Normal families of holomorphic functions and multiple values*

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Abstract

Let $\mathcal{F}$ be a family of holomorphic functions defined in $D \subset C$, and let k, m, n, p be four positive integers with $\frac{k+p+1}{m} + \frac{p+1}{n} < 1$. Let $\psi(\not\equiv 0, \infty)$ be a meromorphic function in D and which has zeros only of multiplicities at most p. Suppose that, for every function $f \in \mathcal{F}$, (i) f has zeros only of multiplicities at least m; (ii) all zeros of $f^{(k)} - \psi(z)$ have multiplicities at least n; (iii) all poles of ψ have multiplicities at most k, and (iv) $\psi(z)$ and $f(z)$ have no common zeros, then $\mathcal{F}$ is normal in D.

1 Introduction

In this paper, we shall use the standard notations of value distribution theory, which can be found in ([6],[13],[17], etc.). We denote by $S(r, f)$ any function satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$, possibly outside a set with finite linear measure.

Let D be a domain in C, and $\mathcal{F}$ be a family of meromorphic functions defined on D. $\mathcal{F}$ is said to be normal on D, in the sense of Montel, if for any sequence $f_n \in \mathcal{F}$ there exists a subsequence f_{n_j}, such that f_{n_j} converges spherically locally uniformly on D, to a meromorphic function or ∞ (see [6],[13],[17]).

One of the most celebrated results in the theory of normal families is the following Gu's normality criterion (see [5], the holomorphic case is due to Miranda [9]), which is a conjecture of Hayman [7].

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Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and let k be a positive integer. If for every function $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} \neq 1$, then $\mathcal{F}$ is normal on D.*

This result has undergone various extensions(see [1], [2], [10], [11], [14], [15], etc.). Yang and Zhang proved that the conditions $f \neq 0$ and $f^{(k)} \neq 1$ are all can be weakened in the holomorphic case. In fact, they proved the following result(see [17]).

Theorem B. *Let $\mathcal{F}$ be a family of holomorphic functions defined in D, and let k, m, n be three positive integers. If for every function $f \in \mathcal{F}$, f has zeros only of multiplicities at least m, $f^{(k)} - 1$ has zeros only of multiplicities at least n and $\frac{k+1}{m} + \frac{1}{n} < 1$, then $\mathcal{F}$ is normal in D.*

A natural problem arises: what can we say if we replace the constant 1 by a holomorphic function $\psi(\not\equiv 0)$ in Theorem B? In this paper, we prove the following result.

Theorem 1. *Let $\mathcal{F}$ be a family of holomorphic functions defined in $D \subset C$, and let k, m, n, p be four positive integers with $\frac{k+p+1}{m} + \frac{p+1}{n} < 1$. Let $\psi(\not\equiv 0)$ be a holomorphic function in D and which has zeros only of multiplicities at most p. Suppose that, for every function $f \in \mathcal{F}$,*
(i) f has zeros only of multiplicities at least m in D;
(ii) $f^{(k)} - \psi(z)$ has zeros only of multiplicities at least n in D; and
(iii) $\psi(z)$ and $f(z)$ have no common zeros in D,
then $\mathcal{F}$ is normal in D.

In fact, we prove the following more general result.

Theorem 2. *Let $\mathcal{F}$ be a family of holomorphic functions defined in $D \subset C$, and k, m, n, p be four positive integers with $\frac{k+p+1}{m} + \frac{p+1}{n} < 1$. Let $\psi(\not\equiv 0)$, $a_0, a_1, ..., a_{k-1}$ be holomorphic functions in D, where $\psi(z)$ has zeros only of multiplicities at most p. Suppose that, for every function $f \in \mathcal{F}$,*
(i) f has zeros only of multiplicities at least m in D;
(ii) $f^{(k)}(z) + a_{k-1}(z)f^{(k-1)}(z) + ... + a_1(z)f'(z) + a_0(z)f(z) - \psi(z)$ has zeros only of multiplicities at least n in D; and
(iii) $\psi(z)$ and $f(z)$ have no common zeros in D,
then $\mathcal{F}$ is normal in D.

Furthermore, it is natural to ask: whether or not the above result holds if we extend $\psi(z)$ to the meromorphic case? We first prove the following result.

Theorem 3. *Let $\mathcal{F}$ be a family of holomorphic functions defined in $D \subset C$, let $\psi(\not\equiv 0, \not\equiv \infty)$ be a meromorphic function in D, and let k, m, n be three positive integers with $\frac{k+1}{m} + \frac{1}{n} < 1$. If, for every function $f \in \mathcal{F}$,*
(i) f has zeros only of multiplicities at least m in D;
(ii) all zeros of $f^{(k)} - \psi(z)$ have multiplicities at least n in D; and
(iii) all poles of ψ have multiplicities at most k in D,
then $\mathcal{F}$ is normal in D.

Since normality is a local property, combining Theorem 1 and Theorem 3, we obtain the following theorem.

Theorem 4. *Let $\mathcal{F}$ be a family of holomorphic functions defined in $D \subset C$, and let k, m, n, p be four positive integers with $\frac{k+p+1}{m} + \frac{p+1}{n} < 1$. Let $\psi(\not\equiv 0, \infty)$ be a meromorphic function in D and which has zeros only of multiplicities at most p. Suppose that, for every function $f \in \mathcal{F}$,*
(i) f has zeros only of multiplicities at least m in D;
(ii) all zeros of $f^{(k)} - \psi(z)$ have multiplicities at least n in D;
(iii) all poles of ψ have multiplicities at most k in D; and
(iv) $\psi(z)$ and $f(z)$ have no common zeros in D,
then $\mathcal{F}$ is normal in D.

2 Some lemmas

The well-known Zalcman's lemma is a very important tool in the study of normal families. It has also undergone various extensions and improvements. The following is one-to-date local version, which is due to Pang and Zalcman(see [12]).

Lemma 1. *Let k be a positive integer and let $\mathcal{F}$ be a family of holomorphic function in a domain D, such that each function $f \in \mathcal{F}$ has zeros only of multiplicities at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0, f \in \mathcal{F}$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then for each α, $0 \leq \alpha \leq k$, there exist a sequence of points $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathcal{F}$ such that*

$$g_n(\xi) = \frac{f_n(z_n+\rho_n\xi)}{\rho_n^{\alpha}} \to g(\xi)$$

locally uniformly with respect to the spherical metric, where $g(\xi)$ is a nonconstant holomorphic function on C, all of whose zeros have multiplicity at least k, such that $g^{\#}(\xi) \leq g^{\#}(0) = kA + 1$. Moreover, $g(\xi)$ has order at most 1.

Here, as usual, $g^{\#}(\xi) = |g'(\xi)|/(1 + |g(\xi)|^2)$ is the spherical derivative.

Lemma 2. *Let $\mathcal{F}$ be a family of holomorphic functions defined in $D \subset C$, and k, m, n, p be four positive integers. Let $b(z)(\neq 0)$, $a_0, a_1, ..., a_{k-1}$ be holomorphic functions in D. Suppose that, for every function $f \in \mathcal{F}$, f has zeros only of multiplicities at least m, $f^{(k)}(z) + a_{k-1}(z)f^{(k-1)}(z) + ... + a_1(z)f'(z) + a_0(z)f(z) - b(z)$ has zeros only of multiplicities at least n and $\frac{k+1}{m} + \frac{1}{n} < 1$, then $\mathcal{F}$ is normal in D.*

Proof. Without loss of generality, we may assume $D = \Delta = \{z : |z| < 1\}$. Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. By Lemma 1, there exist a sequence of points $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathcal{F}$ such that

$$g_n(\xi) = \frac{f_n(z_n+\rho_n\xi)}{\rho_n^{k}} \to g(\xi)$$

locally uniformly with respect to the spherical metric, where $g(\xi)$ is a nonconstant holomorphic function on C, all of whose zeros have multiplicity at least m. we have

$$\begin{aligned} & g_n^{(k)}(\xi)+\sum_{i=0}^{k-1}\rho_n^{k-i}a_i(z_n+\rho_n\xi)g_n^{(i)}(\xi)-b(z_n+\rho_n\xi) \\ = \;& f_n^{(k)}(z_n+\rho_n\xi)+\sum_{i=0}^{k-1}a_i(z_n+\rho_n\xi)f_n^{(i)}(z_n+\rho_n\xi)-b(z_n+\rho_n\xi) \end{aligned}$$

Noting that $a_i(z_n+\rho_n\xi)g_n^{(i)}(\xi)$ is locally bounded on C since $a_i(z_n+\rho_n\xi)g_n^{(i)}(\xi)\to a_i(z_0)g^{(i)}(\xi)$, on every compact subset of C, we have

$$g_n^{(k)}(\xi)+\sum_{i=0}^{k-1}\rho_n^{k-i}a_i(z_n+\rho_n\xi)g_n^{(i)}(\xi)-b(z_n+\rho_n\xi)\to g^{(k)}(\xi)-b(z_0) \tag{2.1}$$

Since $f_n^{(k)}(z_n+\rho_n\xi)+a_{k-1}(z_n+\rho_n\xi)f_n^{(k-1)}(z_n+\rho_n\xi)+...+a_1(z_n+\rho_n\xi)f_n'(z_n+\rho_n\xi)+a_0(z_n+\rho_n\xi)f_n(z_n+\rho_n\xi)-b(z_n+\rho_n\xi)$ has zeros only of multiplicities at least n, from (2.1), Hurwitz's theorem yields that $g^{(k)}(\xi)-b(z_0)$ has zeros only of multiplicities at least n, by Milloux's inequality and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} T(r,g) &\le \overline{N}(r,g)+N(r,\frac{1}{g})+N(r,\frac{1}{g^{(k)}-b(z_0)})-N(r,\frac{1}{g^{(k+1)}})+S(r,g) \\ &\le (k+1)\overline{N}(r,\frac{1}{g})+\overline{N}(r,\frac{1}{g^{(k)}-b(z_0)})+S(r,g) \\ &\le \frac{k+1}{m}N(r,\frac{1}{g})+\frac{1}{n}N(r,\frac{1}{g^{(k)}-b(z_0)})+S(r,g) \\ &\le \frac{k+1}{m}T(r,g)+\frac{1}{n}(T(r,g)+k\overline{N}(r,g))+S(r,g) \\ &\le (\frac{k+1}{m}+\frac{1}{n})T(r,g)+S(r,g) \end{aligned}$$

In above, we have used the fact that $g(\xi)$ is entire function in both the second and last inequalities. This is contradicts the fact that $g(\xi)$ is a nonconstant holomorphic function on C and $\frac{k+1}{m}+\frac{1}{n}<1$. Lemma 2 is proved. ■

Lemma 3. *Let $\mathcal{F}=\{f_n\}$ be a family of holomorphic functions defined in $D\subset C$, and let k,m,n be three positive integers with $\frac{k+1}{m}+\frac{1}{n}<1$. Let $\varphi_n(z)$ be a sequence of holomorphic functions on D such that $\varphi_n\to\varphi$ locally uniformly on D, where $\varphi(z)(\neq 0)$ is a holomorphic function on D. If all zeros of f_n have multiplicities at least m, $f_n^{(k)}(z)-\varphi_n(z)$ has zeros only of multiplicities at least n, then $\mathcal{F}$ is normal in D.*

Proof. We omit the proof since it can be carried out in the line of prove of Lemma 2. ■

3 Proof of Theorem 2

Proof. Since normality is a local property, without loss of generality, we may assume $D = \Delta = \{z : |z| < 1\}$, and $\psi(z) = z^l\varphi(z)(z \in \Delta)$, where l is a non-negative integer with $l \leq p$, $\varphi(0) = 1, \varphi(z) \neq 0$ on $\Delta' = \{z : 0 < |z| < 1\}$.If $l = 0$, then by lemma 2 we know that Theorem 2 is valid. If l is a positive integer with $l \leq p$, also by lemma 2, we only need to prove that $\mathcal{F}$ is normal at $z = 0$. Consider the family $\mathcal{G} = \{g(z) = \frac{f(z)}{\psi(z)} : f \in \mathcal{F}, z \in \Delta\}$. Since $\psi(z)$ and $f(z)$ have no common zeros for each $f \in \mathcal{F}$, we get $g(0) = \infty$ for each $g \in \mathcal{G}$. we first prove that $\mathcal{G}$ is normal in Δ. Suppose, on the contrary, that $\mathcal{G}$ is not normal at $z_0 \in \Delta$. By lemma 1, there exist a sequence of functions $g_n \in \mathcal{G}$, a sequence of complex numbers $z_n \to z_0$ and a sequence of positive numbers $\rho_n \to 0$, such that

$$G_n(\xi) = \frac{g_n(z_n+\rho_n\xi)}{\rho_n^k} = \frac{f_n(z_n+\rho_n\xi)}{\rho_n^k\psi(z_n+\rho_n\xi)} \to G(\xi)$$

converges spherically uniformly on compact subsets of C, where $G(\xi)$ is a non-constant meromorphic function on C, and all of whose zeros have multiplicity at least m. We distinguish two cases:

Case1. $z_n/\rho_n \to \infty$. Since $G_n(-z_n/\rho_n) = g_n(0)/\rho_n^k$, then the pole of G_n corresponding to that of g_n at 0 drifts off to infinity , $G(\xi)$ has no poles.
By a simple calculation, for $0 \leq i \leq k$, we have

$$g_n^{(i)}(z) = \frac{f_n^{(i)}(z)}{\psi(z)} - \sum_{j=1}^{i}\begin{pmatrix} i \\ j \end{pmatrix} g_n^{(i-j)}(z)\frac{\psi^{(j)}(z)}{\psi(z)} =$$

$$\frac{f_n^{(i)}(z)}{\psi(z)} - \sum_{j=1}^{i}[\begin{pmatrix} i \\ j \end{pmatrix} g_n^{(i-j)}(z)\sum_{t=0}^{j} A_{jt}\frac{1}{z^{j-t}}\frac{\varphi^{(t)}(z)}{\varphi(z)}] \quad (3.1)$$

where $A_{jt} = l(l-1)...(l-j+t+1)\begin{pmatrix} j \\ t \end{pmatrix}$ if $l \geq j$, $A_{jt} = 0$ if $l < j$, for $t = 0, 1, ..., j-1$ and $A_{jj} = 1$. Thus, from (3.1) we have

$$\begin{aligned}
\rho_n^{k-i}G_n^{(i)}(\xi) &= g_n^{(i)}(z_n + \rho_n\xi) \\
&= \frac{f_n^{(i)}(z_n + \rho_n\xi)}{\psi(z_n + \rho_n\xi)} - \sum_{j=1}^{i}[\begin{pmatrix} i \\ j \end{pmatrix} g_n^{(i-j)}(z_n + \rho_n\xi) \\
&\qquad \sum_{t=0}^{j} A_{jt}\frac{1}{(z_n + \rho_n\xi)^{j-t}}\frac{\varphi^{(t)}(z_n + \rho_n\xi)}{\varphi(z_n + \rho_n\xi)}] \\
&= \frac{f_n^{(i)}(z_n + \rho_n\xi)}{\psi(z_n + \rho_n\xi)} - \sum_{j=1}^{i}[\begin{pmatrix} i \\ j \end{pmatrix} \frac{g_n^{(i-j)}(z_n + \rho_n\xi)}{\rho_n^j} \\
&\qquad \sum_{t=0}^{j} A_{jt}\frac{1}{(z_n/\rho_n + \xi)^{j-t}}\frac{\rho_n^j\varphi^{(t)}(z_n + \rho_n\xi)}{\varphi(z_n + \rho_n\xi)}]
\end{aligned}$$

On the other hand, we have

$$\lim_{n\to\infty}\frac{1}{(z_n/\rho_n + \xi)} = 0$$

and

$$\lim_{n\to\infty} \frac{\rho_n^j \varphi^{(t)}(z_n+\rho_n\xi)}{\varphi(z_n+\rho_n\xi)} = 0$$

for $t \geq 1$. Noting that $g_n^{(i-j)}(z_n+\rho_n\xi)/\rho_n^j$ is locally bounded on C since $g_n(z_n+\rho_n\xi)/\rho_n^k \to G(\xi)$. Therefore, on every compact subset of C, we have

$$\frac{f_n^{(k)}(z_n+\rho_n\xi)}{\psi(z_n+\rho_n\xi)} \to G^{(k)}(\xi)$$

and

$$\frac{f_n^{(i)}(z_n+\rho_n\xi)}{\psi(z_n+\rho_n\xi)} \to 0,$$

for $i = 0, 1, ..., k-1$, and thus

$$\frac{f_n^{(k)}(z_n+\rho_n\xi) + \sum_{i=0}^{k-1} a_i(z_n+\rho_n\xi) f_n^{(i)}(z_n+\rho_n\xi) - \psi(z_n+\rho_n\xi)}{\psi(z_n+\rho_n\xi)} \to G^{(k)}(\xi) - 1,$$

since $a_0, a_1, ..., a_{k-1}$ are analytic in D.

Noting that $f_n^{(k)}(z_n+\rho_n\xi) + \sum_{i=0}^{k-1} a_i(z_n+\rho_n\xi) f_n^{(i)}(z_n+\rho_n\xi) - \psi(z_n+\rho_n\xi)$ has zeros only of multiplicity at least n, and $\psi(z_n+\rho_n\xi)$ has zeros only at $\xi = -\frac{z_n}{\rho_n} \to \infty$. Therefore, we have $G^{(k)}(\xi) - 1$ has zeros only of multiplicity at least n. Next we can arrive at a contradiction by the same argument as in the latter part of proof of Lemma 2 since $\frac{k+1}{m} + \frac{1}{n} < \frac{k+p+1}{m} + \frac{p+1}{n} < 1$.

Case2. $z_n/\rho_n \to \alpha$, a finite complex number. Then

$$\frac{g_n(\rho_n\xi)}{\rho_n^k} = \frac{g_n(z_n+\rho_n(\xi - z_n/\rho_n))}{\rho_n^k} = G_n(\xi - z_n/\rho_n) \to G(\xi-\alpha) = \tilde{G}(\xi)$$

spherically uniformly on compact subsets of C. Clearly, $\tilde{G}(\xi)$ has zeros only of multiplicity at least m, and $\tilde{G}(\xi)$ has a pole only at $\xi = 0$. We claim that $\tilde{G}(\xi)$ has a pole only at $\xi = 0$ of multiplicity l. Since $\frac{g_n(\rho_n\xi)}{\rho_n^k} = \frac{f_n(\rho_n\xi)}{\psi(\rho_n\xi)\rho_n^k} = \frac{f_n(\rho_n\xi)}{\xi^l\varphi(\rho_n\xi)\rho_n^{k+l}}$, $f_n(\xi)$ and $\psi(\xi)$ don't have common zeros and $\rho_n \to 0$, thus there exist $r > 0(< 1)$ such that $f_n(\rho_n\xi)$ don't have zeros in Δ_r when n is large enough. Thus $\frac{\rho_n^k}{g_n(\rho_n\xi)}$ is holomorphic in Δ_r and $\xi = 0$ is the only zero of $\frac{\rho_n^k}{g_n(\rho_n\xi)}$ of multiplicity l. On the other hand, since $\tilde{G}(\xi)$ has a pole only at $\xi = 0$, we have $\frac{1}{\tilde{G}(\xi)}$ has a zero only at $\xi = 0$. Therefore, there exist $\varepsilon_0 > 0$ such that $|\frac{1}{\tilde{G}(\xi)}| > \varepsilon_0$ when $|\xi| = r'$, where $0 < r' < r$, and $|\frac{\rho_n^k}{g_n(\rho_n\xi)} - \frac{1}{\tilde{G}(\xi)}| < \varepsilon_0$ when n is large enough. By Rouche's theorem we obtain $\frac{1}{\tilde{G}(\xi)}$ has a zero only at $\xi = 0$ of multiplicity l. Thus we have proved

the claim.
Set

$$H_n(\xi) = \frac{f_n(\rho_n\xi)}{\rho_n^{k+l}} \tag{3.2}$$

Then

$$H_n(\xi) = \frac{\psi(\rho_n\xi)}{\rho_n^l}\frac{f_n(\rho_n\xi)}{\rho_n^k\psi(\rho_n\xi)} = \frac{\psi(\rho_n\xi)}{\rho_n^l}\frac{g_n(\rho_n\xi)}{\rho_n^k}.$$

Noting that $\frac{\psi(\rho_n\xi)}{\rho_n^l} \to \xi^l$, thus $H_n(\xi) \to \xi^l\tilde{G}(\xi) = H(\xi)$ uniformly on compact subsets of C. Since $\tilde{G}(\xi)$ has a pole only at $\xi = 0$ of multiplicity l, we have $H(0) \neq 0$ and $H(0) \neq \infty$, so $H(\xi)$ is holomorphic in C and which has zeros only of multiplicity at least m. From(3.2), we get

$$H_n^{(i)}(\xi) = \frac{f_n^{(i)}(\rho_n\xi)}{\rho_n^{k+l-i}} \to H^{(i)}(\xi),$$

spherically uniformly on compact subsets of C. As the above, on every compact subsets of C, we have

$$\frac{f_n^{(k)}(\rho_n\xi) + \sum_{i=0}^{k-1} a_i(\rho_n\xi)f_n^{(i)}(\rho_n\xi) - \psi(\rho_n\xi)}{\rho_n^l} \to H^{(k)}(\xi) - \xi^l \tag{3.3}$$

locally uniformly on C. By the assumption of Theorem 2 and (3.3), Hurwitz's theorem implies that all zeros of $H^{(k)}(\xi) - \xi^l$ have multiplicity at least n.

If $H(\xi)$ is a transcendental function, then $T(r, H^{(k)} - \xi^l) = T(r, H^{(k)}) + S(r, H)$. By Nevanlinna's first fundamental theorem, we have

$$\begin{aligned}
& m(r, \frac{1}{H}) + m(r, \frac{1}{H^{(k)} - \xi^l}) \\
= \;& m(r, \frac{1}{H} + \frac{1}{H^{(k)} - \xi^l}) + S(r, H) \\
\leq \;& m(r, \frac{1}{H^{(k+l+1)}}) + S(r, H) \\
\leq \;& T(r, H^{(k+l+1)}) - N(r, \frac{1}{H^{(k+l+1)}}) + S(r, H) \\
\leq \;& T(r, H^{(k)}) + (l+1)\overline{N}(r, H^{(k)}) - N(r, \frac{1}{H^{(k+l+1)}}) + S(r, H)
\end{aligned}$$

both sides add $N(r,\frac{1}{H})+N(r,\frac{1}{H^{(k)}-\xi^l})$, we have

$$\begin{aligned}T(r,H) &\le (l+1)\overline{N}(r,H^{(k)})+N(r,\frac{1}{H})+N(r,\frac{1}{H^{(k)}-\xi^l})\\ &\qquad -N(r,\frac{1}{H^{(k+l+1)}})+S(r,H)\\ &\le (k+l+1)\overline{N}(r,\frac{1}{H})+(l+1)\overline{N}(r,\frac{1}{H^{(k)}-\xi^l})+S(r,H)\\ &\le \frac{k+l+1}{m}N(r,\frac{1}{H})+\frac{l+1}{n}N(r,\frac{1}{H^{(k)}-\xi^l})+S(r,H)\\ &\le \frac{k+l+1}{m}N(r,\frac{1}{H})+\frac{l+1}{n}(T(r,H)+k\overline{N}(r,H))+S(r,H)\\ &\le (\frac{k+l+1}{m}+\frac{l+1}{n})T(r,H)+S(r,H)\end{aligned}$$

In above, we have used the fact that $H(\xi)$ is a entire function in both the second and last inequalities. This is a contradiction since $\frac{k+p+1}{m}+\frac{p+1}{n}<1$ and $l\le p$.

If $H(\xi)$ is a constant, then we have $H^{(k)}(\xi)-\xi^l=-\xi^l$. This is a contradiction since $H^{(k)}(\xi)-\xi^l$ has zeros only of multiplicity at least n.

Therefore, $H(\xi)$ is a nonconstant polynomial. Set

$$H(\xi)=a(\xi-\alpha_1)^{n_1}(\xi-\alpha_2)^{n_2}...(\xi-\alpha_t)^{n_t} \tag{3.4}$$

$$H^{(k)}(\xi)-\xi^l=b(\xi-\beta_1)^{m_1}(\xi-\beta_2)^{m_2}...(\xi-\beta_s)^{m_s} \tag{3.5}$$

where a, b are two nonzero constants, and $n_i\ge m$, $m_j\ge n$ are both positive integers for $i=1,2,...,t,\ j=1,2,...,s$. Set $N=\deg H$, then

$$N=n_1+n_2+...+n_t, \tag{3.6}$$

and

$$\deg(H^{(k)}(\xi)-\xi^l)=N-k,$$

$$m_1+m_2+...+m_s=N-k. \tag{3.7}$$

If $\alpha_i=\beta_j$, then $H(\beta_j)=0$, since $H(\xi)$ has zeros only of multiplicity at least m, we have $H^{(k)}(\beta_j)=0$. Thus, from (3.5) we have $\beta_j=0$, without loss of generality, we may assume $j=1$. On the other hand, from (3.5) we have

$$H^{(k+l)}(\xi)-l!=\xi^{m_1-l}p(\xi) \tag{3.8}$$

where $p(\xi)$ is a nonconstant polynomial and $p(0)\ne 0$. This is a contradiction.

Therefore, $\alpha_i\ne\beta_j$ for $i=1,2,...,t,\ j=1,2,...,s$ and that they are all zeros of $H^{(k+l+1)}$ of multiplicity $n_i-(k+l+1),\ m_j-(l+1)$ for $i=1,2,...,t,\ j=1,2,...,s$.

Since

$$\deg(H^{(k+l+1)}(\xi))=\deg H(\xi)-(k+l+1)=N-(k+l+1)$$

So

$$N - (k + l + 1)t + N - k - (l + 1)s \leq N - (k + l + 1) \tag{3.9}$$

From (3.9), we have

$$N \leq (k + l + 1)t + (l + 1)(s - 1) \tag{3.10}$$

Noting that $n_i \geq m$, from (3.6) we have $t \leq \frac{N}{m}$. Noting that $m_i \geq n$, from (3.7) we have $s \leq \frac{N-k}{n}$. Therefore, we have

$$(1 - \frac{k + l + 1}{m} - \frac{l + 1}{n})N \leq -\frac{l + 1}{n}k$$

This is a contradiction. Thus, we have proved that $\mathcal{G}$ is normal in Δ.

It remains to show that $\mathcal{F}$ is normal at $z = 0$. Since $\mathcal{G}$ is normal on Δ, then the family $\mathcal{G}$ is equicontinuous on Δ with respect to the spherical distance. Noting that $g(0) = \infty$ for each $g \in \mathcal{G}$, so there exist $\delta > 0$ such that $|g(z)| \geq 1$ for all $g \in \mathcal{G}$ and each $z \in \Delta_\delta$. On the other hand, since $\mathcal{F}$ is normal in Δ'_δ, then $\mathcal{F}$ $_1 = \{1/f : f \in \mathcal{F}\}$ is normal in Δ'_δ, but it is not normal in Δ_δ. Therefore, there exist a sequence $\{1/f_n\} \subset \mathcal{F}_1$ which converges locally uniformly on Δ'_δ, but it is not on Δ_δ. Since $f(z) \neq 0$ for every $f \in \mathcal{F}$, then $\mathcal{F}_1$ is a holomorphic function family. The maximum modulus principle implies that $1/f_n \to \infty$ on Δ'_δ, and hence so does $\{g_n\} \subset \mathcal{G}$, where $g_n = f_n/\psi$. But $|g_n(z)| \geq 1$ for $z \in \Delta_\delta$, a contradiction. This finally completes the proof of Theorem 2. ∎

4 Proof of Theorem 3

Proof. Without loss of generality, we may assume $D = \Delta = \{z : |z| < 1\}$, and $\psi(z) = \frac{\varphi(z)}{z^l}$ $(z \in \Delta)$, where l is a non-negative integer with $l \leq k$, $\varphi(0) = 1, \varphi(z) \neq 0, \infty$ on $\Delta' = \{z : 0 < |z| < 1\}$. If $l = 0$, then by Theorem 1 we know that Theorem 3 is valid. If l is a positive integer with $l \leq k$, also by Theorem 1, it is enough to show that $\mathcal{F}$ is normal at $z = 0$.

Suppose, on the contrary, that $\mathcal{F}$ is not normal at $z = 0$. By lemma 1(with $\alpha = k - l$), there exist a sequence of functions $f_n \in \mathcal{F}$, a sequence of complex numbers $z_n \to 0$ and a sequence of positive numbers $\rho_n \to 0$, such that

$$F_n(\xi) = \frac{f_n(z_n + \rho_n \xi)}{\rho_n^{k-l}} \to F(\xi) \tag{4.1}$$

converges spherically uniformly on compact subsets of C, where $F(\xi)$ is a non-constant holomorphic function on C, and all of whose zeros have multiplicity at least m. Now we distinguish two cases:

Case1. $z_n/\rho_n \to \infty$.

Set

$$g_n(\xi) = z_n^{l-k} f_n(z_n(1 + \xi))$$

Clearly, all zeros of g_n have multiplicity at least m. Since

$$\begin{aligned} g_n^{(k)}(\xi) - \frac{\varphi(z_n(1+\xi))}{(1+\xi)^l} &= z_n^l[f_n^{(k)}(z_n(1+\xi)) - \frac{\varphi(z_n(1+\xi))}{(z_n(1+\xi))^l}] \\ &= z_n^l[f_n^{(k)}(z_n(1+\xi)) - \psi(z_n(1+\xi))] \end{aligned}$$

by the assumption of Theorem 3 and Hurwits's theorem, we know that all zeros of $g_n^{(k)}(\xi) - \frac{\varphi(z_n(1+\xi))}{(1+\xi)^l}$ have multiplicity at least n in Δ. On the other hand, $\frac{\varphi(z_n(1+\xi))}{(1+\xi)^l}$ is holomorphic in Δ for each n, and

$$\frac{\varphi(z_n(1+\xi))}{(1+\xi)^l} \to \frac{1}{(1+\xi)^l}(\neq 0)$$

for $\xi \in \Delta$. Then, by Lemma 3, $\{g_n\}$ is normal in Δ.

So we can find a subsequence $\{g_{n_j}\} \subset \{g_n\}$ and a function g such that

$$g_{n_j}(\xi) = z_{n_j}^{l-k} f_{n_j}(z_{n_j}(1+\xi)) \to g(\xi) \tag{4.2}$$

converges spherically locally on Δ.

If $g(0) \neq \infty$, from (4.1) and (4.2), and noting $z_n/\rho_n \to \infty$, we have

$$\begin{aligned} F^{(k-l)}(\xi) &= \lim_{j\to\infty} f_{n_j}^{(k-l)}(z_{n_j} + \rho_{n_j}\xi) = \lim_{j\to\infty} f_{n_j}^{(k-l)}(z_{n_j} + z_{n_j}(\frac{\rho_{n_j}}{z_{n_j}}\xi) \\ &= \lim_{j\to\infty} g_{n_j}^{(k-l)}(\frac{\rho_{n_j}}{z_{n_j}}\xi) = g^{(k-l)}(0) \end{aligned} \tag{4.3}$$

It follows from (4.3) that $F^{(k-l)}(\xi)$ must be a finite constant, and then $F(\xi)$ is a polynomial with degree at most $k-l$. But this is impossible since all zeros of $F(\xi)$ have multiplicity at least m.

If $g(0) = \infty$, then

$$g_{n_j}(\frac{\rho_{n_j}}{z_{n_j}}\xi) = z_{n_j}^{l-k} f_{n_j}(z_{n_j} + \rho_{n_j}\xi) \to g(0) = \infty$$

and therefore

$$F(\xi) = \lim_{j\to\infty} \frac{f_{n_j}(z_{n_j} + \rho_{n_j}\xi)}{\rho_{n_j}^{k-l}} = \lim_{j\to\infty} (\frac{z_{n_j}}{\rho_{n_j}})^{k-l} z_{n_j}^{l-k} f_{n_j}(z_{n_j} + \rho_{n_j}\xi) = \infty$$

which is impossible since F is a nonconstant holomorphic function.

Case2. $z_n/\rho_n \to \alpha$, a finite complex number. Then

$$F_n^{(k)}(\xi) - \frac{\rho_n^l \varphi(z_n + \rho_n\xi)}{(z_n + \rho_n\xi)^l} \to F^{(k)}(\xi) - \frac{1}{(\alpha+\xi)^l}$$

on $C - \{-\alpha\}$. Noting that

$$F_n^{(k)}(\xi) - \frac{\rho_n^l \varphi(z_n + \rho_n\xi)}{(z_n + \rho_n\xi)^l} = \rho_n^l (f_n^{(k)}(z_n + \rho_n\xi) - \psi(z_n + \rho_n\xi))$$

and all zeros of $f_n^{(k)}(z_n+\rho_n\xi)-\psi(z_n+\rho_n\xi)$ have multiplicity at least n, Hurwitz's theorem implies that all zeros of $F^{(k)}(\xi)-\frac{1}{(\alpha+\xi)^l}$ have multiplicity at least n.

By Nevanlinna's first and second fundamental theorems (for small functions), we obtain

$$\begin{aligned} T(r,F^{(k)}) &\le \overline{N}(r,F^{(k)})+\overline{N}(r,\frac{1}{F^{(k)}})+\overline{N}(r,\frac{1}{F^{(k)}-1/(\alpha+\xi)^l})+S(r,F^{(k)}) \\ &\le \frac{1}{m-k}N(r,\frac{1}{F^{(k)}})+\frac{1}{n}N(r,\frac{1}{F^{(k)}-1/(\alpha+\xi)^l})+S(r,F^{(k)}) \\ &\le \frac{k+1}{m}N(r,\frac{1}{F^{(k)}})+\frac{1}{n}N(r,\frac{1}{F^{(k)}-1/(\alpha+\xi)^l})+S(r,F^{(k)}) \\ &\le (\frac{k+1}{m}+\frac{1}{n})T(r,F^{(k)})+S(r,F^{(k)}) \end{aligned}$$

In above, we have used the fact that $\frac{k+1}{m}-\frac{1}{m-k}=\frac{[m-(k+1)]k}{m(m-k)}$ and noting that $\frac{k+1}{m}+\frac{1}{n}<1$, hence $\frac{k+1}{m}>\frac{1}{m-k}$. From the last inequalities and noting that $\frac{k+1}{m}+\frac{1}{n}<1$, we know that $F(\xi)$ is not transcendental. So $F(\xi)$ is a nonconstant polynomial. Set

$$F(\xi)=a(\xi-\alpha_1)^{n_1}(\xi-\alpha_2)^{n_2}...(\xi-\alpha_t)^{n_t} \tag{4.4}$$

$$F^{(k)}(\xi)-\frac{1}{(\alpha+\xi)^l}=\frac{b(\xi-\beta_1)^{m_1}(\xi-\beta_2)^{m_2}...(\xi-\beta_s)^{m_s}}{(\alpha+\xi)^l} \tag{4.5}$$

where a,b are two nonzero constants, and $n_i\ge m$, $m_j\ge n$ are both positive integers for $i=1,2,...,t$, $j=1,2,...,s$. Set $N=\deg F$, then

$$N=n_1+n_2+...+n_t, \tag{4.6}$$

and

$$m_1+m_2+...+m_s=N+l-k. \tag{4.7}$$

If $\alpha_i=\beta_j$, then $F(\beta_j)=0$, since $F(\xi)$ has zeros only of multiplicity at least m, we have $F^{(k)}(\beta_j)=0$. Thus, from(4.5) we have $1/(\alpha+\beta_j)^l=0$, which is impossible. Therefore, $\alpha_i\ne\beta_j$ for $i=1,2,...,t$, $j=1,2,...,s$.

From (4.5), we have

$$(\alpha+\xi)^lF^{(k)}(\xi)-1=b(\xi-\beta_1)^{m_1}(\xi-\beta_2)^{m_2}...(\xi-\beta_s)^{m_s}$$

Hence

$$l(\alpha+\xi)^{l-1}F^{(k)}(\xi)+(\alpha+\xi)^lF^{(k+1)}(\xi)=(\xi-\beta_1)^{m_1-1}...(\xi-\beta_s)^{m_s-1}g(\xi) \tag{4.8}$$

where $g(\xi)$ is a polynomial of $\deg g=s-1$.

Since $-\alpha$, α_i are both the zeros of left side of (4.8) of multiplicity $l-1$, $n_i-(k+1)$ for $i=1,2,...,s$. From (4.8), we have $-\alpha$, α_i are both the zeros of $g(\xi)$ of multiplicity $l-1$, $n_i-(k+1)$ for $i=1,2,...,s$. Thus

$$l-1+N-(k+1)t\le s-1$$

So

$$N \leq (k+1)t + s - l \leq \frac{k+1}{m}N + \frac{N+l-k}{n} - l \tag{4.9}$$

From(4.9), we have

$$(1 - \frac{k+1}{m} - \frac{1}{n})N \leq -(\frac{k-l}{n} + l)$$

This is a contradiction. Thus, we have proved that $\mathcal{F}$ is normal in Δ. Theorem 3 is proved. ■

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