On Doi-Hopf modules and Yetter-Drinfeld modules in symmetric monoidal categories

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Abstract

We study entwining structures on a monoidal category $\mathcal{C}$ and their corresponding categories of entwined modules. Examples can be constructed from lax Doi-Koppinen and lax Yetter-Drinfeld structures in $\mathcal{C}$. If $\mathcal{C}$ is symmetric then lax Yetter-Drinfeld structures appear as special cases of lax Doi-Koppinen structures, at least if we work over a so-called lax Hopf algebra. In this case the corresponding categories of entwined modules are isomorphic, and this generalizes a well-known result of Caenepeel, Militaru and Zhu [15]. In particular, our theory applies to Doi-Koppinen and Yetter-Drinfeld structures in symmetric monoidal categories. We present some examples of entwining structures in monoidal categories coming from actions and coactions of a weak Hopf algebra.

1 Introduction

Various notions of modules appear in Hopf algebra theory. Doi-Hopf modules and Yetter-Drinfeld modules are defined as vector spaces with an action and a coaction, with a compatibility relation that is different in both cases. In [15], it

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was shown that Yetter-Drinfeld modules are special cases of Doi-Hopf modules. This allows to transport properties of Doi-Hopf modules to Yetter-Drinfeld modules, and, in particular, it leads to generalizations of the Drinfeld double construction. Similar results have been applied for Hopf-group coalgebras [8], weak Hopf algebras [16], quasi-Hopf algebras in [10] and weak π-coalgebras in [20].

The aim of this paper is two-fold. First, we will generalize this result to Hopf algebras in braided monoidal categories. Some of the results mentioned above appear as special cases. It also leads to new results, if we apply it to Hopf-group coalgebras and monoidal Hom-Hopf algebras, as these are Hopf algebras in a suitable symmetric monoidal category, see [14, 13]. On the other hand, we want to construct non-trivial examples of entwining structures in monoidal categories or, equivalently, of monoidal wreath structures, see [9].

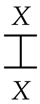
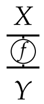
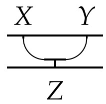
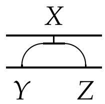
In Section 2, we recall the diagrammatic notations for the structure of a braided Hopf algebra and for the (co)action of a (co)algebra on an object in a module category. In Section 3 we introduce the notion of (right) entwining structure in a monoidal category and show that giving an entwining structure is equivalent to giving a (co)algebra structure in a suitable monoidal category of transfer morphisms through a (co)algebra (Proposition 3.2). Then we show that particular examples of entwining structures can be obtained from lax Doi-Koppinen structures, abbreviated as DK-structures. These are triples (B, A, C) consisting of an algebra A, a coalgebra C and an object B which is at the same time an algebra and a coalgebra and that acts on C and coacts on A in such a way that the structure morphisms are respectively coalgebra and algebra morphisms in $\mathcal{C}$. In the situation where B is a bialgebra, C is a B-module coalgebra and A is a B-comodule algebra, we recover the classical notion of DK structure. Furthermore, particular examples of lax DK structures can be obtained from lax Yetter-Drinfeld structures (abbreviated YD structures) over a lax Hopf algebra. A lax Hopf algebra is an object B admitting an algebra and a coalgebra structure in $\mathcal{C}$ such that Id_B has an inverse S_B in the convolution algebra $\mathrm{Hom}(B, B)$ that is an anti-algebra and an anti-coalgebra endomorphism of B. Actually, a simple inspection shows that this condition is not needed in the proof of the fact that any lax YD structure induces a lax DK structure (Proposition 3.9). In turn, we need this extra condition in the proof of our main result, namely Theorem 4.6.

We point out that our definition of an entwined module is given in the framework of module categories. This new approach will be used in Section 5, where we will show that the category of Doi-Hopf modules over a DK (monoidal) structure can be identified with the category of weak Doi-Hopf modules introduced in [2]. Unfortunately, in the weak case Theorem 4.6 does not apply since we don't know whether a weak Hopf algebra is a lax Hopf algebra in a suitable symmetric monoidal category. Nevertheless, Theorem 4.6 can be used as a source of inspiration for the definition of a weak Yetter-Drinfeld module over a weak bialgebra and then, moving backwards, we can prove a weak version of Theorem 4.6. Finally, note that the theory performed in the weak case gives particular non-trivial examples of entwining structures in monoidal categories of bimodules, and this achieves our second aim.

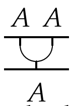
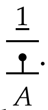
2 Preliminaries

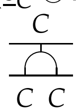
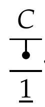
2.1 Hopf algebras in braided monoidal categories

A monoidal category is a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the tensor product, an object $\underline{1} \in \mathcal{C}$ called the unit object, and natural isomorphisms $a : \otimes \circ (\otimes \times \mathrm{Id}) \to \otimes \circ (\mathrm{Id} \times \otimes)$ (the associativity constraint), $l : \otimes \circ (\underline{1} \times \mathrm{Id}) \to \mathrm{Id}$ (the left unit constraint) and $r : \otimes \circ (\mathrm{Id} \times \underline{1}) \to \mathrm{Id}$ (the right unit constraint). In addition, a has to satisfy the pentagon axiom, and l and r have to satisfy the triangle axiom. We refer to [19, XI.2] for a detailed discussion. In the sequel, for any object $X \in \mathcal{C}$ we will identify $\underline{1} \otimes X \cong X \cong X \otimes \underline{1}$ using l_X and r_X. In addition, all the results will be proved for strict monoidal categories (i.e., for monoidal categories for which all a, l are r are the identity morphisms). Then the results remain valid in the case of an arbitrary monoidal category, since every monoidal category is equivalent to a strict one, see [19] for more detail. Also

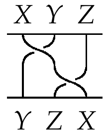
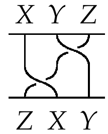
 ,  , $\mu =$  and $\nu =$ 

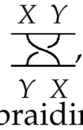
will be the diagrammatic notation for the following morphisms in $\mathcal{C}$: $\mathrm{Id}_X : X \to X$, $f : X \to Y$, $\mu : X \otimes Y \to Z$ and $\nu : X \to Y \otimes Z$.

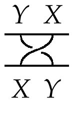
In a monoidal category $\mathcal{C}$ we can define algebras and coalgebras. An algebra in $\mathcal{C}$ is an object A of $\mathcal{C}$ endowed with a multiplication $\underline{m}_A : A \otimes A \to A$ and unit morphism $\underline{\eta}_A : \underline{1} \to A$ such that $\underline{m}_A$ is associative up to the associativity constraint a of $\mathcal{C}$ and $\underline{m}_A \circ (\underline{\eta}_A \otimes \mathrm{Id}_A) = \underline{m}_A \circ (\mathrm{Id}_A \otimes \underline{\eta}_A) = \mathrm{Id}_A$. We will write $\underline{m}_A =$  and $\underline{\eta}_A =$ .

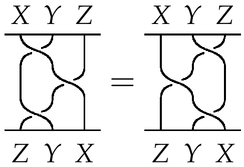
Similarly, a coalgebra in $\mathcal{C}$ is an object C of $\mathcal{C}$ together with a comultiplication $\underline{\Delta}_C : C \to C \otimes C$ and a counit $\underline{\varepsilon}_C : C \to \underline{1}$ such that $\underline{\Delta}_C$ is coassociative up to the coassociativity constraint a and $(\underline{\varepsilon}_C \otimes \mathrm{Id}_C) \circ \underline{\Delta}_C = \mathrm{Id}_C = (\mathrm{Id}_C \otimes \underline{\varepsilon}_C) \circ \underline{\Delta}_C$. We use the diagrammatic notation $\underline{\Delta}_C =$  and $\underline{\varepsilon}_C =$ .

The switch functor $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is defined by $\tau(X, Y) = (Y, X)$. A prebraiding on a monoidal category $\mathcal{C}$ is a natural transformation $c : \otimes \to \otimes \circ \tau$, satisfying the conditions

(a) $c_{X,Y\otimes Z} =$  and (b) $c_{X\otimes Y,Z} =$  (2.1)

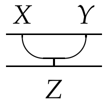
(see [19, XIII.1]), where $c_{X,Y} :=$ , for any two objects X and Y of $\mathcal{C}$.

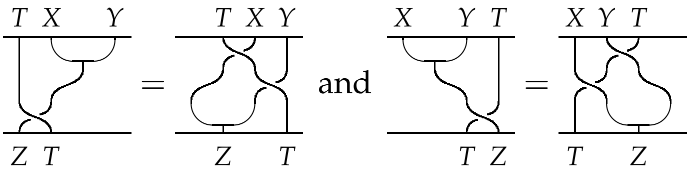
A prebraiding c is called a braiding if it is a natural isomorphism. In this case we denote $c^{-1}_{X,Y} :=$  and call $\mathcal{C}$ a braided monoidal category. One of the main properties of a braiding c on a monoidal category $\mathcal{C}$ is given by the equality



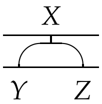
(2.2)

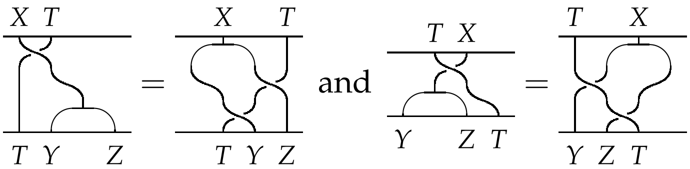
which holds for any objects X, Y and Z of $\mathcal{C}$. It is considered as the categorical version of the Yang-Baxter equation.

The naturality of c means that $(g \otimes f)c_{M,N} = c_{U,V}(f \otimes g)$, for any $f : M \to U$, $g : N \to V$ in $\mathcal{C}$. In particular, for $T \in \mathcal{C}$ and  $: X \otimes Y \to Z$ in $\mathcal{C}$, we have

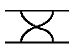
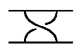
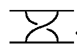


(2.3)

In a similar way, for a morphism  between X and $Y \otimes Z$, we have that



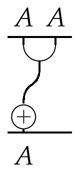
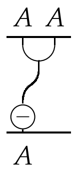
(2.4)

For a braided monoidal category $\mathcal{C}$, let $\mathcal{C}^{\rm in}$ be equal to $\mathcal{C}$ as a monoidal category, with the mirror-reversed braiding $\underline{c}_{X,Y} = c^{-1}_{Y,X}$. It is well known that $\mathcal{C}^{\rm in}$ is also a braided monoidal category. We call $\mathcal{C}$ symmetric if $\mathcal{C} = \mathcal{C}^{\rm in}$, as braided monoidal categories. When $\mathcal{C}$ is symmetric we denote  :=  = .

If A and A' are two algebras in a braided monoidal category then there are two algebra structures in $\mathcal{C}$ on $A \otimes A'$. Namely, we denote by $A \otimes_+ A'$ the object $A \otimes A'$ of $\mathcal{C}$ endowed with the multiplication $(\underline{m}_A \otimes \underline{m}_{A'}) \circ (\mathrm{Id}_A \otimes c_{A',A} \otimes \mathrm{Id}_{A'})$ and tensor product unit morphism. Then, with this structure, $A \otimes_+ A'$ becomes an algebra in $\mathcal{C}$ (see, for instance, [21, Lemma 2.1]). The second algebra structure in $\mathcal{C}$ on $A \otimes A'$, denoted in what follows by $A \otimes_- A'$, is obtained by considering A and A' algebras in $\mathcal{C}^{\rm in}$. Since $\mathcal{C}^{\rm in} = \mathcal{C}$ as a monoidal category we obtain that $A \otimes_- A'$ is an algebra in $\mathcal{C}$ with the multiplication $(\underline{m}_A \otimes \underline{m}_{A'}) \circ (\mathrm{Id}_A \otimes c^{-1}_{A,A'} \otimes \mathrm{Id}_{A'})$ and tensor product unit morphism.

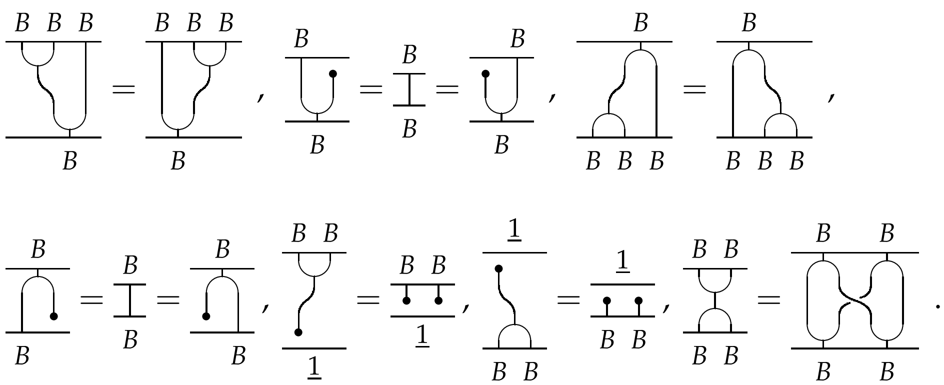
There is a notion of opposite algebra in a braided monoidal category. More precisely, if A is an algebra in a braided monoidal category $\mathcal{C}$ then $A^{\rm op+}$ is the object A endowed with the new multiplication $\underline{m}_A \circ c_{A,A}$ and original unit morphism. Then one can easily see that $A^{\rm op+}$ is an algebra in $\mathcal{C}$, called the c-opposite algebra associated to A. Replacing $\mathcal{C}$ by $\mathcal{C}^{\rm in}$ we obtain $A^{\rm op-}$, the object A endowed with the multiplication $\underline{m}_A \circ c^{-1}_{A,A}$ and the same unit morphism as that of

A, and we call it the c^{-1}-opposite algebra associated to A. For further use, we denote

$$\underline{m}_{A^{\mathrm{op}+}} = \underline{m}_A \circ c_{A,A} =$$  and $$\underline{m}_{A^{\mathrm{op}-}} = \underline{m}_A \circ c^{-1}_{A,A} =$$  . (2.5)

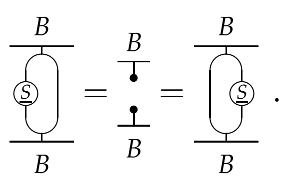
Likewise, if C and C' are two coalgebras in a braided monoidal category then $C \otimes C'$ has two coalgebra structures in $\mathcal{C}$. We denote by $C \otimes^{+} C'$ the coalgebra in $\mathcal{C}$ having the comultiplication given by $(\mathrm{Id}_C \otimes c_{C,C'} \otimes \mathrm{Id}_{C'}) \circ (\underline{\Delta}_C \otimes \underline{\Delta}_{C'})$ and the tensor product counit morphism. Analogously, $C \otimes^{-} C'$ is the coalgebra in $\mathcal{C}$ with comultiplication $(\mathrm{Id}_C \otimes c^{-1}_{C',C} \otimes \mathrm{Id}_{C'}) \circ (\underline{\Delta}_C \otimes \underline{\Delta}_{C'})$ and the tensor product counit morphism as counit.

Next, we recall the notions of a bialgebra and Hopf algebra in a braided monoidal category. A bialgebra in $\mathcal{C}$ is a fivetuple $(B, \underline{m}_B, \underline{\eta}_B, \underline{\Delta}_B, \underline{\varepsilon}_B)$, such that $(B, \underline{m}_B, \underline{\eta}_B)$ is an algebra and $(B, \underline{\Delta}_B, \underline{\varepsilon}_B)$ is a coalgebra such that $\underline{\Delta}_B : B \to B \otimes B$ and $\underline{\varepsilon}_B : B \to \underline{1}$ are algebra morphisms. $B \otimes B$ has the tensor product algebra structure (using the braiding on $\mathcal{C}$), and $\underline{1}$ is considered as an algebra in $\mathcal{C}$ with the multiplication and unit map both equal to the identity morphism of $\underline{1}$. For later reference, we give explicit formulas for the axioms of a bialgebra B: $\underline{\varepsilon}_B \underline{\eta}_B = \mathrm{Id}_{\underline{1}}$, and

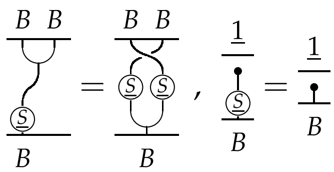
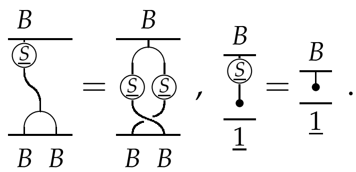


(2.6)

A Hopf algebra in a braided monoidal category $\mathcal{C}$ is a bialgebra B in $\mathcal{C}$ together with a morphism $\underline{S} : B \to B$ in $\mathcal{C}$ (the antipode) satisfying the axioms

 . (2.7)

It is well-known, see [21, Lemma 2.3], that the antipode $\underline{S}$ of a Hopf algebra B in a braided monoidal category $\mathcal{C}$ is an anti-algebra and anti-coalgebra morphism, in the sense that

(a)  and (b)  . (2.8)

2.2 Modules and comodules in module categories

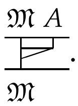
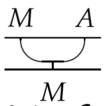
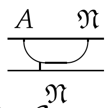
Let $\mathcal{C}$ be a monoidal category. A right $\mathcal{C}$-category (also known as a module category over $\mathcal{C}$) is a quadruple $(\mathcal{D}, \diamond, \Psi, \mathbf{r})$, where $\mathcal{D}$ is a category, $\diamond : \ \mathcal{D} \times \mathcal{C} \to \mathcal{D}$ is a functor, and $\Psi : \ \diamond \circ (\diamond \times \mathrm{Id}) \to \diamond \circ (\mathrm{Id} \times \otimes)$ and $\mathbf{r} : \ \diamond \circ (\mathrm{Id} \times \underline{1}) \to \mathrm{Id}$ are natural isomorphisms such that the diagrams

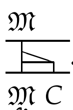
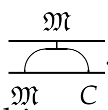
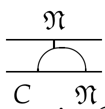
$$\begin{tikzcd}
((\mathfrak{M} \diamond X) \diamond Y) \diamond Z \arrow[r, "\Psi_{\mathfrak{M}\diamond X,Y,Z}"] \arrow[d, "\Psi_{\mathfrak{M},X,Y}\diamond \mathrm{Id}_Z"'] & (\mathfrak{M} \diamond X) \diamond (Y \otimes Z) \\
(\mathfrak{M} \diamond (X \otimes Y)) \diamond Z \arrow[r, "\Psi_{\mathfrak{M},X\otimes Y,Z}"] & \mathfrak{M} \diamond ((X \otimes Y) \otimes Z) \arrow[u, "\mathrm{Id}_{\mathfrak{M}}\diamond a_{X,Y,Z}"']
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
(\mathfrak{M} \diamond \underline{1}) \diamond X \arrow[r, "\Psi_{\mathfrak{M},\underline{1},X}"] \arrow[dr, "\mathbf{r}_{\mathfrak{M}}\diamond \mathrm{Id}_X"'] & \mathfrak{M} \diamond (\underline{1} \otimes X) \arrow[d, "\mathrm{Id}_{\mathfrak{M}}\diamond l_X"] \\
 & X
\end{tikzcd}$$

commute, for all $\mathfrak{M} \in \mathcal{D}$ and $X, Y, Z \in \mathcal{C}$. Obviously $\mathcal{C}$ is itself a right $\mathcal{C}$-category, with $\diamond = \otimes$, and Ψ and $\mathbf{r}$ the natural identities (recall that we assumed that $\mathcal{C}$ is strict). In fact, the above mentioned coherence theorem can be extended to $\mathcal{C}$-categories, and this enables us to assume, without loss of generality, that Ψ and $\mathbf{r}$ are natural identities.

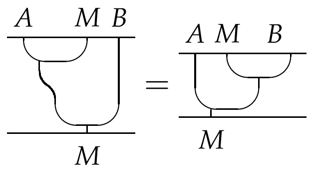
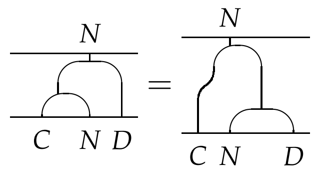
Let $\mathcal{D}$ be a right $\mathcal{C}$-category, and consider an algebra A in $\mathcal{C}$. A right module in $\mathcal{D}$ over A is an object $\mathfrak{M} \in \mathcal{D}$ together with a morphism $\nu_{\mathfrak{M}} : \mathfrak{M} \diamond A \to \mathfrak{M}$ such that $\nu_{\mathfrak{M}} \circ (\mathrm{Id}_{\mathfrak{M}} \diamond \underline{\eta}_A) = \mathbf{r}_{\mathfrak{M}}$ and such that the diagram

$$\begin{tikzcd}
(\mathfrak{M} \diamond A) \diamond A \arrow[rr, "\nu_{\mathfrak{M}}\diamond \mathrm{Id}_A"] \arrow[d, "\psi_{\mathfrak{M},A,A}"'] & & \mathfrak{M} \diamond A \arrow[d, "\nu_{\mathfrak{M}}"] \\
\mathfrak{M} \diamond (A \otimes A) \arrow[r, "\mathrm{Id}_{\mathfrak{M}}\diamond \underline{m}_A"'] & \mathfrak{M} \diamond A \arrow[r, "\nu_{\mathfrak{M}}"'] & \mathfrak{M}
\end{tikzcd}$$

commutes. Let $\mathcal{D}_A$ be the category of right modules and right linear maps in $\mathcal{D}$ over A. The right module structure on $\mathfrak{M} \in \mathcal{D}_A$ will be written symbolically as $\nu_{\mathfrak{M}} =$ . When $\mathcal{D} = \mathcal{C}$,  will be a shorter notation for the right A-module structure morphism of $\mathfrak{M}$ in $\mathcal{C}$. Furthermore, in this case one can define left A-modules, too. For simplicity, we denote by  the morphism that defines on an object $\mathfrak{N}$ of $\mathcal{C}$ a left A-module structure in $\mathcal{C}$.

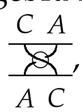
We can also define the dual notion of right comodule in a right $\mathcal{C}$-category $\mathcal{D}$ over a coalgebra C in $\mathcal{C}$. The category of right comodules and right colinear maps in $\mathcal{D}$ over C will be denoted as $\mathcal{D}^C$. The right comodule structure on $\mathfrak{M} \in \mathcal{D}^C$ will be written as $\rho_{\mathfrak{M}} =$ . When $\mathcal{C} = \mathcal{D}$ we denote the morphism $\rho_{\mathfrak{M}}$ by . Likewise, we can define left C-comodules in a monoidal category $\mathcal{C}$. In this case we will denote by  the morphism that defines on an object $\mathfrak{N}$ of $\mathcal{C}$ a a left C-comodule structure in $\mathcal{C}$.

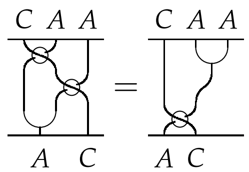
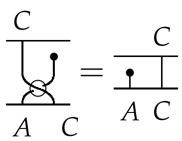
Finally, if A, B are algebras in a monoidal category $\mathcal{C}$ (respectively if C, D are coalgebras in $\mathcal{C}$) then we can define (A, B)-bimodules (respectively (C, D)-comodules) in $\mathcal{C}$. They are left A and right B-modules (respectively, left C and right D-comodules) such that

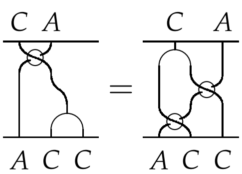
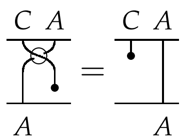
(a)  , resp. (b)  . (2.9)

3 Entwining structures defined by lax Doi-Koppinen and lax Yetter-Drinfeld structures

Throughout this section, $\mathcal{C}$ is a monoidal or a braided monoidal category. As we have seen, we can assume that $\mathcal{C}$ is strict, without loss of generality. Entwined structures over a field were introduced by Brzeziński in [5]; in [18], this notion was generalized to symmetric monoidal categories. In fact the assumption that $\mathcal{C}$ has a symmetry is not needed.

Definition 3.1. A right entwining structure in $\mathcal{C}$ is a triple (A, C, ψ), where A is an algebra in $\mathcal{C}$, C is a coalgebra in $\mathcal{C}$, and $\psi : C \otimes A \to A \otimes C$ is a morphism in $\mathcal{C}$, which we denote by $\psi =$ , such that the following equalities hold:

(a)  , (b)  ,

(c)  and (d)  . (3.1)

We call an entwining structure (A, C, ψ) in $\mathcal{C}$ trivial if $A = \underline{1}$ and $\psi = \mathrm{Id}_C$, or if $C = \underline{1}$ and $\psi = \mathrm{Id}_A$.

Next, we show that any entwining structure in $\mathcal{C}$ can be viewed as a trivial entwining structure in a different monoidal category. The next result is essentially due to Schauenburg [22] and has its roots in a paper of Tambara [23]. Also, our result is slightly more general than in [22] because we do not assume that the entwining map ψ is an isomorphism in $\mathcal{C}$.

Let us start by presenting some concepts and constructions. Similar to [22, Definition 4.2] we define the category of transfer morphisms through an algebra (or a coalgebra) as follows.

For an algebra A in $\mathcal{C}$, we consider the category $\mathfrak{T}_A$ of right transfer morphisms through A. The objects are pairs $(X, \psi_{X,A})$ with $X \in \mathcal{C}$ and $\psi_{X,A} : X \otimes A \to A \otimes X$ a morphism in $\mathcal{C}$ such that (3.1.a) and (3.1.b) are fulfilled (with C replaced by X). A morphism in $\mathfrak{T}_A$ between $(X, \psi_{X,A})$ and $(Y, \psi_{Y,A})$ is a morphism $\mu : X \to Y$ in $\mathcal{C}$ such that $(\mathrm{Id}_A \otimes \mu) \circ \psi_{X,A} = \psi_{Y,A} \circ (\mu \otimes \mathrm{Id}_A)$. $\mathfrak{T}_A$ is a strict monoidal category, with unit object $(\underline{1}, \mathrm{Id}_A)$ and tensor product

$$(X,\psi_{X,A})\underline{\otimes}(Y,\psi_{Y,A}) = (X\otimes Y,\psi_{X\otimes Y,A}),$$
$$\text{with } \psi_{X\otimes Y,A} := (\psi_{X,A}\otimes \mathrm{Id}_Y)(\mathrm{Id}_X\otimes \psi_{Y,A}). \quad (3.2)$$

We leave it to the reader to introduce the monoidal category $\mathfrak{T}^C$ of right transfer morphisms through the coalgebra C in $\mathcal{C}$.

Proposition 3.2. *Let $\mathcal{C}$ be a monoidal category.*

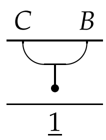
i) If A is an algebra in $\mathcal{C}$ then $(C,\psi_{C,A})$ is a coalgebra in $\mathfrak{T}_A$ if and only if $(A,C,\psi_{C,A})$ is a right entwining structure in $\mathcal{C}$.

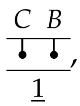
ii) If C is a coalgebra in $\mathcal{C}$ then $(A,\psi_{C,A})$ is an algebra in $\mathfrak{T}^C$ if and only if $(A,C,\psi_{C,A})$ is a right entwining structure in $\mathcal{C}$.

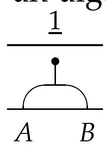
Consequently, (A,C,ψ) is a right entwining structure in $\mathcal{C}$ if and only if $((\underline{1},\mathrm{Id}_A),(C,\psi),\mathrm{Id})$ is a trivial entwining structure in $\mathcal{T}_A$ or, equivalently, $((A,\psi),(\underline{1},\mathrm{Id}_C),\mathrm{Id})$ is a trivial entwining structure in $\mathcal{T}^C$.

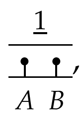
Proof. Straightforward. The verification of all these details is left to the reader. ∎

For $\mathcal{C} = {}_k\mathcal{M}$, the category of vector spaces, we obtain the classical definition of an entwining structure [5]. In this case, it is well known that a class of examples is given by the Doi-Koppinen structures (*DK* for short) over k. This can be generalized to entwining structures in braided monoidal categories.

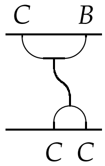
Assume that $\mathcal{C}$ is a braided monoidal category, and that B is an object in $\mathcal{C}$ which is both an algebra and a coalgebra (but not necessarily a bialgebra) in $\mathcal{C}$. A right $c^{\pm 1}$-module coalgebra over B is a coalgebra C in $\mathcal{C}$ which is also a right B-module such that the structure map $C\otimes B\to C$ is a coalgebra morphism from $C\otimes^{\pm} B$ to C. Similarly, a right $c^{\pm 1}$-comodule algebra over C is an algebra A in $\mathcal{C}$ which is also a right B-comodule such that the structure morphism $A\to A\otimes B$ is an algebra morphism from A to $A\otimes_{\pm} B$. For example, in the c-case, C is a coalgebra and a right B-module in $\mathcal{C}$, respectively A is an algebra and a right B-comodule in $\mathcal{C}$, such that 

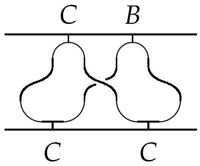
= 

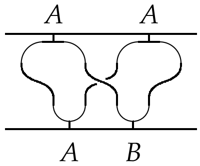
, respectively 

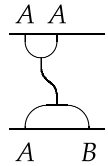
= 

, and the following equalities hold:

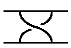
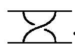
(a) 

= 

, resp. (b) 

= 

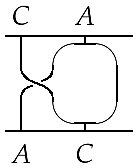
. (3.3)

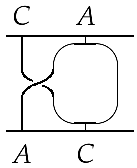
Note that for the c^{-1}-case we have to replace in the above equalities the braiding  with its inverse .

In a similar way, we can define the notions of left $c^{\pm 1}$-module coalgebra and left $c^{\pm 1}$-left comodule algebra over B.

Definition 3.3. Let $\mathcal{C}$ be a braided monoidal category and B an object of $\mathcal{C}$ which has both an algebra and a coalgebra structure in $\mathcal{C}$. If A is a right $c^{\pm 1}$-comodule algebra over B and C is a right $c^{\pm 1}$-module coalgebra over B, respectively, then we call the triple (B, A, C) a lax $c^{\pm 1}$-right Doi-Koppinen (DK for short)-structure in $\mathcal{C}$. If B is a bialgebra in $\mathcal{C}$ then we simply say that (B, A, C) is a $c^{\pm 1}$-right DK-structure in $\mathcal{C}$.

Proposition 3.4. *Let $\mathcal{C}$ be a braided monoidal category, B an object of $\mathcal{C}$ which has both an algebra and a coalgebra structure in $\mathcal{C}$, and $A \in \mathcal{C}^B$ and $C \in \mathcal{C}_B$. Define*

$\psi_+ :=$ 

, and $\psi_- :=$ 

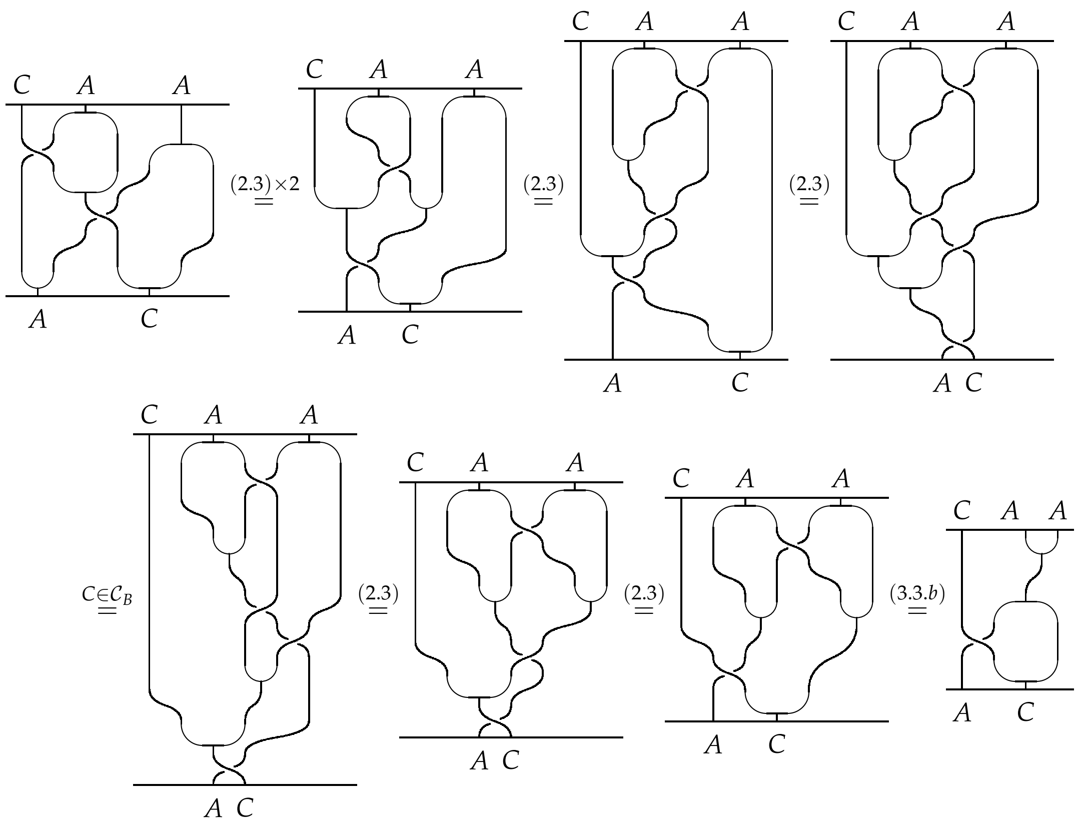
. (3.4)

Then the following assertions hold:

(i) If (B, A, C) is a lax c-right DK-structure in $\mathcal{C}$ then (A, C, ψ_+) is a right entwining structure in $\mathcal{C}$.

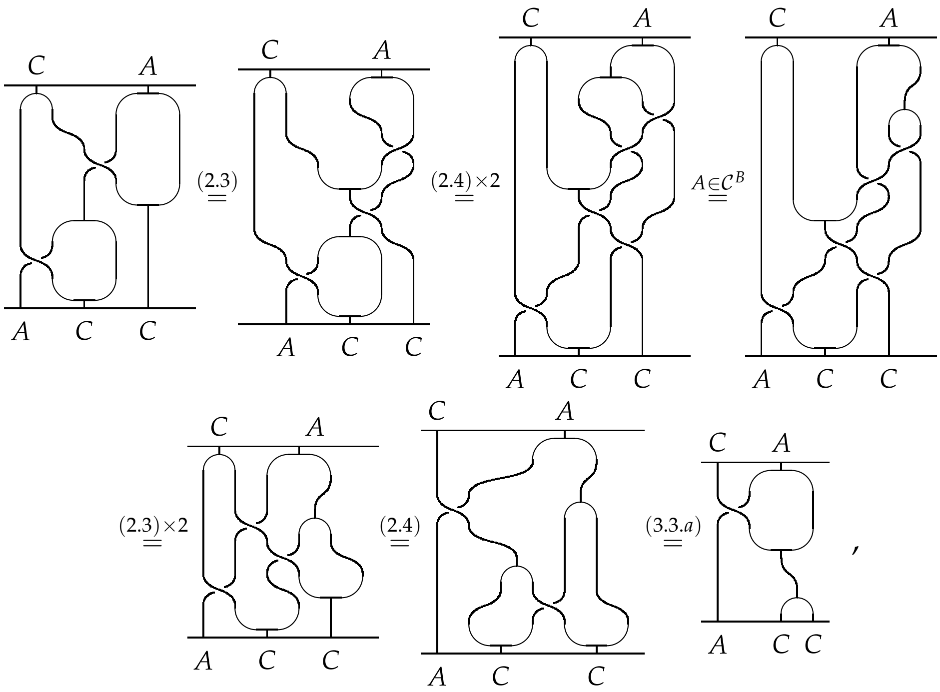
(ii) If (B, A, C) is a lax c^{-1}-right DK-structure in $\mathcal{C}$ then (A, C, ψ_-) is a right entwining structure in $\mathcal{C}$.

Proof. It suffices to prove (i), as (ii) is (i) applied to $\mathcal{C}^{\text{in}}$. The computation



,

shows that (3.1.a) is satisfied. In a similar way, we have that



and this shows that (3.1.c) is satisfied. It is easy to see that (A, C, ψ_+) satisfies (3.1.b,d), and this completes the proof. ∎

If $\mathcal{C}$ is symmetric then lax DK-structures in $\mathcal{C}$ can be obtained from lax Yetter-Drinfeld structures (abbreviated as YD-structures) in $\mathcal{C}$ defined over lax Hopf algebras.

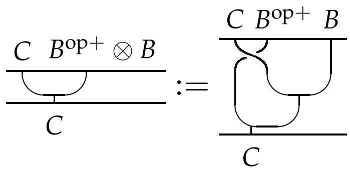
Definition 3.5. Let $\mathcal{C}$ be a braided monoidal category. A lax Hopf algebra in $\mathcal{C}$ is a sextuple $(B, \underline{m}_B, \underline{\eta}_B, \underline{\Delta}_B, \underline{\eta}_B, \underline{S}_B)$ consisting in an algebra $(B, \underline{m}_B, \underline{\eta}_B)$ and a coalgebra $(B, \underline{\Delta}_B, \underline{\varepsilon}_B)$ structure on an object B of $\mathcal{C}$, and an anti-algebra and anti-coalgebra endomorphism $\underline{S}_B$ of B satisfying (2.7).

It is clear that a braided Hopf algebra is a lax Hopf algebra. Let $\mathcal{C}$ be a braided monoidal category and B an object of $\mathcal{C}$ which is both an algebra and coalgebra in $\mathcal{C}$. A (c, c)-bimodule coalgebra over B is a coalgebra C in $\mathcal{C}$ which has a B-bimodule structure such that C is both a left and right c-module coalgebra over B. Similarly, we can define (c, c^{-1}), (c^{-1}, c) and (c^{-1}, c^{-1})-bimodule coalgebras over B in $\mathcal{C}$, respectively. Likewise, a (c, c)-bicomodule algebra in $\mathcal{C}$ is an algebra A of $\mathcal{C}$ endowed with a B-bicomodule structure such that A is simultaneously a left and right c-comodule algebra over B. In a similar manner one can define (c, c^{-1}), (c^{-1}, c) and (c^{-1}, c^{-1})-bicomodule algebras over B in $\mathcal{C}$, respectively.

Definition 3.6. Let $\mathcal{C}$ be a braided monoidal category and $B \in \mathcal{C}$ such that B is both an algebra and a coalgebra in $\mathcal{C}$. If C is a (c, c)-bimodule coalgebra and A is a (c, c)-bicomodule algebra over B in $\mathcal{C}$, then we call the triple (B, C, A) a lax $\begin{pmatrix} c & c \\ c & c \end{pmatrix}$ YD-structure in $\mathcal{C}$ (the first row of the matrix refers to the bicomodule algebra structure, while the second row refers to the bimodule coalgebra structure over B). In a similar way we define lax $\begin{pmatrix} c & c \\ c^{-1} & c \end{pmatrix}, \cdots, \begin{pmatrix} c^{-1} & c^{-1} \\ c^{-1} & c^{-1} \end{pmatrix}$ YD-structures in $\mathcal{C}$. If B is a bialgebra in $\mathcal{C}$ then the word lax will be omitted. Also, in the particular case when $\mathcal{C}$ is symmetric monoidal we simply say that (B, C, A) is a (lax) YD-structure in $\mathcal{C}$.

First, we show that any (c^{-1}, c)-bimodule coalgebra can be viewed as a c-right module coalgebra.

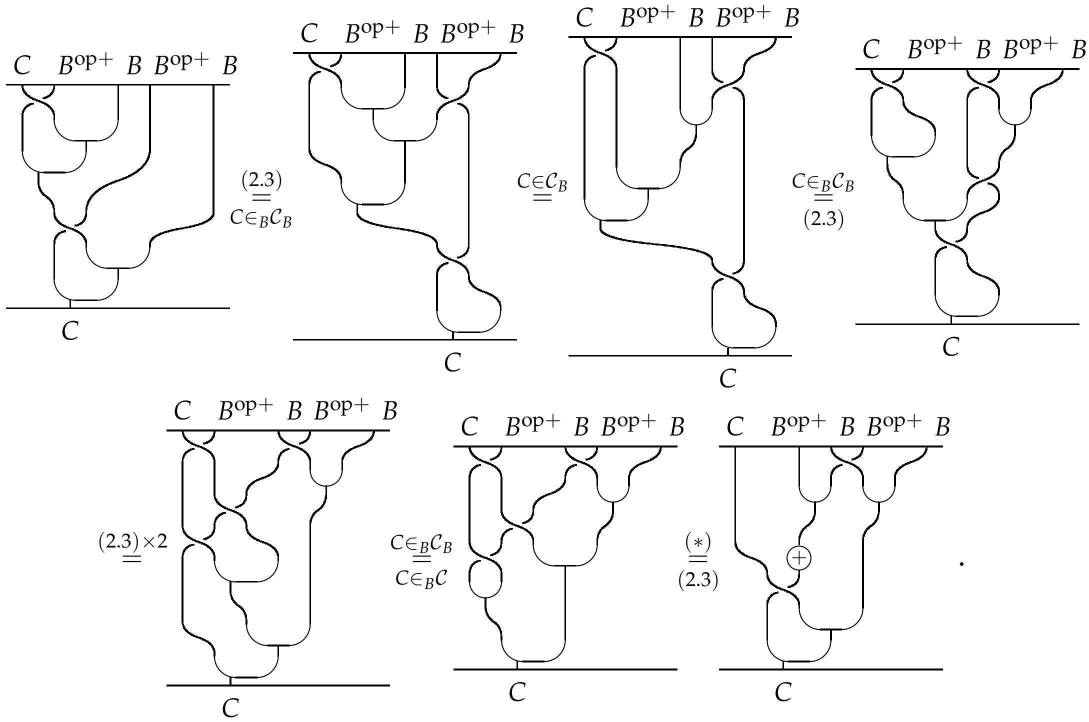
Lemma 3.7. *Let $\mathcal{C}$ be a braided monoidal category, B an object of $\mathcal{C}$ which has both an algebra and a coalgebra structure in $\mathcal{C}$, and C a (c^{-1}, c)-bimodule coalgebra over B in $\mathcal{C}$. Then C with structure defined by*



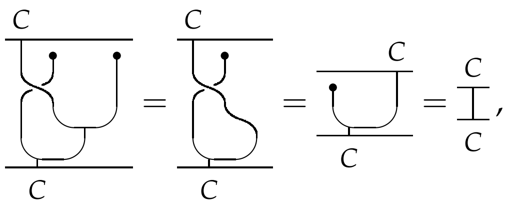
(3.5)

is a c-right module coalgebra over $B^{\mathrm{op}+} \otimes^{+}_{+} B$ in $\mathcal{C}$.

Proof. C is a right $B^{\mathrm{op}+} \otimes_{+} B$-module since

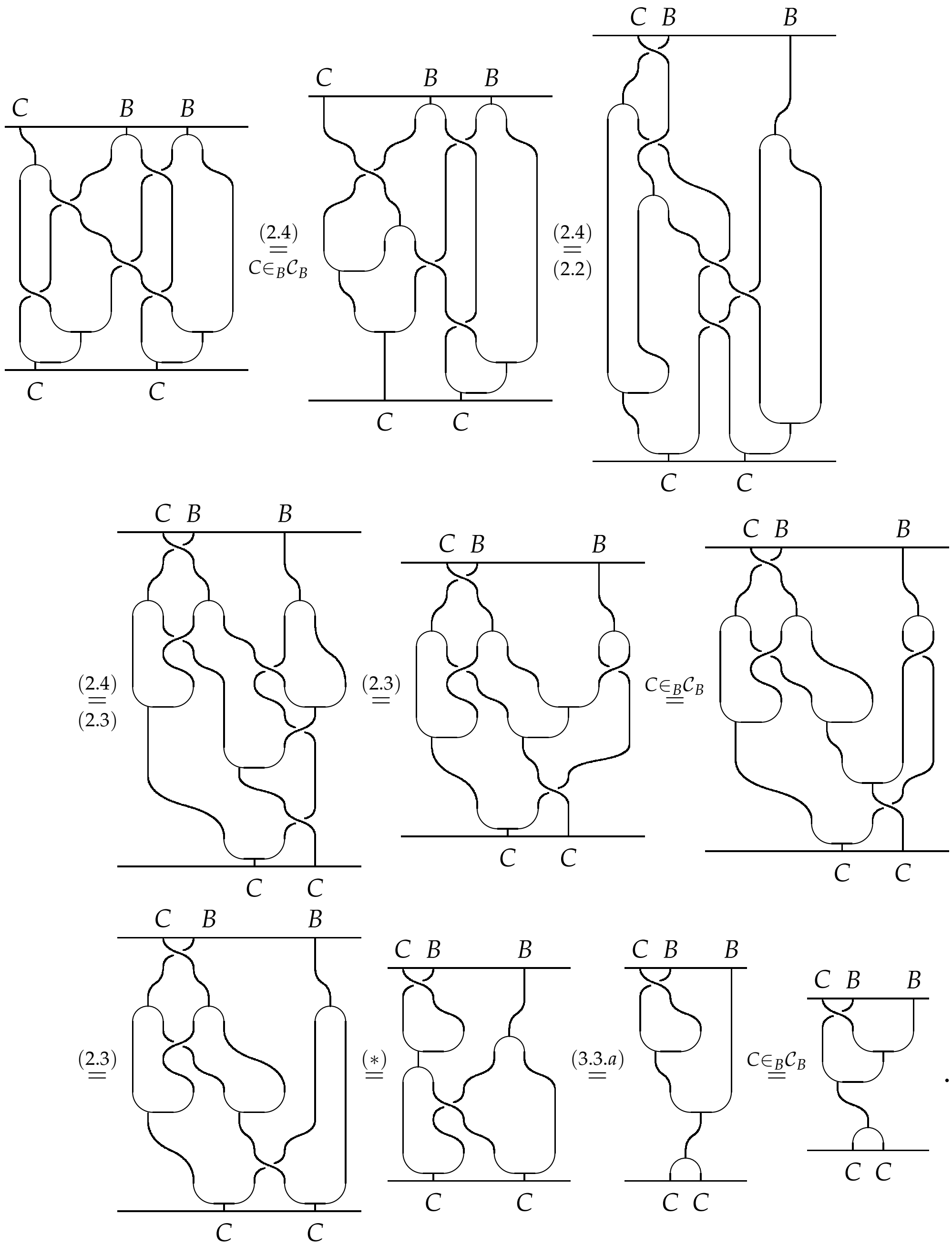


In the equality $(*)$, we used the definition of $\underline{\mathrm{m}}^{\mathrm{op}+}$. We also have that

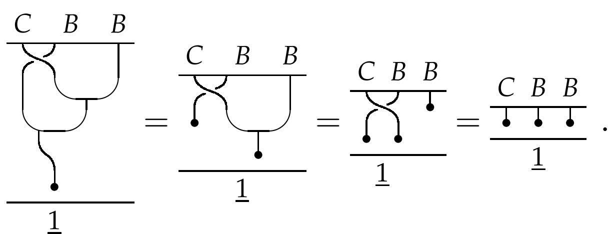


since $C \in {}_B\mathcal{C}_B$ and $c_{X,\underline{1}} = c_{\underline{1},X} = \mathrm{Id}_X$, for any object X of $\mathcal{C}$ (see [19, Prop. XIII.1.2]).

Next, we prove that the structure map $C \otimes^+ (B \otimes^+ B) \to C$ defined above is a coalgebra morphism in $\mathcal{C}$. This follows from our next computation. At the very beginning we use (2.1) (a), and in $(*)$, we use the fact that C is a c^{-1}-left module coalgebra over B.



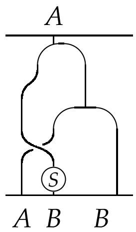
In a similar way, we have



Here we used successively that C is a left c^{-1}-module coalgebra, C is a right c-module coalgebra, and $c_{C,\underline{1}} = \mathrm{Id}_C$. ■

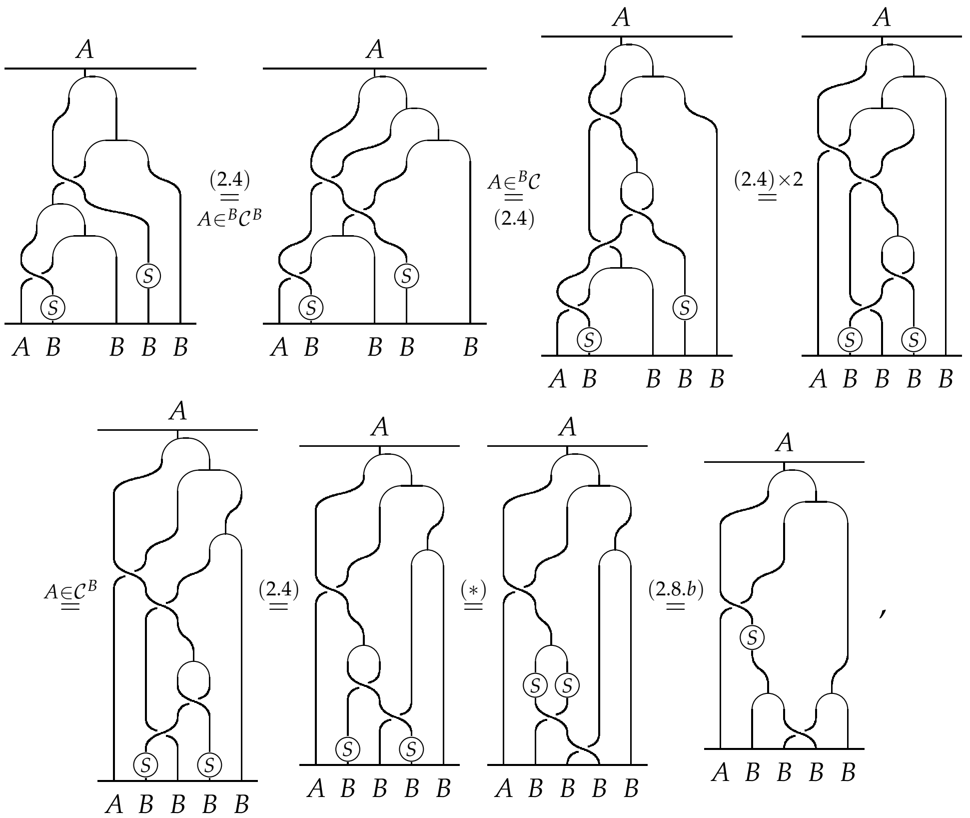
Our next aim is to show that a bicomodule algebra over a lax Hopf algebra gives rise to a right comodule algebra.

Lemma 3.8. *Let $\mathcal{C}$ be a braided monoidal category, B a lax Hopf algebra in $\mathcal{C}$ and A a (c^{-1}, c)-bicomodule algebra over B. Then A is a c-right comodule algebra over $B^{\mathrm{op}+} \otimes_{-}^{+} B$ in $\mathcal{C}$ via the structure morphism $\underline{\rho}_A : A \to A \otimes B^{\mathrm{op}+} \otimes B$ given by*

$\underline{\rho}_A :=$ 

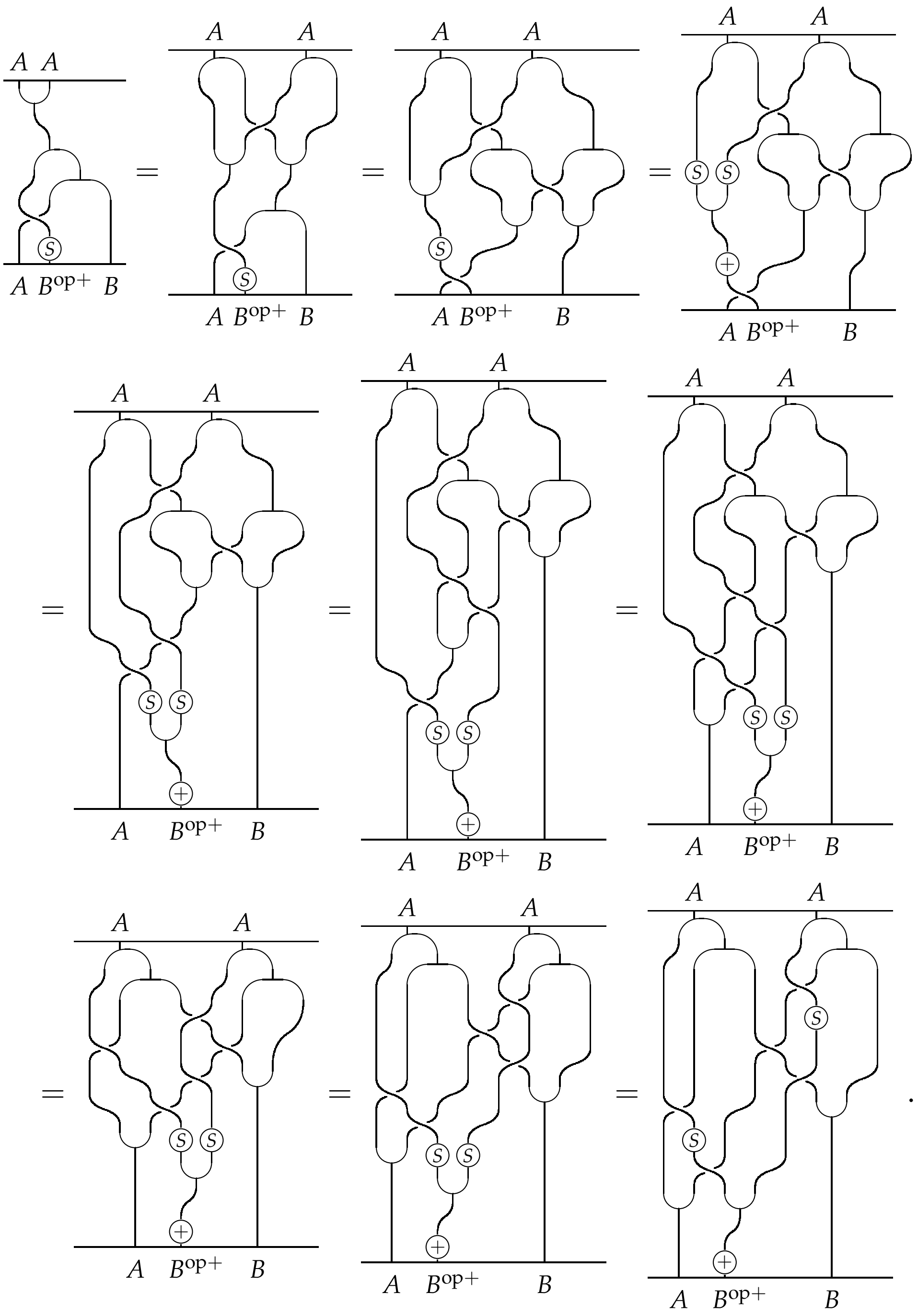
. (3.6)

Proof. We first check that A is a right $B \otimes^{+} B$-comodule.



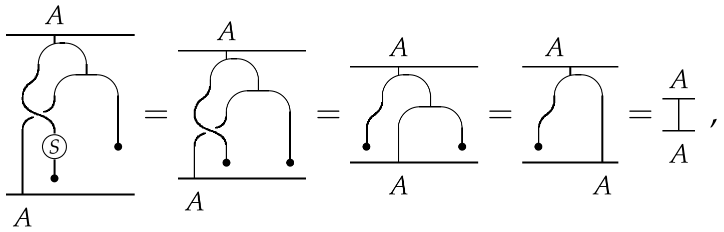
and this is exactly what we need. Note that in $(*)$ we used the naturality of the braiding c twice.

We show next that $\underline{\rho}_A$ is an algebra map from A to $A \otimes_+ (B^{\mathrm{op}+} \otimes_- B)$. It is easy to check that $\underline{\rho}_A$ respects the unit morphisms. We also have that



In the first equality we used the fact that A is a c^{-1}-left comodule algebra over B; in the second equality we used the naturality of the braiding c and the fact that A is a c-right comodule algebra over B; in the third equality we used (2.8.a), the naturality of the braiding c and the definition of the multiplication of $B^{\mathrm{op}+}$ in (2.5); in the fourth, fifth and sixth equality we used (2.3); in the seventh equality we used (2.4); in the eight equality we used (2.2); in the final equality th naturality of c was used twice. Finally, using (2.8.b), the naturality of the braiding c, $c_{\underline{1},A} = \mathrm{Id}_A$

and the fact that A is a B-bicomodule, we compute



and this finishes the proof. ■

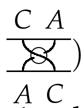
Applying Lemmas 3.7, 3.8 and Proposition 3.4 we obtain the following result.

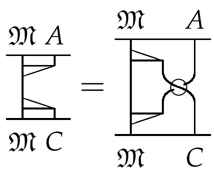
Proposition 3.9. *Let B be a lax Hopf algebra in a symmetric monoidal category $\mathcal{C}$. To any right lax YD-structure (B, C, A) in $\mathcal{C}$ we can associate a right lax DK-structure in $\mathcal{C}$, namely $(B^{\mathrm{op}} \otimes B, C, A)$. Consequently, any lax YD-structure (B, C, A) over a lax Hopf algebra B produces a right entwining structure (A, C, ψ) in $\mathcal{C}$, where ψ can be explicitly computed using (3.4), (3.5) and (3.6).*

4 Entwined modules - a module categorical approach

Let A be an algebra in a monoidal category $\mathcal{C}$. To A we have associated the monoidal category $\mathfrak{T}_A$, called the category of transfer morphisms through A. Then it came out that there is a bijective correspondence between right entwining structures in $\mathcal{C}$ and coalgebras in $\mathfrak{T}_A$. On the other hand, to a right entwining structure (A, C, ψ) in $\mathcal{C}$, we can associate the category of entwined modules $\mathcal{C}(\psi)_A^C$. The objects are right A-modules and right C-comodules in $\mathcal{C}$ for which the right C-comodule morphism structure is right A-linear or, equivalently, for which the right A-module morphism structure is right C-colinear. The morphisms in $\mathcal{C}(\psi)_A^C$ are the morphisms in $\mathcal{C}$ which are right A-linear and right C-colinear.

Based on these observations and with the help of the notion of $\mathcal{C}$-category we will see that entwined modules can be viewed as comodules over a coalgebra in the category of transfer morphisms through an algebra in $\mathcal{C}$. First we generalize the classical notion of entwined module.

Definition 4.1. Let $\mathcal{C}$ be a monoidal category, let $\mathcal{D}$ be a right $\mathcal{C}$-category and let $(A, C, \psi =$ $)$ be a right entwining structure in $\mathcal{C}$. An object $\mathfrak{M} \in \mathcal{D}$ is called a right entwined module with entwining map ψ if $\mathfrak{M}$ is a right module in $\mathcal{D}$ over A, a right comodule in $\mathcal{D}$ over C and the following compatibility relation holds:



(4.1)

$\mathcal{D}(\psi)_A^C$ will be the category of right entwined modules with entwining map ψ and right A-module and right C-colinear morphisms.

Remark 4.2. Observe that Definition 4.1 does not cover all categories of Hopf modules defined over Hopf algebras and their generalization. For example, the category of Doi-Hopf modules over a quasi-Hopf algebra H (see [7]) has not such a description. This is because in the quasi-Hopf case a right DK-structure consists of a right H-comodule algebra $\mathfrak{A}$ (in the sense of Hausser and Nill [17]) which is associative and a right H-module coalgebra C (in the sense of [12]) which is coassociative up to the reassociator of H which is, in general, not trivial. Therefore $(H, \mathfrak{A}, C)$ does not produce a right entwining structure in a certain monoidal category. Nevertheless, in the forthcoming paper [11] we will see that the notion of entwining structure in a monoidal category can be generalized. Then, using the point of view suggested by the result below we will define a more general category of entwined modules, unifying in this way most of the Doi-Hopf module categories known so far.

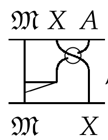
Proposition 4.3. *Let $\mathcal{C}$ be a monoidal category, (A, C, ψ) a right entwining structure in $\mathcal{C}$ and $\mathcal{D}$ a right $\mathcal{C}$-category.*

(i) If $\mathfrak{M} \in \mathcal{D}_A$ and $(X, \psi_{X,A}) \in \mathfrak{T}_A$ then $\mathfrak{M} \diamond X \in \mathcal{D}_A$ with the structure

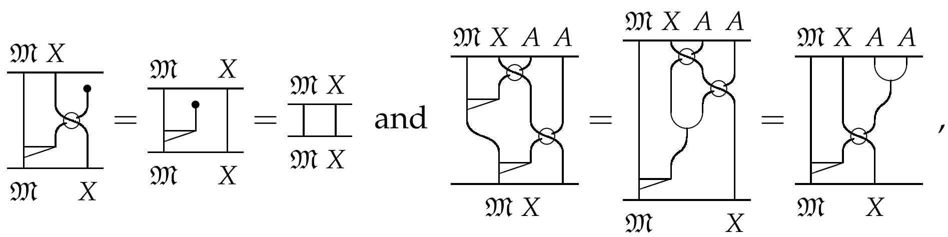
$$\nu_{\mathfrak{M}\diamond X} : \Big((\mathfrak{M} \diamond X) \diamond A \xrightarrow{\Psi_{\mathfrak{M},X,A}} \mathfrak{M} \diamond (X \otimes A) \xrightarrow{\mathrm{Id}\diamond\psi_{X,A}} \mathfrak{M} \diamond (A \otimes X) \xrightarrow{\Psi^{-1}_{\mathfrak{M},A,X}} (\mathfrak{M} \diamond A) \diamond X \xrightarrow{\nu_{\mathfrak{M}}\diamond \mathrm{Id}} \mathfrak{M} \diamond X \Big).$$

The associated functor from $\mathcal{D}_A \times \mathfrak{T}_A$ to $\mathcal{D}_A$ turns $\mathcal{D}_A$ into a right $\mathfrak{T}_A$-category.

(ii) The category of entwined modules $\mathcal{D}(\psi)^C_A$ coincides with the category of right comodules in $\mathcal{D}_A$ over the coalgebra (C, ψ) in $\mathfrak{T}_A$.

Proof. In diagrammatic notation, we have that $\nu_{\mathfrak{M}\diamond X} =$ 

, hence

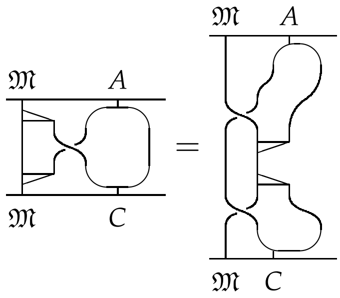


by (3.1.a). All other details are left to the reader. ∎

Example 4.4. Let (B, A, C) be a lax c-right DK-structure in a braided monoidal category $\mathcal{C}$, and let $\mathcal{D}$ be a right $\mathcal{C}$-category. The category $\mathcal{D}_{\mathrm{lax}}(B)^C_A$ of entwined modules in $\mathcal{D}$ corresponding to the right entwining structure in $\mathcal{C}$ defined by (3.4) is called the category of lax right Doi-Hopf modules in $\mathcal{D}$ over B.

Now we introduce the braided version of the category of Yetter-Drinfeld modules.

Definition 4.5. Let (B, A, C) be a lax $\begin{pmatrix} c & c \\ c & c \end{pmatrix}$ YD-structure in a braided monoidal category $\mathcal{C}$, and let $\mathcal{D}$ be a right $\mathcal{C}$-category. An object $\mathfrak{M}$ of $\mathcal{D}$ is called a lax right Yetter-Drinfeld module if $\mathfrak{M} \in \mathcal{D}_A$, $\mathfrak{M} \in \mathcal{D}^C$ and the following compatibility relation between these structures holds:



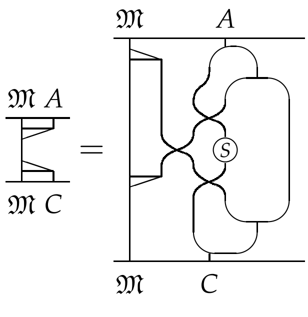
(4.2)

Then $\mathcal{YD}_{\mathrm{lax}}(B)^C_A$ will be the notation for the category of lax right Yetter-Drinfeld modules in $\mathcal{D}$ over B, and right A-linear and right C-colinear morphisms.

Theorem 4.6 is the braided version of the main result in [15], and tells us that $\mathcal{YD}_{\mathrm{lax}}(B)^C_A$ is a category of entwined modules, at least if we work over a symmetric monoidal category $\mathcal{C}$.

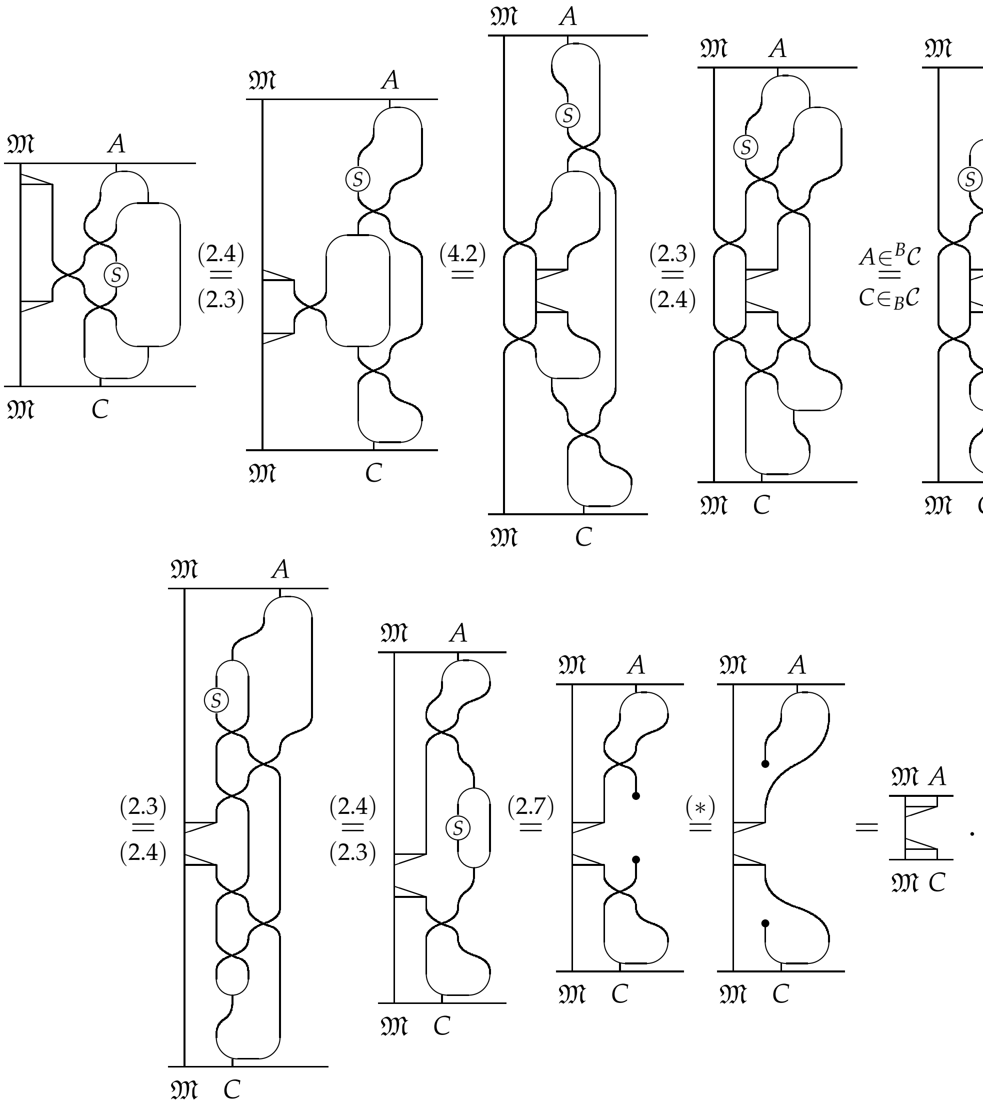
Theorem 4.6. *Let $\mathcal{D}$ be a right $\mathcal{C}$-category with $\mathcal{C}$ symmetric monoidal. If B is a lax Hopf algebra in $\mathcal{C}$ and (B, A, C) is a lax YD-structure in $\mathcal{C}$ then* $\mathcal{YD}_{\mathrm{lax}}(B)^C_A = \mathcal{D}_{\mathrm{lax}}(B^{\mathrm{op}} \otimes B)^C_A$.

Proof. By Example 4.4 and Proposition 3.9 an object $\mathfrak{M}$ of $\mathcal{D}_{\mathrm{lax}}(B^{\mathrm{op}} \otimes B)^C_A$ is a right module in $\mathcal{D}$ over A and a right comodule in $\mathcal{D}$ over C such that

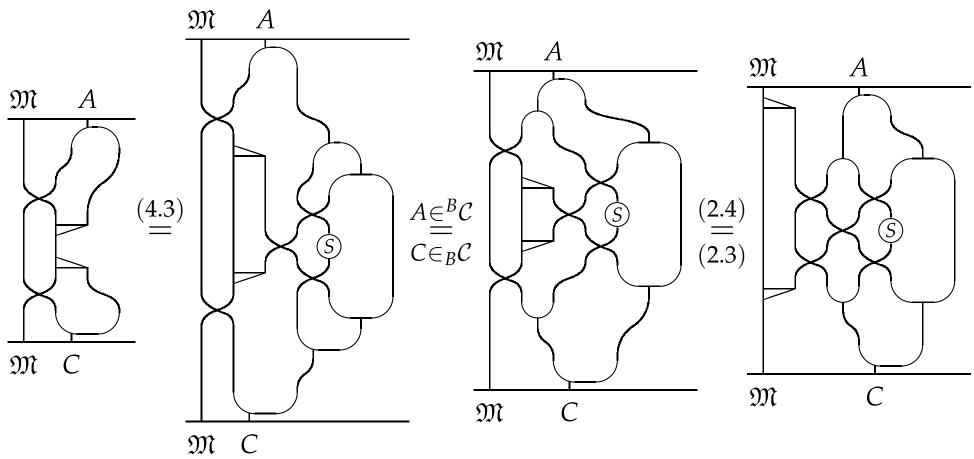


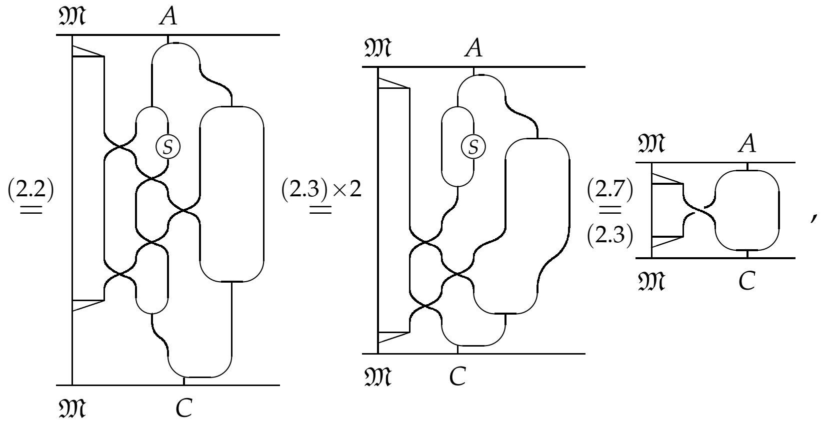
(4.3)

Therefore it suffices to show that (4.2) and (4.3) are equivalent. The following computation shows that (4.2) implies (4.3). Observe that $(*)$ follows from the naturality of the braiding c and the fact that $c_{\underline{1},X} = c_{X,\underline{1}} = \mathrm{Id}_X$, and that the last equality follows from the unit and counit axioms.



Conversely, assume that (4.3) holds. We then compute:





and this ends the proof. In the last equality we also used the fact that $c_{\underline{1},X} = c_{X,\underline{1}} = \mathrm{Id}_X$, for all $X \in \mathcal{C}$. ■

5 Monoidal entwining structures defined by weak Hopf algebra actions and coactions

The aim of this Section is to present examples of entwining structures in monoidal categories obtained from actions and coactions of a weak bialgebra. Then we relate our results to some existing results on Doi-Hopf modules and Yetter-Drinfeld modules over a weak Hopf algebra.

Let k be a field. Recall from [4] that a weak bialgebra is a k-module H together with a k-algebra structure (H, m, u) and a k-coalgebra structure (H, Δ, ε) such that Δ is multiplicative and the following relations hold

$$\mathbf{1}_1 \otimes \mathbf{1}_2 \otimes \mathbf{1}_3 = \mathbf{1}_1 \otimes \mathbf{1}_2\mathbf{1}_{1'} \otimes \mathbf{1}_{2'} = \mathbf{1}_1 \otimes \mathbf{1}_{1'}\mathbf{1}_2 \otimes \mathbf{1}_{2'}, \tag{5.1}$$

$$\varepsilon(ghl) = \varepsilon(gh_1)\varepsilon(h_2l) = \varepsilon(gh_2)\varepsilon(h_1l), \ \forall\ g, h, l \in H. \tag{5.2}$$

$\mathbf{1}_{1'} \otimes \mathbf{1}_{2'}$ is a second copy of $\Delta(\mathbf{1})$, $\mathbf{1}$ is the unit of H and $\Delta(h) = h_1 \otimes h_2$, $h \in H$. It is known that the category of right H-(co)representations is monoidal; this can be explained easily using the following arguments, as presented in [3].

The endomorphisms $\varepsilon_s, \varepsilon_t : H \to H$, $\varepsilon_s(h) = \varepsilon(h\mathbf{1}_2)\mathbf{1}_1$, $\varepsilon_t(h) = \varepsilon(\mathbf{1}_1h)\mathbf{1}_2$ are idempotent. Their images

$$H_t := \{h \in H \mid \Delta(h) = \mathbf{1}_1h \otimes \mathbf{1}_2\} \ \text{ and } \ H_s := \{h \in H \mid \Delta(h) = \mathbf{1}_1 \otimes h\mathbf{1}_2\},$$

called the target and source subspace of H, are subalgebras of H.

If H a weak k-bialgebra then so is H^{op}, and it can be easily seen that

$$\varepsilon_s^{\mathrm{op}}(h) = \varepsilon(\mathbf{1}_2h)\mathbf{1}_1 := \overline{\varepsilon}_s(h) \ \text{ and } \ \varepsilon_t^{\mathrm{op}}(h) = \varepsilon(h\mathbf{1}_1)\mathbf{1}_2 := \overline{\varepsilon}_t(h),$$

for all $h \in H$. Note that the map ε_t restricts to an anti-algebra isomorphism from H_s to H_t with inverse $\overline{\varepsilon}_s$, while ε_s restricts to an anti-algebra isomorphism from H_t to H_s with inverse $\overline{\varepsilon}_t$.

Now, if M is a right H-module then M becomes an H_s-bimodule via $r \cdot m \cdot r' = m\overline{\varepsilon}_t(r)r' = mr'\overline{\varepsilon}_t(r)$, for all $m \in M$ and $r, r' \in H_s$. Furthermore, H_s is a right H-module via $r \triangleleft h = \varepsilon_s(rh)$, for all $r \in H_s$ and $h \in H$. Then $(\mathcal{M}_H, \otimes_{H_s}, H_s)$ is monoidal in such a way that the forgetful functor from $\mathcal{M}_H$ to ${}_{H_s}\mathcal{M}_{H_s}$ turns into a monoidal functor.

We call a coalgebra C in $\mathcal{M}_H$ a right H-module coalgebra. The next result says that this notion is equivalent to the notion of weak H-module coalgebra, as introduced in [2].

Proposition 5.1. *If H is a weak k-bialgebra then C is a right H-module coalgebra if and only if C has a right H-module structure and a coalgebra structure $(C, \Delta_C, \varepsilon_C)$ in ${}_k\mathcal{M}$ that are compatible in the following sense,*

$$\Delta_C(c \cdot h) = c_1 \cdot h_1 \otimes c_2 \cdot h_2, \ \text{ and } \tag{5.3}$$

$$\varepsilon_C(c \cdot \varepsilon_t(h)) = \varepsilon_C(c \cdot h), \ \forall \ c \in C, \ h \in H. \tag{5.4}$$

Proof. First let $(C, \underline{\Delta}, \underline{\varepsilon})$ be a coalgebra in $\mathcal{M}_H$. We have a well-defined map $\Delta_C : C \to C \otimes C$ given by

$$\Delta(c) = c_{\underline{1}} \cdot \overline{\varepsilon}_s(\mathbf{1}_2) \otimes \mathbf{1}_1 \cdot c_{\underline{2}} = c_{\underline{1}} \cdot \overline{\varepsilon}_s(\mathbf{1}_2) \otimes c_{\underline{2}} \cdot \overline{\varepsilon}_t(\mathbf{1}_1) = c_{\underline{1}} \cdot \mathbf{1}_1 \otimes c_{\underline{2}} \cdot \mathbf{1}_2,$$

for all $c \in C$. Moreover, $(C, \Delta_C, \varepsilon_C := \varepsilon\underline{\varepsilon})$ is a k-coalgebra, a right H-module and the conditions in the statement are satisfied.

Conversely, if $(C, \Delta_C, \varepsilon_C)$ is a k-coalgebra and a right H-module obeying (5.3) and (5.4) then C becomes a right H-module coalgebra via the comultiplication $\underline{\Delta} : C \xrightarrow{\Delta_C} C \otimes C \longrightarrow C \otimes_{H_s} C$ and the counit $\underline{\varepsilon} : C \to H_s$ given by $\underline{\varepsilon}(c) = \varepsilon_C(c \cdot \mathbf{1}_2)\mathbf{1}_1$, for all $c \in C$. ∎

The monoidal structure on the category $\mathcal{M}^H$ is defined in a similar manner. In this case the key observation is the fact that H is an H_s-coring. Indeed, it is well-known that

$$\widetilde{\Delta} : H \xrightarrow{\Delta} H \otimes H \longrightarrow H \otimes_{H_s} H$$

together with $\widetilde{\varepsilon} := \varepsilon_s$ defines a coalgebra structure on H within the monoidal category ${}_{H_s}\mathcal{M}_{H_s}$. This fact allows us to define a right corepresentation over a weak bialgebra:

Definition 5.2. A right comodule over a weak k-bialgebra H consists in a right comodule over the H_s-coring H. Otherwise stated, a right comodule over H is a right H_s-module M together with a right H_s-module map $\rho^M : M \to M \otimes_{H_s} H$ such that, via the canonical identifications in ${}_{H_s}\mathcal{M}_{H_s}$, we have

$$(\rho^M \otimes_{H_s} \mathrm{Id}_H)\rho^M = (\mathrm{Id}_M \otimes_{H_s} \widetilde{\Delta})\rho^M \ \text{ and } \ (\mathrm{Id}_M \otimes_{H_s} \varepsilon_s)\rho^M = \mathrm{Id}_M.$$

A morphism $f : M \to N$ between two right comodules over H is a right H_s-module map satisfying $(f \otimes_{H_s} \mathrm{Id}_H)\rho^N = \rho^M f$. The category of right comodules over H and right H-comodule maps is denoted by $\mathcal{M}^H$.

Although a right comodule M over H is not necessarily a left H_s-module, one can easily see that $M \otimes_{H_s} H$ is a left H_s-module by defining $r \cdot (m \otimes_{H_s} h) = m \otimes_{H_s} rh$, for all $r \in H_s$, $m \in M$ and $h \in H$. Observe that this action is well-defined because of the H_s-bimodule structure of H. Using this observation and specializing [1, Proposition 1.1] for a weak bialgebra we obtain the following result.

Proposition 5.3. *Let H be a weak bialgebra and M a right comodule over H via the structure morphism $\rho : M \to M \otimes_{H_s} H$, $\rho^M(m) := m_{(0)} \otimes_{H_s} m_{(1)}$, for all $m \in M$. Then there is a unique left H_s-module structure on M making ρ^M a left H_s-module morphism. Namely, $r \cdot m = m_{(0)} \cdot \varepsilon_s(rm_{(1)})$, for all $r \in H_s$ and $m \in M$.*

Furthermore, with this additional structure,

(1) M becomes an H_s-bimodule;

(2) ρ^M becomes an H_s-bimodule map;

(3) $\mathrm{Im}(\rho^M) \subseteq \Big\{\sum_i x_i \otimes_{H_s} y_i \in M \otimes_{H_s} H \mid \sum_i r \cdot x_i \otimes_{H_s} y_i = \sum_i x_i \otimes_{H_s} \overline{\varepsilon}_t(r) y_i, \ \forall\ r \in H_s\Big\}$;

(4) any morphism in $\mathcal{M}^H$ becomes an H_s-bimodule map.

We are now able to describe the monoidal structure of $\mathcal{M}^H$, when H is a weak k-bialgebra.

For $X, Y \in \mathcal{M}^H$ we have seen that X, Y are H_s-bimodules, and so we can define their tensor product as being $X \otimes_{H_s} Y$, the tensor product in the monoidal category of H_s-bimodules. If we endow $X \otimes_{H_s} Y$ with the right H-coaction $\rho^{X \otimes_{H_s} Y}$ given by

$$X \otimes_{H_s} Y \ni x \otimes_{H_s} y \mapsto (x_{(0)} \otimes_{H_s} y_{(0)}) \otimes_{H_s} x_{(1)} y_{(1)} \in (X \otimes_{H_s} Y) \otimes_{H_s} H$$

then this coaction is well-defined and determines on $X \otimes_{H_s} Y$ a right comodule structure over H. Furthermore, in this way we have a monoidal category $(\mathcal{M}^H, \otimes_{H_s}, H_s)$, where the unit object H_s is a right comodule over H via the trivial coaction $H_s \ni r \mapsto r \otimes_{H_s} \mathbf{1} \in H_s \otimes_{H_s} H$. This monoidal structure is designed in such a way that the forgetful functor from $\mathcal{M}^H$ to ${}_{H_s}\mathcal{M}_{H_s}$ becomes a monoidal functor.

An algebra in the monoidal category $\mathcal{M}^H$ will be called a right H-comodule algebra. As the reader might expect, this notion is equivalent to the notion of right weak comodule algebra over H, in the sense of [2]. The next result is the (improved) right version of [6, Proposition 3.9].

Proposition 5.4. *Let H be a weak k-bialgebra. Then to give a right H-comodule algebra is equivalent to give a k-algebra A with unit 1 such that A is a right H-comodule in ${}_k\mathcal{M}$, the comodule structure morphism $\rho : A \to A \otimes H$ is multiplicative and $\rho(1) \in A \otimes H_t$.*

Proof. We sketch the proof, leaving further detail to the reader.

If A is a right comodule algebra over H then H is a k-algebra with multiplication

$$A \otimes A \longrightarrow A \otimes_{H_s} A \xrightarrow{\underline{m}_A} A$$

and unit 1, where $(A, \underline{m}_A, 1)$ stands for the algebra structure of A in $\mathcal{M}^H$. In addition, if $A \ni a \mapsto \underline{\rho}^A(a) = a_{\underline{\langle 0 \rangle}} \otimes_{H_s} a_{\underline{\langle 1 \rangle}} \in A \otimes_{H_s} H$ denotes the right coaction of the H_s-coring H on A then $A \ni a \mapsto \rho^A(a) := a_{\underline{\langle 0 \rangle}} \cdot \mathbf{1}_1 \otimes a_{\underline{\langle 1 \rangle}} \cdot \mathbf{1}_2 \in A \otimes H$ is well-defined and satisfies the requirements in the statement.

Conversely, let A be a k-algebra and $A \ni a \mapsto \rho^A(a) = a_{\langle 0 \rangle} \otimes a_{\langle 1 \rangle} \in A \otimes H$ a multiplicative map that endows A with a right H-comodule structure in ${}_k\mathcal{M}$ such that $\rho^A(1) \in A \otimes H_t$. Then A is a right H_s-module via $a \cdot r = \varepsilon(a_{\langle 1 \rangle} r) a_{\langle 0 \rangle}$, for $a \in A$ and $r \in H_s$, and a right H-comodule via the structure morphism $A \ni a \mapsto \underline{\rho}^A(a) := a_{\langle 0 \rangle} \otimes_{H_s} a_{\langle 1 \rangle} \in A \otimes_{H_s} H$. ∎

Let H be a weak k-bialgebra, A a right H-comodule algebra H and C a right H-module coalgebra. We call (H, A, C) a right Doi-Koppinen (DK for short) structure over H.

Proposition 5.5. *If (H, A, C) is a right DK structure over a weak k-bialgebra H then (A, C, ψ) with $\psi : C \otimes_{H_s} A \to A \otimes_{H_s} C$ defined by*

$$\psi(c \otimes_{H_s} a) = a_{\underline{\langle 0\rangle}} \otimes_{H_s} c \cdot a_{\underline{\langle 1\rangle}}, \ \ \forall \ a \in A, \ \ c \in C,$$

is a right entwining structure in ${}_{H_s}\mathcal{M}_{H_s}$.

Proof. This follows after we specialize [9, Proposition 5.13] to a weak bialgebra. ■

It is immediate that the bifunctor $\otimes_{H_s}$ defines a right ${}_{H_s}\mathcal{M}_{H_s}$-category structure on $\mathcal{M}_{H_s}$. If we consider the associated category of entwined modules $\mathcal{M}_{H_s}(\psi)_A^C$ as in Definition 4.1 then by the comments made after the proof of [9, Prop. 5.13], see also [6, Theorem 3.11 & Prop. 4.1], we obtain that $\mathcal{M}_{H_s}(\psi)_A^C$ is isomorphic to the category of weak Doi-Hopf modules $\mathcal{M}(H)_A^C$ in the sense of [2] and to the category of Doi-Koppinen modules over (H, A, C) in the sense of [6].

An alternative approach to Doi-Hopf modules over a weak bialgebra is the following.

Proposition 5.6. *Let H be a weak k-bialgebra and A a right H-comodule algebra. Then $\mathcal{M}_A$, the category of right A-modules in ${}_k\mathcal{M}$ is a right $\mathcal{M}_H$-category via the functor $\diamond : \mathcal{M}_A \times \mathcal{M}_H \to \mathcal{M}_A$ defined as follows. If $M \in \mathcal{M}_A$ and $X \in \mathcal{M}_H$ then $M \diamond X := M \otimes_{H_s} X$, where M is a right H_s-module via $m * r = m \cdot (1 \cdot r)$, for all $m \in M$ and $r \in H_s$, and where X inherits the left H_s-module of H, i.e., $r \cdot x = x \cdot \overline{\varepsilon}_t(r)$, for all $r \in H_s$ and $x \in X$. $M \diamond X \in \mathcal{M}_A$ with the right A-action, $m \in M$, $x \in X$ and $a \in A$,*

$$(m \otimes_{H_s} x) \cdot a = m \cdot a_{\underline{\langle 0\rangle}} \otimes_{H_s} x \cdot a_{\underline{\langle 1\rangle}},$$

where $\underline{\rho}^A(a) = a_{\underline{\langle 0\rangle}} \otimes_{H_s} a_{\underline{\langle 1\rangle}}$. $\mathbf{r}$ is defined by the right unit constraint r of $\mathcal{M}_H$.

Furthermore, if C is a coalgebra in $\mathcal{M}_H$, that is, a right H-module coalgebra, then a right comodule over C in $\mathcal{M}_A$ is precisely a right Doi-Hopf module over H in the sense of [6].

Proof. We only prove that $\diamond$ is well-defined. To this end we compute

$$\begin{aligned}(m \otimes_{H_s} r \cdot x) \cdot a &= (m \otimes_{H_s} x \cdot \overline{\varepsilon}_t(r)) \cdot a = m \cdot a_{\langle \underline{0}\rangle} \otimes_{H_s} x \cdot \overline{\varepsilon}_t(r) a_{\langle \underline{1}\rangle} \\ &= m \cdot (r \cdot a_{\langle \underline{0}\rangle}) \otimes_{H_s} x \cdot a_{\langle \underline{1}\rangle} = m \cdot ((1^A \cdot r) a_{\langle \underline{0}\rangle}) \otimes_{H_s} x \cdot a_{\langle \underline{1}\rangle} = \\ &\qquad (m * r) \cdot a_{\langle \underline{0}\rangle} \otimes_{H_s} x \cdot a_{\langle \underline{1}\rangle},\end{aligned}$$

for all $m \in M$, $x \in X$ and $r \in H_s$. Note that we used in the fourth equality the fact that the multiplication of A is H_s-balanced.

It follows now that $M \diamond X$ is a right A-module and that for any $f : M \to M'$ in $\mathcal{M}_A$ and $g : X \to X'$ in $\mathcal{M}_H$ the morphism $f \diamond g := f \otimes_{H_s} g$ is in $\mathcal{M}_A$, as required.

Now let C be a coalgebra in $\mathcal{M}_H$. Then a right comodule over C in $\mathcal{M}_A$ is a k-vector space M equipped with the following structure:

- M is a right A-module in ${}_k\mathcal{M}$ inheriting the right H_s-module structure from the right A-action, that is, $m * r = m \cdot (1 \cdot r)$, for all $r \in H_s$ and $m \in M$;
- C coacts on M to the right in the sense that there exists $\underline{\rho}^M : M \to M \otimes_{H_s} C$ in $\mathcal{M}_{H_s}$ such that the following relations hold,

$$(\underline{\rho}^M \otimes_{H_s} \mathrm{Id}_C)\underline{\rho}^M = (\mathrm{Id}_M \otimes_{H_s} \underline{\Delta})\underline{\rho}^M, \ \ (\mathrm{Id}_M \otimes_{H_s} \underline{\varepsilon})\underline{\rho}^M = \mathrm{Id}_M, \text{ and}$$
$$\underline{\rho}^M(m \cdot a) = m_{\underline{(0)}} \cdot a_{\underline{\langle 1 \rangle}} \otimes_{H_s} m_{\underline{(1)}} \cdot a_{\underline{\langle 1 \rangle}},$$

for all $m \in M$ and $a \in A$, where we denoted $\underline{\rho}^M(m) = m_{\underline{(0)}} \otimes_{H_s} m_{\underline{(1)}}$, for all $m \in M$. But this is nothing else than a right (H, A, C) Hopf-module in the sense of [6]. ∎

We will now show that particular examples of DK structures over a weak bialgebra H can be constructed from YD structures over H, at least if H has an antipode. This means that there is a k-linear map $S : H \to H$ such that

$$S(h_1)h_2 = \varepsilon_s(h) \ , \ h_1 S(h_2) = \varepsilon_t(h) \text{ and } S(h_1)h_2 S(h_3) = S(h),$$

for all $h \in H$. To this end we need the notions of bicomodule algebra and bimodule coalgebra over a weak bialgebra. A left comodule algebra over a weak bialgebra H is an algebra in the monoidal category ${}^H\mathcal{M}$ of left H-corepresentations. But this time we have to deal with $\otimes_{H_t}$ rather than $\otimes_{H_s}$. Therefore we have to move to the category of vector spaces in order to unify the left and right version. Proceeding as in Proposition 5.4, we can show that giving a left H-comodule algebra A is equivalent to giving a k-algebra with unit 1 and a left H-comodule structure on A in ${}_k\mathcal{M}$ such that the comodule morphism structure $A \ni a \mapsto \lambda_A(a) = a_{[-1]} \otimes a_{[0]} \in H \otimes A$ is multiplicative and satisfies $\lambda_A(1) \in H_s \otimes A$. Thus by an H-bicomodule algebra we mean a k-algebra A with unit 1 which is an H-bicomodule via some morphisms $\lambda_A : A \to H \otimes A$ and $\rho_A : A \to A \otimes H$ that are multiplicative and such that $\lambda_A(1) \in H_s \otimes A$ and $\rho_A(1) \in A \otimes H_t$.

Likewise, by an H-bimodule coalgebra C we mean a k-coalgebra C that is an H-bimodule, and for which Δ is an H-bilinear morphism and the counit satisfies $\varepsilon(h \cdot c) = \varepsilon(\varepsilon_s(h) \cdot c)$ and $\varepsilon(c \cdot h) = \varepsilon(c \cdot \varepsilon_t(h))$, for all $c \in C$ and $h \in H$.

If H is a weak bialgebra, A an H-bicomodule algebra and C an H-bimodule coalgebra then we call the triple (H, A, C) a right weak YD structure over H.

Proposition 5.7. *Let H be a weak Hopf algebra and (H, A, C) a right weak YD structure over H. Then A with $A \ni a \mapsto \rho(a) := a_{(0)} \otimes (S(a_{(-1)}) \otimes a_{(1)}) \in A \otimes (H^{\mathrm{op}} \otimes H)$ is a right weak $H^{\mathrm{op}} \otimes H$-comodule algebra and C with the action given by $c \cdot (h' \otimes h) = h' \cdot c \cdot h$, for all $c \in C$ and $h, h' \in H$, is a right weak $H^{\mathrm{op}} \otimes H$-module coalgebra. Consequently, a right weak (H, A, C) YD structure defines a right weak DK structure $(H^{\mathrm{op}} \otimes H, A, C)$, and therefore a right DK structure $(H^{\mathrm{op}} \otimes H, A, C)$.*

Proof. It is well-known that the antipode of a weak Hopf algebra is an anti-algebra and an anti-coalgebra endomorphism of H. From here it is immediate that ρ defines a right $H^{\mathrm{op}} \otimes H$-comodule structure on A, and that ρ is multiplicative. Since $S(H_s) \subseteq H_t$, $(H^{\mathrm{op}} \otimes H)_t = H_t \otimes H_t$ and

$$\rho(1) = 1_{\langle 0 \rangle_{[0]}} \otimes (S(1_{\langle 0 \rangle_{[-1]}}) \otimes 1_{\langle 1 \rangle}) = 1_{[0]_{\langle 0 \rangle}} \otimes (S(1_{[-1]}) \otimes 1_{[0]_{\langle 1 \rangle}}),$$

it follows that $\rho(1) \in (H^{\rm op} \otimes H)_t$. Hence A is a right weak $H^{\rm op} \otimes H$-comodule algebra.

It is easy to verify that C is a right $H^{\rm op} \otimes H$-module, and that Δ is right $H^{\rm op} \otimes H$-linear. We have also that ε_t of $H^{\rm op} \otimes H$ is $\overline{\varepsilon}_t \otimes \varepsilon_t$, and since

$$\begin{aligned}\varepsilon(c \cdot (\overline{\varepsilon}_t(h') \otimes \varepsilon_t(h))) &= \varepsilon(\overline{\varepsilon}_t(h') \cdot c \cdot \varepsilon_t(h)) = \\ &\varepsilon(\varepsilon_s(\overline{\varepsilon}_t(h')) \cdot c \cdot h) = \varepsilon(h' \cdot c \cdot h) = \varepsilon(c \cdot (h' \cdot h)),\end{aligned}$$

we conclude that C is a right weak $H^{\rm op} \otimes H$-module coalgebra. ∎

Definition 5.8. Let H be a weak Hopf algebra and (H, A, C) a right weak YD structure over H. The category of entwined modules corresponding to the right weak DK structure $(H^{\rm op} \otimes H, A, C)$ will be denoted by $\mathcal{YD}(H)_A^C$ and called the category of right (A, C)-Yetter-Drinfeld modules over H.

For the sake of simplicity, we will describe the weak version of $\mathcal{YD}(H)_A^C$, see the comments made after the proof of Proposition 5.4. A right weak (A, C)-Yetter-Drinfeld module over H is a k-vector space M that is at the same time a right A-module and a right C-comodule such that

$$(m \cdot a)_{\{0\}} \otimes (m \cdot a)_{\{1\}} = m_{\{0\}} \cdot a_{(0)} \otimes S(a_{(-1)}) \cdot m_{\{1\}} \cdot a_{(1)}\ , \ \forall\, m \in M\ ,\ a \in A\ , \quad (5.5)$$

where $M \ni m \mapsto m_{\{0\}} \otimes m_{\{1\}} \in M \otimes C$ is the right coaction of C on M.

Proposition 5.9. *For a vector space M that is at the same time a right A-module and a right C-comodule, (5.5) is equivalent to the following two relations:*

$$(m \cdot a_{[0]})_{\{0\}} \otimes a_{[-1]} \cdot (m \cdot a_{[0]})_{\{1\}} = m_{\{0\}} \cdot a_{\langle 0\rangle} \otimes m_{\{1\}} \cdot a_{\langle 1\rangle}\ ,\ \forall\, m \in M\ ,\ a \in A; \quad (5.6)$$
$$m_{\{0\}} \otimes m_{\{1\}} = m_{\{0\}} \cdot 1_{\langle 0\rangle} \otimes m_{\{1\}} \cdot 1_{\langle 1\rangle}\ ,\ \forall\, m \in M. \quad (5.7)$$

Proof. In the definition of a weak left H-comodule algebra A the condition $\lambda_A(1) \in H_s \otimes A$ is equivalent to $(\mathrm{Id}_H \otimes \lambda_A)\lambda_A(1) = \mathbf{1}_1 \otimes 1_{[-1]}\mathbf{1}_2 \otimes 1_{[0]}$ and to $(\mathrm{Id}_H \otimes \lambda_A)\lambda_A(1) = \mathbf{1}_1 \otimes \mathbf{1}_2 1_{[-1]} \otimes 1_{[0]}$. We then have

$$\begin{aligned}a_{[0]} \otimes \varepsilon_s(a_{[-1]}) &= a_{[0]} \otimes \varepsilon(a_{[-1]}\mathbf{1}_2)\mathbf{1}_1 = a_{[0]}1_{[0]} \otimes \varepsilon(a_{[-1]}1_{[-1]}\mathbf{1}_2)\mathbf{1}_1 \\ &= a_{[0]}1_{[0]} \otimes \varepsilon(a_{[-1]}1_{[-1]})1_{[-2]} = (a1_{[0]})_{[0]} \otimes \varepsilon((a1_{[0]})_{[-1]})1_{[-1]} = a1_{[0]} \otimes 1_{[-1]},\end{aligned}$$

for all $a \in A$. In a similar way, $a_{[0]} \otimes \overline{\varepsilon}_s(a_{[-1]}) = 1_{[0]}a \otimes 1_{[-1]}$, for all $a \in A$.

Now assume that (5.5) holds. Using the identity $S \circ \overline{\varepsilon}_s = \varepsilon_t$, we compute that

$$\begin{aligned}(m \cdot a_{[0]})_{\{0\}} \otimes a_{[-1]} \cdot (m \cdot a_{[0]})_{\langle 1\rangle} &\overset{(5.5)}{=} m_{\{0\}} \cdot a_{(0)} \otimes a_{(-2)}S(a_{(-1)}) \cdot m_{\{1\}} \cdot a_{(1)} \\ &= m_{\{0\}} \cdot a_{\{0\}} \otimes \varepsilon_t(a_{(-1)}) \cdot m_{\{1\}} \cdot a_{(1)} = m_{\{0\}} \cdot a_{\{0\}} \otimes S(\overline{\varepsilon}_s(a_{(-1)})) \cdot m_{\{1\}} \cdot a_{(1)} \\ &= m_{\{0\}} \cdot 1_{(0)}a_{\langle 0\rangle} \otimes S(1_{(-1)}) \cdot m_{\{1\}} \cdot 1_{(1)}a_{\langle 1\rangle} \overset{(5.5)}{=} m_{\{0\}} \cdot a_{\langle 0\rangle} \otimes m_{\{1\}} \cdot a_{\langle 1\rangle},\end{aligned}$$

for all $m \in M$ and $a \in A$, as desired. With the help of this equality and of the fact that $\lambda_A(1) \in H_s \otimes A$ we deduce that

$$\begin{aligned}m_{\{0\}} \otimes m_{\{1\}} &= (m \cdot 1)_{\{0\}} \otimes (m \cdot 1)_{\{1\}} \\ &\overset{(5.5)}{=} m_{\{0\}} \cdot 1_{[0]_{\langle 0\rangle}} \otimes S(1_{[-1]}) \cdot m_{\{1\}} \cdot 1_{[0]_{\langle 1\rangle}}\end{aligned}$$

$$\begin{aligned}
&= (m\cdot 1_{[0]})_{\{0\}}\otimes S(1_{[-2]})1_{[-1]}\cdot(m\cdot 1_{[0]})_{\{1\}}\\
&= (m\cdot 1_{[0]})_{\{0\}}\otimes \varepsilon_s(1_{[-1]})\cdot(m\cdot 1_{[0]})_{\{1\}}\\
&= (m\cdot 1_{[0]})_{\{0\}}\otimes 1_{[-1]}\cdot(m\cdot 1_{[0]})_{\{1\}} = m_{\{0\}}\cdot 1_{\langle 0\rangle}\otimes m_{\{1\}}\cdot 1_{\langle 1\rangle},
\end{aligned}$$

and this finishes the proof of the direct implication. The converse can be proved as follows:

$$\begin{aligned}
&m_{\{0\}}\cdot a_{(0)}\otimes S(a_{(-1)})\cdot m_{\{1\}}\cdot a_{(1)} = (m\cdot a_{[0]})_{\{0\}}\otimes S(a_{[-2]})a_{[-1]}\cdot(m\cdot a_{[0]})_{\{1\}}\\
&= (m\cdot a_{[0]})_{\{0\}}\otimes \varepsilon_s(a_{[-1]})\cdot(m\cdot a_{[0]})_{\{1\}} = (m\cdot a1_{[0]})_{\{0\}}\otimes 1_{[-1]}\cdot(m\cdot a1_{[0]})_{\{1\}}\\
&= (m\cdot a)_{\{0\}}\cdot 1_{\langle 0\rangle}\otimes(m\cdot a)_{\{1\}}\cdot 1_{\langle 1\rangle} = (m\cdot a)_{\{0\}}\otimes(m\cdot a)_{\{1\}}. \qquad \blacksquare
\end{aligned}$$

Remark 5.10. Theorem 4.6 was our source of inspiration for the definition of Yetter-Drinfeld modules over a weak Hopf algebra. Moving backwards, by Proposition 5.9 it makes sense to define weak right Yetter-Drinfeld modules over a weak bialgebra: all we have to do is to replace (5.5) with (5.6) and (5.7). If we do this then in the case when H is a weak Hopf algebra we can identify the category of right weak Yetter-Drinfeld modules with a category of weak right Doi-Hopf modules. This identification can be regarded as the weak Hopf algebra version of Theorem 4.6 and at the same time as a generalization of [16, Corollary 3.3].

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