Harmonic Functions in Upper Half Space*

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Abstract

In this paper, we prove that if the positive part $u^+(x)$ of a harmonic function $u(x)$ in the upper half space satisfies a fast growing condition, then its negative part $u^-(x)$ can also be dominated by a similar growing condition. Meanwhile, $u(x)$ can be represented in terms of the modified Poisson integral and a harmonic function vanishing on the boundary.

1 Introduction and Results

Let $\mathbf{R}^n (n \geq 3)$ denote the n-dimensional Euclidean space with points $x = (x', x_n)$, where $x' = (x_1, x_2, \cdots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. The boundary and closure of an open set D of $\mathbf{R}^n$ are denoted by ∂D and $\overline{D}$ respectively. The upper half space is the set $H = \{(x', x_n) \in \mathbf{R}^n : \ x_n > 0\}$, whose boundary is ∂H. We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$, writing typical points $x, \ y \in \mathbf{R}^n$ as $x = (x', x_n), \ y = (y', y_n)$, where $y' = (y_1, y_2, \cdots, y_{n-1}) \in \mathbf{R}^{n-1}$ and putting

$$x \cdot y = \sum_{j=1}^{n} x_j y_j = x' \cdot y' + x_n y_n, \ \ |x| = \sqrt{x \cdot x}, \ \ |x'| = \sqrt{x' \cdot x'}.$$

For $r > 0$, let $B(r)$ denote the open ball with center at the origin and radius r in $\mathbf{R}^n$. We use the standard notations $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$ and $[d]$ is the integer part of the positive real number d. In the sense of Lebesgue measure $dx' = dx_1 \cdots dx_{n-1}$ and $dx = dx' dx_n$. Let σ denote $(n-1)$-dimensional surface area measure and $\partial / \partial n$ denote differentiation along the inward normal into H.

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The classical Poisson kernel for H is defined by

$$P(x,y') = \frac{2x_n}{\omega_n |x-y'|^n},$$

where $\omega_n = 2\pi^{\frac{n}{2}}/\Gamma(n/2)$ is the area of the unit sphere in $\mathbf{R}^n$. It has the expansion

$$P(x,y') = \sum_{k=0}^{\infty} \frac{2x_n |x|^k}{\omega_n |y'|^{n+k}} C_k^{\frac{n}{2}} \left(\frac{x \cdot y'}{|x||y'|} \right),$$

where $C_k^{n/2}(t)$ is a Gegenbauer polynomial([7]). The series converges for $|y'| > |x|$. Each term in the series is a harmonic function of x and vanishes on ∂H.

To obtain the integral representations of harmonic functions on H, as in [3, 5, 8], we use the following modified Poisson kernel defined by

$$P_m(x,y') = \begin{cases} P(x,y') & \text{when } |y'| \le 1, \\ P(x,y') - \sum_{k=0}^{m-1} \frac{2x_n|x|^k}{\omega_n|y'|^{n+k}} C_k^{\frac{n}{2}} \left(\frac{x \cdot y'}{|x||y'|} \right) & \text{when } |y'| > 1 \end{cases}$$

for a nonnegative integer m. The new kernel $P_m(x,y')$ will be of order $|y'|^{-(n+m)}$ as $|y'| \to \infty$.

Put

$$U_m(x) = \int_{\partial H} P_m(x,y') u(y') dy',$$

where $u(y')$ is a continuous function on ∂H.

For any nonnegative real number β, we denote by $\mathcal{A}_\beta$ the space of all measurable functions $f(y)$ on H satisfying

$$\int_H \frac{y_n |f(y)| dy}{1 + |y|^{n+\beta+2}} < \infty$$

and $\mathcal{B}_\beta$ the set of all measurable functions $g(y')$ on ∂H such that

$$\int_{\partial H} \frac{|g(y')| dy'}{1 + |y'|^{n+\beta}} < \infty.$$

We also denote by $\mathcal{C}_\beta$ the set of all continuous functions $u(x)$ on $\overline{H}$, harmonic on H with $u^+(y) \in \mathcal{A}_\beta$ and $u^+(y') \in \mathcal{B}_\beta$.

We say that u is of order λ if

$$\lambda = \limsup_{r \to \infty} \frac{\log \left(\sup_{H \cap B(r)} |u| \right)}{\log r}.$$

If $\lambda < \infty$, then u is said to be of finite order (see Hayman-Kennedy [4, Definition 4.1]).

In case $\lambda < \infty$, about the solution of the Dirichlet problem with continuous data in H, we refer readers to the following two results.

Theorem A.(see [1, 6]) If $u(x) \leq 0$ and $u \in \mathcal{C}_\beta$, then there exists a constant $c \leq 0$ such that $u(x) = cx_n + \int_{\partial H} P(x,y')u(y')dy'$ for all $x \in H$.

Using the modified Poisson kernel $P_m(x,y')$, Siegel-Talvila (cf. [5, Corollary 2.1]) proved

Theorem B. If u is a continuous function on ∂H satisfying $\int_{\partial H} |u(y')|(1+|y'|)^{-n-m}dy' < \infty$, then $U_m(x)$ is a classical solution of the Dirichlet problem on H with u.

Motivated by above results, we consider the integral representations for harmonics of infinite order. To do this, we define a nondecreasing and continuously differentiable function $\rho(R) \geq 1$ on the interval $[0,+\infty)$. We assume further that

$$\varepsilon_0 = \limsup_{R\to\infty} \frac{\rho'(R)R\log R}{\rho(R)} < 1. \tag{1.1}$$

Remark. For any ϵ $(0 < \epsilon < 1-\epsilon_0)$, there exists a sufficiently large positive number R such that $r > R$, by (1.1) we have

$$\rho(r) < \rho(e)(\ln r)^{\epsilon_0+\epsilon}.$$

For any positive real number α, we denote by $(LU)_\alpha$ the space of all measurable functions $f(y)$ on H satisfying

$$\int_H \frac{y_n|f(y)|dy}{1+|y|^{\rho(|y|)+n+\alpha+1}} < \infty \tag{1.2}$$

and $(LV)_\alpha$ the set of all measurable functions $g(y')$ on ∂H such that

$$\int_{\partial H} \frac{|g(y')|dy'}{1+|y'|^{\rho(|y'|)+n+\alpha-1}} < \infty. \tag{1.3}$$

We also denote by $(CH)_\alpha$ the set of all continuous functions $u(y)$ on $\overline{H}$, harmonic on H with $u^+(y) \in (LU)_\alpha$ and $u^+(y') \in (LV)_\alpha$.

Now we have

Theorem. If $u \in (CH)_\alpha$, then the following properties hold:

(I) $u \in (LV)_\alpha$.

(II) The integral $U_{[\rho(|y'|)+\alpha]}(x)$ is absolutely convergent. It represents a harmonic function on H and can be continuously extended to $\overline{H}$ such that $U_{[\rho(|y'|)+\alpha]}(z') = u(z')$ for any $z' \in \partial H$;

(III) There exists a harmonic function $h(x)$ which vanishes on ∂H such that $u(x) = h(x) + U_{[\rho(|y'|)+\alpha]}(x)$ for all $x \in \overline{H}$.

2 Lemmas

Lemma 1.(see [5]) There exists a positive constant M such that

$$|P_m(x,y')| \leq Mx_n|x|^m|y'|^{-n-m}$$

for $x \in H$ and $y' \in H$ satisfying $|y'| \geq \max\{1, 2|x|\}$.

The following Lemma (see [9, Lemma 1]) generalizes the Carleman's formula (referring to the holomorphic functions in the half space) to the harmonic functions in H, which is essentially due to T. Carleman (see [2]).

Lemma 2. If $R > 1$ and $u(y)$ is a harmonic function on H with continuous boundary on ∂H, then we have

$$\begin{aligned} m_-(R) &+ \int_{\{y' \in \partial H: 1<|y'|<R\}} u^-(y')(\frac{1}{|y'|^n} - \frac{1}{R^n})dy' \\ &= m_+(R) + \int_{\{y' \in \partial H: 1<|y'|<R\}} u^+(y')(\frac{1}{|y'|^n} - \frac{1}{R^n})dy' - c_1 - \frac{c_2}{R^n}, \end{aligned}$$

where

$$m_\pm(R) = \int_{\{y \in H: |y|=R\}} u^\pm(y)\frac{ny_n}{R^{n+1}}d\sigma(y),$$

$$c_1 = \int_{\{y \in H: |y|=1\}} \left((n-1)y_n u(y) + y_n\frac{\partial u(y)}{\partial n}\right)d\sigma(y),$$

$$c_2 = \int_{\{y \in H: |y|=1\}} \left(y_n u(y) - y_n\frac{\partial u(y)}{\partial n}\right)d\sigma(y).$$

3 Proof of Theorem

Since $u \in (CH)_\alpha$, we obtain by (1.2)

$$\int_1^\infty \frac{m_+(R)}{R^{\rho(R)+\alpha}}dR = n\int_{\{y \in H:\ |y|>1\}} \frac{y_n u^+(y)}{|y|^{\rho(|x|)+n+\alpha+1}}dx < \infty, \tag{3.1}$$

where $m_+(R)$ is defined in Lemma 2.

We have by (1.3)

$$\begin{aligned} \int_1^\infty &\frac{1}{R^{\rho(R)+\alpha}}\int_{\{y' \in \partial H: 1<|y'|<R\}} u^+(y')(\frac{1}{|y'|^n} - \frac{1}{R^n})dy'dR \\ &= \int_{\{y' \in \partial H: |y'| \geq 1\}} u^+(y')\int_{|y'|}^\infty \frac{1}{R^{\rho(R)+\alpha}}(\frac{1}{|y'|^n} - \frac{1}{R^n})dRdy' \\ &\leq \frac{n}{n+1}\int_{\{y' \in \partial H: |y'| \geq 1\}} \frac{u^+(y')}{|y'|^{\rho(|y'|)+n+\alpha-1}}dy' < \infty. \end{aligned} \tag{3.2}$$

From (3.1), (3.2) and Lemma 2, we see that

$$\begin{aligned}
&\int_1^\infty \frac{1}{R^{\rho(R)+\alpha/2}} \int_{\{y'\in\partial H:1<|y'|<R\}} u^-(y')(\frac{1}{|y'|^n}-\frac{1}{R^n})dy'dR \\
&\qquad = \int_{\{y'\in\partial H:|y'|\geq 1\}} u^-(y') \int_{|y'|}^\infty \frac{1}{R^{\rho(R)+\alpha/2}}(\frac{1}{|y'|^n}-\frac{1}{R^n})dRdy' \\
&\qquad \leq \int_1^\infty \frac{1}{R^{\rho(R)+\alpha/2}} m_+(R)dR - \int_1^\infty \frac{1}{R^{\rho(R)+\alpha/2}}(c_1+\frac{c_2}{R^n})dR \\
&\qquad\quad + \int_1^\infty \frac{1}{R^{\rho(R)+\alpha/2}}\left(\int_{\{y'\in\partial H:1<|y'|<R\}} u^+(y')(\frac{1}{|y'|^n}-\frac{1}{R^n})dy'\right)dR < \infty.
\end{aligned}$$

Set

$$I(\alpha) = \lim_{|y'|\to\infty} |y'|^{\rho(|y'|)+n+\alpha-1} \int_{|y'|}^\infty \frac{1}{R^{\rho(R)+\alpha/2}}(\frac{1}{|y'|^n}-\frac{1}{R^n})dR.$$

By the L'hospital's rule and Remark, we have

$$I(\alpha) = +\infty,$$

which yields that there exists $\varepsilon_1 > 0$ such that

$$\int_{|y'|}^\infty \frac{1}{R^{\rho(R)+\alpha/2}}\left(\frac{1}{|y'|^n}-\frac{1}{R^n}\right)dR \geq \frac{\varepsilon_1}{|y'|^{\rho(|y'|)+n+\alpha-1}}$$

for any $|y'| \geq 1$.

Thus

$$\begin{aligned}
&\varepsilon_1 \int_{\{y'\in\partial H:|y'|\geq 1\}} \frac{u^-(y')}{|y'|^{\rho(|y'|)+n+\alpha-1}}dx' \\
&\qquad \leq \int_{\{y'\in\partial H:|y'|\geq 1\}} u^-(y') \int_{|y'|}^\infty \frac{1}{R^{\rho(R)+\alpha/2}}\left(\frac{1}{|y'|^n}-\frac{1}{R^n}\right)dRdy' < \infty.
\end{aligned}$$

Then (I) is proved from $|u| = u^+ + u^-$.

To prove (II). For any $k > k_R = [2R]+1$, there exists a positive constant $M(R)$ dependent only on R such that

$$k^{-\alpha/2}(2R)^{\rho(k+1)+\alpha+1} \leq M(R) \tag{3.3}$$

from Remark.

For any $x \in H$ and $|x| \leq R$, we have by (1.3), Lemma 1 and (3.3)

$$\begin{aligned}
&\sum_{k=k_R}^\infty \int_{\{y'\in\partial H:k\leq|y'|<k+1\}} \frac{(2|x|)^{[\rho(|y'|)+\alpha]+1}}{|y'|^{[\rho(|y'|)+\alpha]+n}}|u(y')|dy' \\
&\qquad \leq \sum_{k=k_R}^\infty \frac{(2R)^{\rho(k+1)+\alpha+1}}{k^{\alpha/2}} \int_{\{y'\in\partial H:k\leq|y'|<k+1\}} \frac{2|u(y')|}{1+|y'|^{\rho(|y'|)+\alpha/2+(n-1)}}dy' \\
&\qquad\qquad \leq 2M(R)\int_{\{y'\in\partial H:|y'|\geq k_R\}} \frac{|u(y')|}{1+|y'|^{\rho(|y'|)+\alpha/2+(n-1)}}dy' < \infty.
\end{aligned}$$

So $U_{[\rho(|y'|)+\alpha]}(x)$ is absolutely convergent. Now we shall prove the boundary behavior of $U_{[\rho(|y'|)+\alpha]}(x)$. For fixed $z' \in \partial H$, we choose a number $t > |z'| + 1$ and write

$$U_{[\rho(|y'|)+\alpha]}(x) = X(x) - Y(x) + Z(x),$$

where

$$X(x) = \int_{\{y' \in \partial H: |y'| \le t\}} P(x, y') u(y') dy'$$

$$Y(x) = \sum_{k=0}^{[\rho(|y'|+\alpha)]-1} \frac{2x_n |x|^k}{\omega_n} \int_{\{y' \in \partial H: 1 < |y'| \le t\}} \frac{1}{|y'|^{n+k}} C_k^{\frac{n}{2}} \left(\frac{x' \cdot y'}{|x||y'|} \right) u(y') dy',$$

$$Z(x) = \int_{\{y' \in \partial H: |y'| > t\}} P_{[\rho(|y'|+\alpha)]}(x, y') u(y') dy'.$$

Note that $X(x)$ is the Poisson integral of $u(y')\chi_{B(t)}(y')$, where $\chi_{B(t)}$ is the characteristic function of $B(t)$. So it tends to $u(z')$ as $x \to z'$. Clearly, $Y(x)$ vanishes on ∂H. Further, $Z(x) = O(x_n)$, which tends to zero as $x \to z'$. Thus the function $U_{[\rho(|y'|)+\alpha]}(x)$ can be continuously extended to $\overline{H}$ such that $U_{[\rho(|y'|)+\alpha]}(z') = u(z')$ for any $z' \in \partial H$. (II) is proved.

To prove (III). Consider the function $u(x) - U_{[\rho(|y'|)+\alpha]}(x)$. Then it follows that this is harmonic on H, vanishes on ∂H and can be continuously extended to $\overline{H}$. Applying Schwarz Reflection Principle ([1, p.68]) to $u(x) - U_{[\rho(|y'|)+\alpha]}(x)$, we obtain that there exists a harmonic function $h(x)$ on H such that $h(x^*) = -h(x) = -(u(x) - U_{[\rho(|y'|)+\alpha]}(x))$ for $x \in \overline{H}$, where $*$ denotes reflection in ∂H just as $x^* = (x', -x_n)$. Thus $u(x) = h(x) + U_{[\rho(|y'|)+\alpha]}(x)$ for all $x \in \overline{H}$, where $h(x)$ is a harmonic function on H and vanishes continuously on ∂H. We complete the proof of Theorem.

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