

where $a>\ln 4$ and the function

$$\bar\omega(t) = 2e^t + \frac{2e^a}{2-e^a}\big(e^t-1\big)$$

satisfies the functional differential inequality

$$\omega'(t) \ge \omega(t)+\omega(a),\ t\in[0,a]. \tag{11}$$

Note that the right hand sides of (10) and (11) do not satisfy the Volterra condition. Moreover we have $\tilde\omega(0)<\bar\omega(0)$ and the inequality

$$\tilde\omega(t)\le\bar\omega(t),\ \ t\in[0,a]$$

is not satisfied.

3 First order partial functional differential equations

For $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, we write $x < y$ if $x_i < y_i$ for $1 \le i \le n$. In a similar way we define the relation $x \le y$.

Suppose that $\alpha, \beta \in C([-b_0, a], \mathbb{R}^n)$ where $a > 0$, $b_0 \in \mathbb{R}_+$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, and $\alpha(t) < \beta(t)$ for $t \in [-b_0, a]$. Write

$$\Xi_0 = \{(t, x) \in \mathbb{R}^{1+n} : \ t \in [-b_0, 0],\ \alpha(t) \le x \le \beta(t)\},$$

$$\Xi = \{\ (t, x) \in \mathbb{R}^{1+n} : \ t \in [0, a],\ \alpha(t) \le x \le \beta(t)\ \}.$$

Solutions of functional differential equations or inequalities will be defined on $\Xi_0 \cup \Xi$, and Ξ_0 will be an initial set for the Cauchy problem. For $(t, x) \in \Xi$ we put

$$A[t, x] = \{(\tau, y) \in \mathbb{R}^{1+n} : \ (t + \tau, x + y) \in \Xi_0 \cup \Xi\}.$$

Then

$$A[t, x] = \{(\tau, y) \in \mathbb{R}^{1+n} : \ -b_0 - t \le \tau \le -t + a,\\ -x + \alpha(t + \tau) \le y \le -x + \beta(t + \tau)\}.$$

Write

$$B[t, x] = \{(\tau, y) \in A[t, x] : \ \tau \le 0\},\ (t, x) \in \Xi.$$

Set

$$c_i = \min\{\alpha_i(t) : \ t \in [-b_0, a]\},\ d_i = \max\{\beta_i(t) : \ t \in [-b_0, a]\},\ 1 \le i \le n,$$

and $c = (c_1, \ldots, c_n)$, $d = (d_1, \ldots, d_n)$. Write

$$D = [-b_0 - a, a] \times [c - d, d - c],\ \ E = [-b_0 - a, 0] \times [c - d, d - c].$$

Then $A[t, x] \subset D$ and $B[t, x] \subset E$ for $(t, x) \in \Xi$.

Given $z : \Xi_0 \cup \Xi \to \mathbb{R}$ and $(t, x) \in \Xi$, define $z_{(t,x)} : B[t, x] \to \mathbb{R}$ and $z_{[t,x]} : A[t, x] \to \mathbb{R}$ by

$$z_{(t,x)}(\tau, y) = z(t + \tau, x + y) \text{ for } (\tau, y) \in B[t, x], \tag{12}$$

and

$$z_{[t,x]}(\tau, y) = z(t + \tau, x + y) \text{ for } (\tau, y) \in A[t, x]. \tag{13}$$

Put $\Omega = \Xi \times C(E, \mathbb{R}) \times C(D, \mathbb{R}) \times \mathbb{R}^n$ and suppose that $f : \Omega \to \mathbb{R}$ is a given function of the variables (t, x, v, w, q), $x = (x_1, \ldots, x_n)$, $q = (q_1, \ldots, q_n)$. Let us denote by z an unknown function of the variables (t, x). Set

$$\mathbf{F}[z](t, x) = f(t, x, z_{(t,x)}, z_{[t,x]}, \partial_x z(t, x))$$

where $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$. We consider the functional differential equation

$$\partial_t z = \mathbf{F}[z] \tag{14}$$

with the initial condition

$$z = \psi \text{ on } \Xi_0, \tag{15}$$

where $\psi : \Xi_0 \to \mathbb{R}$ is a given function. We will say that f satisfies condition (V) if for each $(t, x, q) \in \Xi \times \mathbb{R}^n$ and for $v, \tilde{v} \in C(E, \mathbb{R})$, $w, \tilde{w} \in C(D, \mathbb{R})$ such that $v(\tau, y) = \tilde{v}(\tau, y)$ for $(\tau, y) \in B[t, x]$ and $w(\tau, y) = \tilde{w}(\tau, y)$ for $(\tau, y) \in A[t, x]$ we have $f(t, x, v, w, q) = f(t, x, \tilde{v}, \tilde{w}, q)$. It is clear that condition (V) means that the value of f at $(t, x, v, w, q) \in \Omega$ depends on (t, x, q) and on the restrictions of v and w to the sets $B[t, x]$ and $A[t, x]$ respectively. We assume that f satisfies condition (V) and we consider classical solutions of (14), (15).

Remark 3.1. *Suppose that $G : \Xi \times C(\Xi_0 \cup \Xi, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ is a given function. Let us consider the functional differential equation*

$$\partial_t z(t, x) = G(t, x, z, \partial_x z(t, x)) \tag{16}$$

where z is the functional variable. It is clear that (14) *is a special case of* (16). *We will say that G satisfies the Volterra condition if for each $(t, x, q) \in \Xi \times \mathbb{R}^n$, and for $z, \tilde{z} \in C(\Xi_0 \cup \Xi, \mathbb{R})$ such that $z(\tau, y) = \tilde{z}(\tau, y)$ for $(\tau, y) \in (\Xi_0 \cup \Xi) \cap ([-b_0, t] \times \mathbb{R}^n)$ we have $G(t, x, z, q) = G(t, x, \tilde{z}, q)$.*

Note that the Volterra condition is not satisfied for equation (14).

The results presented in [1], [4], [5], [8], [9], [17] *have the following limitation: functional differential equations or systems considered in these papers satisfy the Volterra condition. In fact, we are not aware of any previous results pertaining to the functional differential equation* (16) *which do not satisfy the Volterra condition.*

With the above motivation we consider problem (14), (15).

Write

$$\Xi_\star = \{(t, x) \in \Xi : \ t \in (0, a] : \ \alpha(t) < x < \beta(t)\},$$
$$\partial_0 \Xi = \big(\Xi \setminus \Xi_\star\big) \cap \big((0, a] \times \mathbb{R}^n\big).$$

A function $z : \Xi_0 \cup \Xi \to \mathbb{R}$ will be called the function of class $C^\star$ if z is continuous on $\Xi_0 \cup \Xi$ and there exist the derivatives $\partial_t z$, $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$ on $\Xi \cap \big((0, a] \times \mathbb{R}^n\big)$ and z is differentiable on $\partial_0 \Xi$. For $z \in C(\Xi_0 \cup \Xi, \mathbb{R})$ we put

$$\Pi_+[z] = \{(t, x) \in \Xi : \ |z(t, x)| \geq |z(t, y)| \text{ for } y \in [\alpha(t), \beta(t)]\}$$

and

$$\|z\|_t = \max\{|z(\tau, y)| : \ (\tau, y) \in \Xi_0 \cup \Xi, \ \tau \leq t\}, \ t \in [-b_0, a]. \tag{17}$$

For each $(t, x) \in \Xi$ there are the sets of natural numbers $I_0[t, x]$, $I_-[t.x]$, $I_+[t, x]$ such that $I_-[t, x] \cap I_+[t, x] = \emptyset$, $I_0[t, x] \cup I_-[t, x] \cup I_+[t, x] = \{1, \ldots, n\}$, and

$$\alpha_i(t) < x_i < \beta_i(t) \text{ for } i \in I_0[t, x],$$

$$x_i = \alpha_i(t) \text{ for } i \in I_-[t, x], \ x_i = \beta_i(t) \text{ for } i \in I_+[t, x].$$

It is clear that there are $(t, x) \in \Xi$ such that $I_0[t, x] = \emptyset$ or $I_-[t, x] = \emptyset$ or $I_+[t, x] = \emptyset$. For $(t, x) \in \partial_0 \Xi$ and $i \in I_-[t, x]$, $j \in I_+[t, x]$ we put

$$x[i, \alpha] = (x_1, \ldots, x_{i-1}, \alpha_i(t), x_{i+1}, \ldots, x_n),$$

$$x[j,\beta] = (x_1, \ldots, x_{j-1}, \beta_j(t), x_{j+1}, \ldots, x_n).$$

Set $e_i = (1, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$ with 1 standing in the $i-$th place. For $(t, x, v) \in \Xi \times C(E, \mathbb{R})$ we define a function $Vv : [-b_0 - t, 0] \to \mathbb{R}_+$ in the following way:

$$(Vv)(\tau) = \max\{|v(\tau, y)| : (\tau, y) \in B[t, x]\}, \; \tau \in [-b_0 - t, 0]. \tag{18}$$

For $w \in C(D, \mathbb{R})$ and $(t, x) \in \Xi$ we define the seminorm

$$\|w\|_{A[t,x]} = \max\{|z(\tau, y)| : (\tau, y) \in A[t, x]\}. \tag{19}$$

If $z \in C(\Xi_0 \cup \Xi, \mathbb{R})$ and $(t, x) \in \Xi$ then we write

$$\|z_{[t,x]}\|_{A[t,x]} = \max\{|z_{[t,x]}(\tau, y)| : (\tau, y) \in A[t, x]\}. \tag{20}$$

For a function $z : \Xi_0 \cup \Xi \to \mathbb{R}$ which is of class $C^\star$ and for a point $(t, x) \in \Xi \cap \big((0, a] \times \mathbb{R}^n]\big)$ we put

$$Y[z](t, x) = \sum_{i \in I_-[t,x]} \alpha_i'(t) |\partial_{x_i} z(t, x)| - \sum_{i \in I_+[t,x]} \beta_i'(t) |\partial_{x_i} z(t, x)|.$$

Assumption $H[\alpha, \beta]$. The functions α, $\beta \in C([-b_0, a], \mathbb{R}^n)$ are such that $\alpha(t) < \beta(t)$ for $t \in [-b_0, a]$ and there exist the derivatives $\alpha'(t)$, $\beta'(t)$ for $t \in [0, a]$ and $\alpha'(t) \geq 0_{[n]}$, $\beta'(t) \leq 0_{[n]}$ for $t \in [0, a]$ where $0_{[n]} = (0, \ldots, 0) \in \mathbb{R}^n$.

Theorem 3.2. *Suppose that Assumptions $H[\sigma, L]$ and $H[\alpha, \beta]$ are satisfied and*

(i) $u : \Xi_0 \cup \Xi \to \mathbb{R}$ *is of class* $C^\star$ *and* $\eta(t) = \|u\|_t$, $t \in [-b_0, 0]$.

(ii) *The functional differential inequality*

$$|\partial_t u(t, x)| \leq \sigma(t, Vu_{(t,x)}) + L(t) \|u_{[t,x]}\|_{A[t,x]} + Y[u](t, x) \tag{21}$$

is satisfied for $(t, x) \in \Pi_+[u]$.

Under these assumptions we have

$$\|u\|_t \leq \omega(t, \eta) \text{ for } t \in [0, a], \tag{22}$$

where $\omega(\,\cdot\,, \eta)$ *is the solution of* (1).

Proof. Write $\zeta(t) = \|u\|_t$, $t \in [-b_0, a]$. Then $\zeta \in C([-b_0, a], \mathbb{R}_+)$ and ζ is nondecreasing. We prove that

$$D_-\zeta(t) \leq \sigma(t, \zeta_t) + L(t)\zeta(a) \text{ for } t \in (0, a]. \tag{23}$$

Suppose that $t \in (0, a]$. There is $(\tau, x) \in \Xi_0 \cup \Xi$ such that $\zeta(t) = |u(\tau, x)|$. If $\tau < t$ then $D_-\zeta(t) = 0$ and (23) is satisfied. Suppose that $\tau = t$. Then *(i)* $\zeta(t) = u(t, x)$ or *(ii)* $\zeta(t) = -u(t, x)$. Let us consider the case (i). It is easily seen that

$$\partial_{x_i} u(t, x) \geq 0 \text{ for } i \in I_+[t, x], \;\; \partial_{x_i} u(t, x) \leq 0 \text{ for } i \in I_-[t, x], \tag{24}$$

and

$$\partial_{x_i} u(t,x) = 0 \text{ for } i \in I_0[t,x]. \tag{25}$$

Let $\gamma : [0,t] \to \mathbb{R}^n$, $\gamma = (\gamma_1, \ldots, \gamma_n)$, be defined by

$$\gamma_i(\tau) = x_i \text{ for } i \in I_0[t,x],$$

$$\gamma_i(\tau) = \alpha_i(\tau) \text{ for } i \in I_+[t,x],\ \gamma_i(\tau) = \beta_i(\tau) \text{ for } i \in I_-[t,x].$$

Set $\lambda(\tau) = u(\tau, \gamma(\tau))$, $\tau \in [0,t]$. Then $\lambda(\tau) \le \zeta(\tau)$ for $\tau \in [0,t]$ and $\lambda(t) = \zeta(t)$. This gives $\lambda'(t) \ge D_-\zeta(t)$. It is clear that $(t,x) \in \Pi_+[u]$. Then we have

$$D_-\zeta(t) \le \lambda'(t) = \partial_t u(t,x) + \sum_{i=1}^n \partial_{x_i} u(t,x)\, \gamma_i'(t)$$

$$\le \sigma(t,\zeta_t) + L(t)\zeta(a) + Y[u](t,x) + \sum_{i=1}^n \partial_{x_i} u(t,x)\, \gamma_i'(t) = \sigma(t,\zeta_t) + L(t)\zeta(a),$$

which proves (23). If the case (ii) is satisfied then (23) can be proved in a similar way. It follows from Theorem 2.3 that $\zeta(t) \le \omega(t,\eta)$ for $t \in [0,a]$. This gives (22) and the proof is complete.

Now we construct estimates of solutions of (14) and we give a result on the uniqueness of solutions of initial problems.

Assumption $H[f,\alpha,\beta]$. The functions $\alpha, \beta : [-b_0, a] \to \mathbb{R}^n$ satisfy Assumption $H[\alpha,\beta]$ and

(i) for $(t,x) \in \partial_0\Xi$, $i \in I_-[t,x]$, $h \in \mathbb{R}$, $h \ne 0$, we have

$$\alpha_i'(t) \ge \Big|\frac{1}{h}\big[f(t,x[i,\alpha],v,w,q+e_ih) - f(t,x[i,\alpha],v,w,q)\big]\Big|$$

where $v \in C(E,\mathbb{R})$, $w \in C(D,\mathbb{R})$, $q = (q_1,\ldots,q_n) \in \mathbb{R}^n$ and $q_j = 0$ for $j \in I_0[t,x]$.

(ii) For $(t,x) \in \partial_0\Xi$, $i \in I_+[t,x]$, $h \in \mathbb{R}$, $h \ne 0$, we have

$$-\beta_i'(t) \ge \Big|\frac{1}{h}\big[f(t,x[i,\beta],v,w,q+e_ih) - f(t,x[i,\beta],v,w,q)\big]\Big|$$

where $v \in C(E,\mathbb{R})$ $w \in C(D,\mathbb{R})$, $q = (q_1,\ldots,q_n) \in \mathbb{R}^n$ and $q_j = 0$ for $j \in I_0[t,x]$.

Theorem 3.3. *Suppose that $f : \Omega \to \mathbb{R}$ satisfies condition (V), and Assumptions $H[\sigma,L]$ and $H[\alpha,\beta]$ hold.*

I. *If the estimate*

$$\big|f(t,x,v,w,0_{[n]}))\big| \le \sigma(t,Vv) + L(t)\|w\|_{A[t,x]} \tag{26}$$

is satisfied for $(t,x,v,w) \in \Xi \times C(E,\mathbb{R}) \times C(D,\mathbb{R})$ and $\tilde z : \Xi_0 \cup \Xi \to \mathbb{R}$ is a solution of (14) *and $\tilde z$ is of class $C^\star$ then*

$$|\tilde z(t,x)| \le \omega(t,\eta) \text{ for } (t,x) \in \Xi, \tag{27}$$

where $\omega(\,\cdot\,,\eta)$ is a solution of (1) *with η given by*

$$\eta(t) = \max\{|\tilde{z}(\tau,y)| : \ (\tau,y) \in \Xi_0, \ \tau \le t\}, \ t \in [-b_0,0].$$

II. *If the estimate*

$$\big|f(t,x,v,w,q) - f(t,x,\tilde{v},\tilde{w},q)\big| \le \sigma(t,V(v-\tilde{v})) + L(t)\|w-\tilde{w}\|_{A[t,x]}$$

is satisfied on Ω and $\sigma(t,\theta) = 0$ for $t \in [0,a]$ then the Cauchy problem (14), (15) *admits at most one solution of class $C^\star$ on $\Xi_0 \cup \Xi$.*

Proof. We apply Theorem 3.2 to prove (27). Suppose that $(t,x) \in \Pi_+[\tilde{z}]$. We prove that

$$|\partial_t \tilde{z}(t,x)| \le \sigma(t, V\tilde{z}_{(t,x)}) + L(t)\|\tilde{z}_{[t,x]}\|_{A[t,x]} + Y[\tilde{z}](t,x). \tag{28}$$

Let us consider the case where $\tilde{z}(t,x) \ge 0$. Then we have

$$\partial_{x_i}\tilde{z}(t,x) \ge 0 \text{ for } i \in I_+[t,x], \ \partial_{x_i}\tilde{z}(t,x) \le 0 \text{ for } i \in I_-[t,x] \tag{29}$$

and

$$\partial_{x_i}\tilde{z}(t,x) = 0 \text{ for } i \in I_0[t,x]. \tag{30}$$

It follows from (26) that

$$\partial_t \tilde{z}(t,x) \le \sigma(t, V\tilde{z}_{(t,x)}) + L(t)\|\tilde{z}_{[t,x]}\|_{A[t,x]} + \Gamma(t,x)$$

where

$$\Gamma(t,x) = \mathbf{F}[\tilde{z}](t,x) - f(t,x,\tilde{z}_{(t,x)},\tilde{z}_{[t,x]},0_{[n]}).$$

We deduce from Assumption $H[f,\alpha,\beta]$ and from (29), (30) that

$$\Gamma(t,x) \le Y[\tilde{z}](t,x),$$

and consequently

$$\partial_t \tilde{z}(t,x) \le \sigma(t, V\tilde{z}_{(t,x)}) + L(t)\|\tilde{z}_{[t,x]}\|_{A[t,x]} + Y[\tilde{z}](t,x).$$

In a similar way we prove that

$$\partial_t \tilde{z}(t,x) \ge -\sigma(t, V\tilde{z}_{(t,x)}) - L(t)\|\tilde{z}_{[t,x]}\|_{A[t,x]} - Y[\tilde{z}](t,x),$$

which completes the proof of (28).

It is a simple matter to show (28) in the case where $\tilde{z}(t,x) < 0$ and $(t,x) \in \Pi_+[\tilde{z}]$. Then all the assumptions of Theorem 3.2 are satisfied and assertion (27) follows.

II. Suppose that $\tilde{z}, \bar{z} : \Xi_0 \cup \Xi \to \mathbb{R}$ are solutions of (14), (15) and they are of class $C^\star$ and $(\tilde{z} - \bar{z})(t,x) \not\equiv 0$ on Ξ. Set $u = \tilde{z} - \bar{z}$. We prove that u satisfies the functional differential inequality (21) for $(t,x) \in \Pi_+[u]$.

Suppose that $(t,x) \in \Pi_+[u]$. Now, there are two cases to be distinguished: (i) $u(t,x) \ge 0$ or (ii) $u(t,x) < 0$. Let us consider the first case. Then relations (24), (25) are satisfied and

$$\partial_t u(t,x) \le \sigma(t, Vu_{(t,x)}) + L(t)\|u_{[t,x]}\|_{A[t,x]} \tag{31}$$

$$+f(t,x,\bar{z}_{(t,x)},\bar{z}_{[t,x]},\partial_x\tilde{z}(t,x)) - \mathbf{F}[\bar{z}](t,x)$$
$$\le \sigma(t,Vu_{t,x)}) + L(t)\|u_{[t,x]}\|_{A[t,x]} + Y[u](t,x).$$

In a similar way we obtain the inequality

$$\partial_t u(t,x) \ge -\sigma(t,Vu_{t,x)}) - L(t)\|u_{[t,x]}\|_{A[t,x]} - Y[u](t,x). \tag{32}$$

Relations (31), (32) imply (21).

The case (ii) can be treated in a similar way. Then all the assumptions of Theorem 3.2 are satisfied with $\eta(t) = 0$ for $t \in [-b_0, 0]$ and the assertion follows.

Now we consider important special cases of Theorem 3.3. Suppose that the sets Ξ and Ξ_0 are given by

$$\Xi = \{(t,x) \in \mathbb{R}^{1+n} : \ t \in [0,a], \ -b + Mt \le x \le b - Mt\},$$
$$\Xi_0 = [-b_0, a] \times [-b, b],$$

where $b = (b_1, \dots, b_n) \in \mathbb{R}^n$, $M = (M_1, \dots, M_n) \in \mathbb{R}^n_+$ and $b > Ma$. For $v \in C(E, \mathbb{R})$ and $(t,x) \in \Xi$ we put

$$\|v\|_{B[t,x]} = \max\{|v(\tau,y)| : \ (\tau,y) \in B[t,x]\}. \tag{33}$$

If $z \in C(\Xi_0 \cup \Xi, \mathbb{R})$ and $(t,x) \in \Xi$ then we write

$$\|z_{(t,x)}\|_{B[t,x]} = \max\{|z_{(t,x)}(\tau,y)| : \ (\tau,y) \in B[t,x]\}.$$

Assumption $H[\tilde{\sigma}, L]$. The functions $\tilde{\sigma} : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$, $L : [0,a] \to \mathbb{R}_+$ satisfy the conditions:

(i) $\tilde{\sigma} \in C([0,a] \times \mathbb{R}_+)$ and $\tilde{\sigma}$ is nondecreasing with respect to the second variable.

(ii) There is $K \in C([0,a], \mathbb{R}_+)$ such that

$$\tilde{\sigma}(t,p) - \tilde{\sigma}(t,\tilde{p}) \le K(t)(p - \tilde{p})$$

where $t \in [0,a]$, p, $\tilde{p} \in \mathbb{R}_+$ and $p \ge \tilde{p}$.

(iii) $L \in C([0,a], \mathbb{R}_+)$ and condition (4) is satisfied.

The following lemmas are important in applications.

Lemma 3.4. *Suppose that $f : \Omega \to \mathbb{R}$ satisfies condition (V) and*

(i) *the derivatives $(\partial_{q_1} f, \dots, \partial_{q_n}) = \partial_q f$ exist and*

$$\big(|\partial_{q_1} f(P)|, \dots, |\partial_{q_n} f(P)|\big) \le M$$

where $P = (t,x,v,w,q) \in \Omega$.

(ii) *There are $\tilde{\sigma} : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ and $L : [0,a] \times \mathbb{R}_+$ such that Assumption $H[\tilde{\sigma}, L]$ is satisfied and*

$$\big|f(t,x,v,w,0_{[n]})\big| \le \tilde{\sigma}(t, \|v\|_{B[t,x]}) + L(t)\|w\|_{A[t,x]}$$

where $(t,x,v,w) \in \Xi \times C(E,\mathbb{R}) \times C(D,\mathbb{R})$.

(iii) $\tilde{z} : \Xi_0 \cup \Xi \to \mathbb{R}$ *is a solution of* (14), (15) *and* $\tilde{z}$ *is of class* $C^\star$.

Under these assumptions we have $\|\tilde{z}\|_t \le \omega(t, \tilde{\eta})$ *for* $t \in [0, a]$, *where* $\omega(\,\cdot\,, \tilde{\eta})$ *is a solution of the Cauchy problem*

$$\omega'(t) = \tilde{\sigma}(t, \omega(t)) + L(t), \quad \omega(0) = \tilde{\eta},$$

and $\tilde{\eta} = \|\tilde{z}\|_0$.

The above lemma is a consequence of Theorem 3.3 part **I.**

Lemma 3.5. *Suppose that* $f : \Omega \to \mathbb{R}$ *satisfies condition* (V) *and*

(i) *Assumption* (i) *of Lemma* 3.4 *holds.*

(ii) *There are* $K,\ L \in C([0, a], \mathbb{R}_+)$ *such that condition* (4) *is satisfied and*

$$\left| f(t, x, v, w, q) - f(t, x, \tilde{v}, \tilde{w}, q) \right| \le K(t)\|v - \tilde{v}\|_{B[t,x]} + L(t)\|w - \tilde{w}\|_{A[t,x]} \text{ on } \Omega.$$

Then the Cauchy problem (14), (15) *admits at most one solution of class* $C^\star$ *on* $\Xi_0 \cup \Xi$.

The above lemma is a consequence of Theorem 3.3 part **II.**

4 Parabolic functional differential problems

Let $Q \subset \mathbb{R}^n$ be a bounded domain with the boundary ∂Q. Write $\Xi = [0, a] \times \overline{Q}$, $\Xi_0 = [-b_0, 0] \times \overline{Q}$ where $a > 0$, $b_0 \in \mathbb{R}_+$, and $\overline{Q}$ is the closure of Q. For $(t, x) \in \Xi$ we put

$$A[t, x] = \{(\tau, y) \in \mathbb{R}^{1+n} : \ (t + \tau, x + y) \in \Xi_0 \cup \Xi\}$$

and

$$B[t, x] = \{(\tau, y) \in A[t, x] : \ \tau \le 0\}.$$

There is $[c, d] \subset \mathbb{R}^n$, $c = (c_1, \ldots, c_n)$, $d = (d_1, \ldots, n)$, such that $A[t, x] \subset [-b_0 - a, a] \times [c, d]$ and $B[t, x] \subset [-b_0, -a, 0] \times [c, d]$. Set $D = [-b_0 - a, a] \times [c, d]$ and $E = [-b_0 - a, 0] \times [c, d]$.

Given $z : \Xi_0 \cup \Xi \to \mathbb{R}$ and $(t, x) \in \Xi$, define $z_{(t,x)} : B[t, x] \to \mathbb{R}$ and $z_{[t,x]} : A[t, x] \to \mathbb{R}$ by (12) and (13) respectively.

Let $M_{n\times n}$ be the class of all $n \times n$ symmetric matrices with real elements. Set $\Omega = \Xi \times C(E, \mathbb{R}) \times C(D, \mathbb{R}) \times \mathbb{R}^n \times M_{n\times n}$ and suppose that $F : \Omega \to \mathbb{R}$ is a given function of the variables (t, x, v, w, q, s) where $x = (x_1, \ldots, x_n)$, $q = (q_1, \ldots, q_n)$ and $s = \left[s_{ij}\right]_{i,j=1,\ldots,n}$. We will say that F satisfies condition (V) if for each $(t, x, q, s) \in \Xi \times \mathbb{R}^n \times M_{n\times n}$ and for $v,\ \tilde{v} \in C(E, \mathbb{R})$, $w,\ \tilde{w} \in C(D, \mathbb{R})$ such that $v(\tau, y) = \tilde{v}(\tau, y)$ for $(\tau, y) \in B[t, x]$ and $w(\tau, y) = \tilde{w}(\tau, y)$ for $(\tau, y) \in A[t, x]$ we have $F(t, x, v, w, q, s) = F(t, x, \tilde{v}, \tilde{w}, q, s)$.

Write $\partial_0\Xi = [0, a] \times \partial Q$ and suppose that $\psi : \Xi_0 \cup \partial_0\Xi \to \mathbb{R}$ is a given function. Set

$$\mathbb{F}[z](t, x) = F(t, x, z_{(t,x)}, z_{[t,x]}, \partial_x z(t, x), \partial_{xx} z(t, x))$$

where $\partial_x z = (\partial_{x_1} z, \dots, \partial_{x_n} z)$ and $\partial_{xx} z = \left[\partial_{x_i x_j} z\right]_{i,j=1,\dots,n}$. We consider the functional differential equation

$$\partial_t z = \mathbb{F}[z] \tag{34}$$

with the initial boundary condition

$$z = \psi \text{ on } \Xi_0 \cup \partial_0 \Xi. \tag{35}$$

We assume that F satisfies condition (V) and we consider classical solutions of (34), (35).

Remark 4.1. *Suppose that $G : \Xi \times C(\Xi_0 \cup \Xi, \mathbb{R}) \times \mathbb{R}^n \times M_{n\times n} \to \mathbb{R}$ is a given function. Let us consider the functional differential equation*

$$\partial_t z(t,x) = G(t,x,z,\partial_x z(t,x), \partial_{xx} z(t,x)) \tag{36}$$

where z is the functional variable. It is clear that (34) *is a special case of* (36)*. We will say that G satisfies the Volterra condition if for each $(t,x) \in \Xi$ and for z, $\tilde{z} \in C(\Xi_0 \cup \Xi, \mathbb{R})$ such that $z(\tau,y) = \tilde{z}(\tau,y)$ for $(\tau,y) \in (\Xi_0 \cup \Xi) \cap ([-b_0, t] \times \mathbb{R}^n)$ we have $G(t,x,z,q,s) = G(t,x,\tilde{z},q,s)$ where $(q,s) \in \mathbb{R}^n \times M_{n\times n}$.*

Note that the Volterra condition is not satisfied for $G(t,x,z,q,s) = F(t,x,z_{(t,x)}, z_{[t,x]}, q, s)$.

The results presented in [12], [13] *and* [15], [16] *have the following limitation: functional differential equations (or systems) considered in these papers satisfy the Volterra condition. In fact, we are not aware of any previous results pertaining to the functional differential equation* (36) *which do not satisfy the Volterra condition.*

With the above motivation we consider problem (34), (35).

A function $z : \Xi_0 \cup \Xi \to \mathbb{R}$ will be called the function of class $C^{1.2}$ if $z \in C(\Xi_0 \cup \Xi, \mathbb{R})$ and $z(\cdot, x) : [0,a] \to \mathbb{R}$ is of class C^1 for $x \in \overline{Q}$ and $z(t, \cdot) : \overline{Q} \to \mathbb{R}$ is of class C^2 for $t \in [0,a]$.

For s, $\tilde{s} \in M_{n\times n}$, where

$$s = \left[s_{ij}\right]_{i,j=1,\dots,n}, \ \tilde{s} = \left[\tilde{s}_{ij}\right]_{i,j=1,\dots,n}$$

we write $s \leq \tilde{s}$ if

$$\sum_{i,j=1}^{n} (s_{ij} - \tilde{s}_{ij}) \lambda_i \lambda_j \leq 0 \text{ for any } \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n.$$

For $v \in C(E, \mathbb{R})$, $w \in C(D, \mathbb{R})$, $z \in C(\Xi_0 \cup \Xi, \mathbb{R})$ and $(t,x) \in \Xi$ we define the function $Vv : [-b_0 - t, 0] \to \mathbb{R}$ and the numbers $\|w\|_{A[t,x]}$, $\|z_{[t,x]}\|_{A[t,x]}$ by (18)–(20) respectively. Let $\|z\|_t$, $t \in [-b_0, a]$, be defined by (17).

Assumption $H[F,\sigma]$. The function $F : \Omega \to \mathbb{R}$ satisfies condition (V) and

(i) F is continuous and there is $\sigma : [0,a] \times C(I, \mathbb{R}_+) \to \mathbb{R}$ such that Assumption $H[\sigma, L]$ is satisfied and

$$|F(t,x,v,w,q,s) - F(t,x,\tilde{v},\tilde{w},q,s)| \leq \sigma(t, V(v-\tilde{v})) + L(t)\|w - \tilde{w}\|_{A[t,x]} \text{ on } \Omega. \tag{37}$$

(ii) If $s, \tilde{s} \in M_{n\times n}$ and $s \leq \tilde{s}$ then for $(t,x,v,w,q) \in \Xi \times C(E,\mathbb{R}) \times C(D,\mathbb{R}) \times \mathbb{R}^n$ we have

$$F(t,x,v,w,q,s) \leq F(t,x,v,w,q,\tilde{s}).$$

We prove a theorem on an estimate of the difference between a solution and an approximate solution of (34), (35).

Theorem 4.2. *Suppose that Assumption $H[F,\sigma]$ is satisfied and*

(i) $\bar{z} : \Xi_0 \cup \Xi \to \mathbb{R}$ *is a solution of* (34), (35) *and $\bar{z}$ is of class $C^{1.2}$.*

(ii) $\tilde{z} : \Xi_0 \cup \Xi \to \mathbb{R}$ *is of class $C^{1.2}$ and $\tilde{z}$ satisfies the boundary condition* (35).

(iii) *The function $\tilde{\Gamma} : \Xi \to \mathbb{R}$ is defined by the relation*

$$\partial_t \tilde{z}(t,x) = \mathbb{F}[\tilde{z}](t,x) + \tilde{\Gamma}(t,x) \tag{38}$$

and $\tilde{\gamma} \in C([0,a],\mathbb{R}_+)$ is given by

$$|\tilde{\Gamma}(t,x)| \leq \tilde{\gamma}(t) \text{ on } \Xi. \tag{39}$$

Under these assumptions we have

$$\|\bar{z} - \tilde{z}\|_t \leq \tilde{\omega}(t) \text{ for } t \in [0,a], \tag{40}$$

where $\tilde{\omega} : [-b_0,a] \to \mathbb{R}_+$ is a solution of the Cauchy problem

$$\omega'(t) = \sigma(t,\omega_t) + L(t)\omega(a) + \tilde{\gamma}(t), \quad \omega(t) = 0 \text{ for } t \in [-b_0,0]. \tag{41}$$

Proof. We apply Theorem 2.3. Set $\varphi(t) = \|\bar{z} - \tilde{z}\|_t$, $t \in [-b_0,a]$. Then $\varphi \in C([-b_0,a],\mathbb{R}_+)$ and φ is nondecreasing. We prove that φ satisfies the functional differential inequality

$$D_-\varphi(t) \leq \sigma(t,\varphi_t) + L(t)\varphi(a) + \tilde{\gamma}(t), \ t \in (0,a]. \tag{42}$$

Suppose that $t \in (0,a]$. There is $(\tau,x) \in \Xi$, $\tau \leq t$, such that $\varphi(t) = |(\bar{z}-\tilde{z})(\tau,x)|$. If $\tau < t$ then $D_-\varphi(t) = 0$ and (42) is satisfied. Suppose that $\tau = t$. Now, there are two cases to be distinguished: (i) $\varphi(t) = (\bar{z}-\tilde{z})(t,x)$ or (ii) $\varphi(t) = -(\bar{z}-\tilde{z})(t,x))$. Let us consider the first case. Since $\bar{z}$ and $\tilde{z}$ satisfy (35) then $x \in Q$ and consequently $\partial_x \bar{z}(t,x) = \partial_x \tilde{z}(t,x)$ and

$$\sum_{i,j=1}^{n} \partial_{x_i x_j}(\bar{z}-\tilde{z})(t,x)\,\lambda_i\lambda_j \leq 0 \ \text{ for } \ \lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n.$$

From (37)–(39) we conclude that

$$\begin{gathered} D_-\varphi(t) \leq \partial_t(\bar{z}-\tilde{z})(t,x) \leq \sigma(t,\varphi_t) + L(t)\varphi(a) \\ +F(t,x,\tilde{z}_{(t,x)},\tilde{z}_{[t,x]},\partial_x\bar{z}(t,x),\partial_{xx}\bar{z}(t,x)) - \mathbb{F}[\tilde{z}](t,x) - \tilde{\Gamma}(t,x) \\ \leq \sigma(t,\varphi_t) + L(t)\varphi(a) + \tilde{\gamma}(t), \end{gathered}$$

which proves (42). In a similar way we prove (42) if the case (ii) holds.

Now all the assumptions of Theorem 2.3 are satisfied with $\eta(t) = 0$ for $t \in [-b_0,0]$, and assertion (40) follows.

Theorem 4.3. *Suppose that Assumptions $H[F,\sigma]$ is satisfied and $\sigma(t,\theta) =$ for $t \in [0,a]$. Then problem* (34), (35) *admits at most one solution of class $C^{1.2}$ on $\Xi_0 \cup \Xi$.*

Proof. If $\bar{z},\ \tilde{z} : \Xi_0 \cup \Xi \to \mathbb{R}$ are solutions of (34), (35) then conditions (38), (39) are satisfied with $\tilde{\Gamma}(t,x) = 0$ on Ξ and $\tilde{\gamma}(t) = 0$ on $[0,a]$. It follows from Theorem 4.2 that $\bar{z} = \tilde{z}$ on Ξ which completes the proof.

For $v \in C(E,\mathbb{R})$ we define the numbers $\|v\|_{B[t,x]}$, $(t,x) \in \Xi$, by (33).

Lemma 4.4. *Suppose that $F : \Omega \to \mathbb{R}$ satisfies condition (V) and*

(i) *F is continuous and there are $K,\ L \in C([0,a],\mathbb{R}_+)$ such that*

$$\big|F(t,x,v,w,q,s) - F(t,x,\tilde{v},\tilde{w},q,s)\big| \leq K(t)\|v-\tilde{v}\|_{B[t,x]} + L(t)\|w-\tilde{w}\|_{A[t,x]} \text{ on } \Omega \tag{43}$$

and inequality (4) *is satisfied.*

(ii) *Condition* (ii) *of Assumption $H[F,\sigma]$ holds.*

(iii) *The functions $\tilde{z},\ \bar{z} : \Xi_0 \cup \Xi \to \mathbb{R}$ satisfy assumption* (i)–(iii) *of Theorem* 4.2.

Under these assumptions we have

$$\|\bar{z} - \tilde{z}\|_t \leq \tilde{\omega}(t) \text{ for } t \in [0,a],$$

where

$$\tilde{\omega}(t) = \int_0^t \big[\tilde{\gamma}(\tau) + A_\star L(\tau)\big] \exp\Big\{\int_\tau^t K(s)ds\Big\}\, d\tau, \tag{44}$$

$$A_\star = \tilde{A}\Big[\int_0^a \tilde{\gamma}(\tau)\exp\Big\{\int_\tau^a K(s)\,ds\Big\}\Big], \tag{45}$$

and $\tilde{A}$ is given by (7)*. Moreover, problem* (34), (35) *admits at most one solution of class $C^{1.2}$ on $\Xi_0 \cup \Xi$.*

The above lemma is a consequence of Theorem 4.2 and Theorem 4.3.

We give examples of equations which can be derived from (34) by specializing the operator F. Suppose that

$$G : \Xi \times \mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R},\ \ \mathbb{A} : \Xi \to M_{n\times n},\ \ \mathbb{A} = \big[a_{ij}\big]_{i,j=1,\ldots n},$$

are given functions. We define $\mathbb{F}$ as follows

$$\mathbb{F}[z](t,x) = G\Big(t,x,\int_{B[t,x]} z_{(t,x)}(\tau,y)\,dyd\tau, \int_{A[t,x]} z_{[t,x]}(\tau,y)\,dyd\tau, \partial_x z(t,x)\Big) + \sum_{i,j=1}^n a_{ij}(t,x)\,\partial_{x_i x_j} z(t,x). \tag{46}$$

Then (34) reduces to the differential integral equation

$$\partial_t z = \mathbb{F}[z]) \tag{47}$$

Assumption $H[G, \mathbb{A}]$. The functions $G : \Xi \times \mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R}$, $\mathbb{A} : \Xi \to M_{n\times n}$ are continuous and

(i) there are $\tilde{K}$, $\tilde{L} \in C([0,a], \mathbb{R}_+)$ such that

$$\big|G(t,x,\zeta,\nu,q) - G(t,x,\tilde{\zeta},\tilde{\nu},q)\big| \le \tilde{K}(t)|\zeta - \tilde{\zeta}| + \tilde{L}|\nu - \tilde{\nu}|.$$

(ii) The matrix $\mathbb{A}$ is symmetric and

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\lambda_i\lambda_j \ge 0, \ (t,x) \in \Xi, \ \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n.$$

We have the following result for the differential integral problem (47), (35).

Lemma 4.5. *Suppose that Assumption $H[G, \mathbb{A}]$ is satisfied and*

(i) $\bar{z} : \Xi_0 \cup \Xi \to \mathbb{R}$ *is a solution of* (47), (35) *and $\bar{z}$ is of class $C^{1.2}$.*

(ii) $\tilde{z} : \Xi_0 \cup \Xi \to \mathbb{R}$ *is of class $C^{1.2}$ and $\tilde{z}$ satisfies condition* (35).

(iii) *The function $\tilde{\Gamma} : \Xi \to \mathbb{R}$ is defined by* (38) *with $\mathbb{F}$ given by* (46), $|\tilde{\Gamma}(t,x)| \le \tilde{\gamma}(t)$ *on Ξ, and $\tilde{\gamma} \in C([0,a], \mathbb{R}_+)$.*

(iv) *The following condition holds:*

$$\kappa\, a \int_0^a \tilde{L}(s)\, \exp\{\kappa \int_s^a \tau\, \tilde{K}(\tau)\, d\tau\}\, ds < 1,$$

where

$$\kappa = \prod_{i=1}^{n}(d_i - c_i).$$

Under these assumptions we have

$$\|\bar{z} - \tilde{z}\|_t \le \tilde{\omega}(t) \text{ for } t \in [0,a],$$

where $\tilde{\omega}$ is given by (44), (45) *with*

$$K(t) = \kappa\, t\tilde{K}(t), \ \ L(t) = a\,\kappa\tilde{L}(t), \tag{48}$$

and $\tilde{A}$ is defined by (7) *with the above given K and L. Moreover, problem* (47), (35) *admits at most one solution of class $C^{1.2}$ on $\Xi_0 \cup \Xi$.*

Proof. It follows from Assumption $H[G, \mathbb{A}]$ that the operator F defined by (46) satisfies condition (ii) of Assumption $H[F, \sigma]$, and the Lipschitz condition (43) holds with K, L given by (48). Then our assertion follows from Lemma 4.4.

Now we consider differential equations with deviated variables. For the above given functions G and $\mathbb{A}$ we put

$$\mathbb{F}[z](t,x) = G\big(t,x,z(\chi(t,x)), z(\vartheta(t,x)), \partial_x z(t,x)\big) + \sum_{i,j=1}^{n} a_{ij}(t,x)\partial_{x_i x_j} z(t,x). \tag{49}$$

where $\chi,\ \vartheta : \Xi \to \mathbb{R}^{1+n}$ and $\chi(t,x),\ \vartheta(t,x) \in \Xi_0 \cup \Xi$ for $(t,x) \in \Xi$. Then (34) reduces to the differential equation with deviated variables

$$\partial_t z = \mathbb{F}[z]. \tag{50}$$

Assumption $H[\chi,\vartheta]$**.** The functions $\chi,\ \vartheta : \Xi \to \mathbb{R}^{1+n}$, $\chi = (\chi_0, \tilde{\chi})$, $\vartheta = (\vartheta_0, \tilde{\vartheta})$, $\tilde{\chi} = (\chi_1, \ldots, \chi_n)$, $\tilde{\vartheta} = (\vartheta_1, \ldots, \vartheta_n)$, satisfy the conditions:

(i) χ, ϑ are continuous and $\tilde{\chi}(t,x),\ \tilde{\vartheta}(t,x) \in \overline{Q}$ for $(t,x) \in \Xi$.

(ii) For $(t,x) \in \Xi$ we have $-b_0 \le \chi_0(t,x) \le t$ and $-b_0 \le \vartheta_0(t,x) \le a$.

We have the following result for the differential problem (49) (35).

Lemma 4.6. *Suppose that Assumption $H[G,\mathbb{A}]$ and $H[\alpha,\beta]$ are satisfied and*

(i) $\bar{z} : \Xi_0 \cup \Xi \to \mathbb{R}$ *is a solution of* (50), (35), *and $\bar{z}$ is of class $C^{1.2}$.*

(ii) $\tilde{z} : \Xi_0 \cup \Xi \to \mathbb{R}$ *is of class $C^{1.2}$ and $\tilde{z}$ satisfies condition* (35).

(iii) *The function $\tilde{\Gamma} : \Xi \to \mathbb{R}$ is defined by* (38) *with $\mathbb{F}$ given by* (49), *and $|\tilde{\Gamma}(t,x)| \le \tilde{\gamma}(t)$ on Ξ where $\tilde{\gamma} \in C([0,a],\mathbb{R}_+)$.*

(iv) *Condition* (4) *is satisfied for $K = \tilde{K}$ and $L = \tilde{L}$.*

Under these assumptions we have

$$\|\bar{z} - \tilde{z}\|_t \le \tilde{\omega}(t) \text{ for } t \in [0,a],$$

where $\tilde{\omega}$ is given by (44), (45) *with $K = \tilde{K}$, $L = \tilde{L}$, and $\tilde{A}$ defined by* (7) *with the above given K, L. Moreover, problem* (49), (35) *admits at most one solution of class $C^{1.2}$ on $\Xi_0 \cup \Xi$.*

The above result is a consequence of Lemma 4.4.

It is clear that more complicated examples of differential equations with deviated variables or differential integral equations can be derived from (34).

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