Representation of surfaces in 3-dimensional lightlike cone

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Abstract

In this paper we give Weierstrass type representation formulas for spacelike surfaces and maximal spacelike surfaces in 3-dimensional lightlike cone $\mathbb{Q}^3$. Then we discuss some properties and structures of spacelike surface and its associated surface.

1 Introduction.

In General Relativity, null submanifolds usually appear to be some smooth parts of the achronal boundaries, for example, event horizons of the Kruskal and Kerr black holes and the compact Cauchy horizons in Taub-NUT spacetime, and their properties are manifested in the proofs of several theorems concerning black holes and singularities. Degenerate submanifolds of Lorentzian manifolds may be useful to study the intrinsic structure of manifolds with degenerate metric and to have a better understanding of the relation between the existence of the null submanifolds and the spacetime metric ([4]).

It is well known that there are three kinds of pseudo Riemannian space forms, namely, the pseudo Euclidean space $\mathbb{E}^n_q$, the pseudo Riemannian sphere $\mathbb{S}^n_q(c,r)$ and the pseudo Riemannian hyperbolic space $\mathbb{H}^n_q(c,r)$. They are nondegenerated complete pseudo Riemannian hypersurface of pseudo Euclidean space with zero,

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positive, or negative constant sectional curvature, respectively ([13]). However, for the degenerate hypersurface $\mathbb{Q}^n_q(c)$ in $\mathbb{E}^{n+1}_q$, it should be considered also as the forth kind of pseudo Riemannian space form, or called degenerate pseudo Riemannian space form. It is meaningful to study the geometry or the geometry of submanifolds of degenerate pseudo Riemannian space form ([5]-[11]).

The concept of trapped surfaces plays extremely important role in general relativity and cosmology. It is considered as a cornerstone for the achievement of the singularity theorems, the analysis of gravitational collapse, the cosmic censorship hypothesis, the Penrose inequality, etc. ([2], [3]). Prof. B. Y. Chen proved that the cone surface of $\mathbb{Q}^3$ is marginally trapped in $\mathbb{E}^4_1$ if and only if the surface is flat ([2], Proposition 4.1). From [6] we know that the surface in $\mathbb{Q}^3$ is flat if and only if the surface is maximal ([6], (1.13) and (2.7); or [7]).

In this paper we consider spacelike surfaces and maximal spacelike surfaces in 3-dimensional lightlike cone $\mathbb{Q}^3 \subset \mathbb{E}^4_1$. Using the complex function theory and differential equation theory we give the representation formulas for the spacelike surfaces and maximal spacelike surfaces in 3-dimensional lightlike cone $\mathbb{Q}^3$. Mainly, we have

Theorem A. (Representation formula of spacelike surface in $\mathbb{Q}^3$) *Let $x = x(u,v) : \mathbf{M} \to \mathbb{Q}^3 \subset \mathbb{E}^4_1$ be a spacelike surface in $\mathbb{Q}^3$ with the isothermal parameter $z = u + iv$. Then $x(u,v) = (x_1, x_2, x_3, x_4)$ can be written as*

$$\begin{cases} x_1(u,v) = \rho(u,v)\left\{f(z) + \overline{f(z)}\right\}, \\ x_2(u,v) = -i\rho(u,v)\left\{f(z) - \overline{f(z)}\right\}, \\ x_3(u,v) = \rho(u,v)\left\{1 - f(z)\overline{f(z)}\right\}, \\ x_4(u,v) = \rho(u,v)\left\{1 + f(z)\overline{f(z)}\right\}. \end{cases} \tag{1.1}$$

For some holomorphic function $f(z)$ and real function $\rho(u,v) = \rho(z,\bar{z})$.

Theorem B. (Representation formula of maximal spacelike surface in $\mathbb{Q}^3$) *Let $x = x(u,v) : \mathbf{M} \to \mathbb{Q}^3 \subset \mathbb{E}^4_1$ be a maximal spacelike surface in $\mathbb{Q}^3$ with the isothermal parameter $z = u + iv$, Then $x(u,v) = (x_1, x_2, x_3, x_4)$ can be written as*

$$\begin{cases} x_1(u,v) = \rho(z)\overline{\rho(z)}\left\{f(z) + \overline{f(z)}\right\}, \\ x_2(u,v) = -i\rho(z)\overline{\rho(z)}\left\{f(z) - \overline{f(z)}\right\}, \\ x_3(u,v) = \rho(z)\overline{\rho(z)}\left\{1 - f(z)\overline{f(z)}\right\}, \\ x_4(u,v) = \rho(z)\overline{\rho(z)}\left\{1 + f(z)\overline{f(z)}\right\}. \end{cases} \tag{1.2}$$

For some holomorphic function $f(z)$ and complex function $\rho(z)$.

With these representation formulas we discuss some properties and structures of the spacelike surface and its associated surface.

2 Surfaces in lightlike cone $\mathbb{Q}^3$.

We follow the notations and conceptions as in [6]. Let **M** be a connected, oriented 2-dimensional differential manifold and $x : \mathbf{M} \to \mathbb{Q}^3 \subset \mathbb{E}^4_1$ be a surface in 3-dimensional lightlike cone $\mathbb{Q}^3 \subset \mathbb{E}^3_1$ with isothermal parameter $\{u,v\}$. In this case, the surface x is always spacelike ([6], [7]). The induced metric (simply the metric) of the surface $x(u,v)$ is given by (cf. [6])

$$G = \langle dx, dx\rangle = 2e^{w}(du^2 + dv^2) = e^{w}(dz \otimes d\bar{z} + d\bar{z} \otimes dz), \tag{2.1}$$

where $z = u + iv$. We use the Cauchy-Riemann operators

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

and denote $x_z = \partial_z x = \partial x/\partial z$. Then we have

$$\langle x, x\rangle = \langle x, x_z\rangle = \langle x, x_{\bar{z}}\rangle = \langle x_z, x_z\rangle = \langle x_{\bar{z}}, x_{\bar{z}}\rangle = 0, \quad \langle x_z, x_{\bar{z}}\rangle = e^{w}. \tag{2.2}$$

From (2.2) we get

$$\begin{cases} \langle x_z, x_{zz}\rangle = \langle x_z, x_{z\bar{z}}\rangle = \langle x_{\bar{z}}, x_{\bar{z}\bar{z}}\rangle = \langle x_{\bar{z}}, x_{z\bar{z}}\rangle = \langle x, x_{zz}\rangle = \langle x, x_{\bar{z}\bar{z}}\rangle = 0, \\ \langle x_{\bar{z}}, x_{zz}\rangle = e^{w}w_z, \quad \langle x_z, x_{\bar{z}\bar{z}}\rangle = e^{w}w_{\bar{z}}, \quad \langle x, x_{z\bar{z}}\rangle = -e^{w}. \end{cases} \tag{2.3}$$

The Laplacian Δ of the metric G and the Gaussian curvature κ of the surface $x(u,v)$ are given by

$$\Delta = 2e^{-w}\partial_z\partial_{\bar{z}} = 2e^{-w}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}, \qquad \kappa = -e^{-w}w_{z\bar{z}}. \tag{2.4}$$

We define

$$y = y(u,v) = -\frac{1}{2}\Delta x - \frac{1}{8}\langle \Delta x, \Delta x\rangle x. \tag{2.5}$$

Then we have

$$\langle y, y\rangle = 0, \quad \langle x, y\rangle = 1, \quad \langle y, x_z\rangle = \langle y, x_{\bar{z}}\rangle = 0.$$

We know that the vector fields $\{x, y, (2e^{w})^{-1/2}x_u, (2e^{w})^{-1/2}x_v\}$ form an asymptotic orthonormal frame on $\mathbb{E}^4_1$ along the surface $x(u,v)$. For the surface $x(u,v)$, we have the following structure equations

$$\begin{cases} x_{zz} = w_z x_z + \varphi x, \\ x_{z\bar{z}} = \lambda x - e^{w}y, \\ y_z = -\lambda e^{-w}x_z - \varphi e^{-w}x_{\bar{z}}. \end{cases} \tag{2.6}$$

The integrability conditions of $x(u,v)$ are

$$\begin{cases} \lambda = -\frac{1}{2}e^{w}\kappa, \\ \\ \varphi_{\bar{z}} = -\frac{1}{2}e^{w}\kappa_z. \end{cases} \tag{2.7}$$

From (2.6) and (2.7) we have

$$\begin{cases} H = \lambda e^{-w} = -\frac{1}{2}\kappa, \\ \Delta x = 2H - 2y. \end{cases} \tag{2.8}$$

Here

$$H = \frac{1}{2}\langle \Delta x, y\rangle$$

is the mean curvature of the cone surface $x(u,v)$. The surface $x(u,v)$ is called *maximal* (or traditionally, minimal or extremal) in $\mathbb{Q}^3$ if and only if $H \equiv 0$ ([6], [7]).

3 Weierstrass type formula of surfaces in $\mathbb{Q}^3$.

Let $x : \mathbf{M} \to \mathbb{Q}^3 \subset \mathbb{E}_1^4$ be a spacelike surface in $\mathbb{Q}^3$ with the isothermal parameter $\{u, v\}$, or $z = u + iv$. Putting $x = (x_1, x_2, x_3, x_4)$ we have

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0.$$

Then from $x_1^2 - (ix_2)^2 = -(x_3^2 - x_4^2)$ we get

$$\frac{x_1 + ix_2}{x_3 + x_4} = -\frac{x_3 - x_4}{x_1 - ix_2}, \qquad \text{or} \qquad \frac{x_1 + ix_2}{x_3 - x_4} = -\frac{x_3 + x_4}{x_1 - ix_2}. \tag{3.1}$$

Without loss of generality we may assume that

$$\frac{x_1 + ix_2}{x_3 + x_4} = -\frac{x_3 - x_4}{x_1 - ix_2} = f(z, \bar{z}), \tag{3.2}$$

$$\frac{x_1 + ix_2}{x_3 - x_4} = -\frac{x_3 + x_4}{x_1 - ix_2} = -\frac{1}{g(z, \bar{z})}, \tag{3.3}$$

and

$$x_3 + x_4 = 2\rho(z, \bar{z}). \tag{3.4}$$

Then from (3.2), (3.3) and (3.4) we get

$$\begin{cases} x_1 + ix_2 = 2\rho f, \\ x_1 - ix_2 = 2\rho g, \\ x_3 + x_4 = 2\rho, \\ x_3 - x_4 = -2\rho f g. \end{cases} \tag{3.5}$$

Therefore we obtain

$$\begin{cases} x_1 = \rho(f + g), \\ x_2 = -i\rho(f - g), \\ x_3 = \rho(1 - fg), \\ x_4 = \rho(1 + fg). \end{cases} \tag{3.6}$$

That is, the surface $x : \mathbf{M} \to \mathbb{Q}^3 \subset \mathbb{E}_1^4$ can be written as

$$x = x(u, v) = x(z, \bar{z}) = (x_1, x_2, x_3, x_4) = \rho(f + g, -i(f - g), 1 - fg, 1 + fg). \tag{3.7}$$

From (3.7) we have

$$\begin{cases} x_z = \rho_z(f + g, -i(f - g), 1 - fg, 1 + fg) + \\ \qquad\qquad \rho(f_z + g_z, -i(f_z - g_z), -f_z g - f g_z, f_z g + f g_z), \\ x_{\bar{z}} = \rho_{\bar{z}}(f + g, -i(f - g), 1 - fg, 1 + fg) + \\ \qquad\qquad \rho(f_{\bar{z}} + g_{\bar{z}}, -i(f_{\bar{z}} - g_{\bar{z}}), -f_{\bar{z}} g - f g_{\bar{z}}, f_{\bar{z}} g + f g_{\bar{z}}), \end{cases}$$

and then

$$\begin{cases} \langle x_z, x_z \rangle = 4\rho^2 f_z g_z, \\ \langle x_{\bar{z}}, x_{\bar{z}} \rangle = 4\rho^2 f_{\bar{z}} g_{\bar{z}}, \\ \langle x_z, x_{\bar{z}} \rangle = 2\rho^2 (f_z g_{\bar{z}} + f_{\bar{z}} g_z). \end{cases} \tag{3.8}$$

Since $\{u,v\}$ is the isothermal parameter of the surface $x(u,v)$, from (2.1) we get

$$\begin{cases} \langle x_z, x_z\rangle = 4\rho^2 f_z g_z = 0, \\ \langle x_{\bar z}, x_{\bar z}\rangle = 4\rho^2 f_{\bar z} g_{\bar z} = 0, \\ \langle x_z, x_{\bar z}\rangle = 2\rho^2 (f_z g_{\bar z} + f_{\bar z} g_z) = e^w. \end{cases} \tag{3.9}$$

Without loss of generality we assume that

$$\begin{cases} f_{\bar z} \equiv 0, \\ g_z \equiv 0. \end{cases} \tag{3.10}$$

That means

$$\begin{cases} f \equiv f(z), \\ g \equiv g(\bar z), \\ e^w = 2\rho^2 f_z g_{\bar z}. \end{cases} \tag{3.11}$$

From (3.2) and (3.3) we get

$$f(z) = \overline{g(\bar z)}, \qquad g(\bar z) = \overline{f(z)}. \tag{3.12}$$

Theorem 3.1. *Let $x = x(u,v) : \mathbf{M} \to \mathbb{Q}^3 \subset \mathbb{E}^4_1$ be a spacelike surface in $\mathbb{Q}^3$ with the isothermal parameter $z = u + iv$. Then $x(u,v) = (x_1, x_2, x_3, x_4)$ can be written as*

$$\begin{cases} x_1(u,v) = \rho(u,v)\left\{f(z) + \overline{f(z)}\right\}, \\ x_2(u,v) = -i\rho(u,v)\left\{f(z) - \overline{f(z)}\right\}, \\ x_3(u,v) = \rho(u,v)\left\{1 - f(z)\overline{f(z)}\right\}, \\ x_4(u,v) = \rho(u,v)\left\{1 + f(z)\overline{f(z)}\right\}. \end{cases} \tag{3.13}$$

For some holomorphic function $f(z)$ and real function $\rho(u,v) = \rho(z,\bar z)$. The metric of $x(u,v)$ is given by

$$G(u,v) = \left(2\rho^2 f_z \overline{f_z}\right)(\mathrm{d}z \otimes \mathrm{d}\bar z + \mathrm{d}\bar z \otimes \mathrm{d}z).$$

The Gaussian curvature of $x(u,v)$ is given by

$$\kappa(u,v) = -2(2\rho^2 f_z \overline{f_z})^{-1} (\log \rho)_{z\bar z} = -\Delta(\log \rho). \tag{3.14}$$

Proof. From (3.6)-(3.11) and the expressions (2.1) and (2.4) of the metric and Gaussian curvature. ■

Definition 1. *The holomorphic function $f(z)$ and real function $\rho(u,v) = \rho(z,\bar z)$ are called structure functions of the spacelike surface $x = x(u,v) : \mathbf{M} \to \mathbb{Q}^3 \subset \mathbb{E}^4_1$. The function $\rho(u,v)$ is called conformal factor of $x(u,v)$ and the holomorphic function $f(z)$ is called harmonic factor of $x(u,v)$.*

Theorem 3.2. *Let $x = x(u, v) : \mathbf{M} \to \mathbb{Q}^3 \subset \mathbb{E}_1^4$ be a maximal spacelike surface in $\mathbb{Q}^3$ with the isothermal parameter $z = u + iv$, Then $x(u, v) = (x_1, x_2, x_3, x_4)$ can be written as*

$$\begin{cases} x_1(u,v) = \rho(z)\overline{\rho(z)}\left\{f(z) + \overline{f(z)}\right\}, \\ x_2(u,v) = -i\rho(z)\overline{\rho(z)}\left\{f(z) - \overline{f(z)}\right\}, \\ x_3(u,v) = \rho(z)\overline{\rho(z)}\left\{1 - f(z)\overline{f(z)}\right\}, \\ x_4(u,v) = \rho(z)\overline{\rho(z)}\left\{1 + f(z)\overline{f(z)}\right\}. \end{cases} \tag{3.15}$$

For some holomorphic function $f(z)$ and complex function $\rho(z)$.

Proof. For the maximal spacelike surface $x(u, v)$, the mean curvature H vanishes identity. From (2.8) we know that the surface is flat. Using (2.4) and (3.11) we have

$$0 = \kappa = -e^{-w}w_{z\bar{z}} = -e^{-w}\left[\log(2\rho^2 f_z\overline{f_z})\right]_{z\bar{z}} = -2e^{-w}(\log\rho)_{z\bar{z}}.$$

Therefore $\rho(z, \bar{z})$ can be written as $\rho(z, \bar{z}) = \rho_1(z)\rho_2(\bar{z})$ and $\rho_1(z) = \overline{\rho_2(\bar{z})}$. Then the surface can be given by (3.15). ∎

Remark 1. From Theorem 3.2, using a holomorphic function $f(z)$ and any complex function $\rho(z)$, we can easy get the maximal spacelike surface in $\mathbb{Q}^3$ with formula (3.15).

4 Structures of spacelike surface and associated surface.

In this section, we consider the properties and structures of the spacelike surfaces and their associated surfaces in $\mathbb{Q}^3$.

Definition 2. *For the spacelike surface $x : \mathbf{M} \to \mathbb{Q}^3 \subset \mathbb{E}_1^4$, define*

$$\tilde{x}(u,v) = y(u,v) = -\frac{1}{2}\Delta x - \frac{1}{8}\langle \Delta x, \Delta x\rangle x. \tag{4.1}$$

Then $\tilde{x}(u, v)$ is also a surface in $\mathbb{Q}^3$ and called the associated surface or duality of the spacelike surface $x(u, v)$.

We define

$$\Phi = \varphi\,\mathrm{d}z^2, \qquad \varphi = \langle x_{zz}, y\rangle, \tag{4.2}$$
$$\Lambda = \lambda\,\mathrm{d}z\mathrm{d}\bar{z}, \qquad \lambda = \langle x_{z\bar{z}}, y\rangle. \tag{4.3}$$

It is easy to verify that Φ and Λ are independent of the choice of the parameters and asymptotic orthonormal frames. Therefore they are globally defined.

Define the components h_{ij} of the second fundamental form II of the spacelike surface $x(u, v)$ by

$$\mathrm{II} = \sum h_{ij}\mathrm{d}u^i\mathrm{d}u^j = (2e^w)^{-1}\sum\langle x_{ij}, y\rangle\mathrm{d}u^i\mathrm{d}u^j = (2e^w)^{-1}\sum\langle x_{u^iu^j}, y\rangle\mathrm{d}u^i\mathrm{d}u^j,$$

where $u = u^1, v = u^2$. Then by a direct calculation we have

$$\begin{cases} \varphi = \frac{1}{2}e^w(h_{11} - h_{22} - 2ih_{12}), \\ \lambda = \frac{1}{2}e^w(h_{11} + h_{22}). \end{cases} \tag{4.4}$$

Proposition 4.1. *Let $x : \mathbf{M} \to \mathbb{Q}^3 \subset \mathbb{E}_1^4$ be a spacelike surface in $\mathbb{Q}^3$. The associated surface (or duality) of the surface $x(u,v)$ is nondegenerated if and only if the second fundamental form of $x(u,v)$ is nondegenerated.*

Proof. From (2.6) we have

$$\begin{aligned} G_y &= \langle \mathrm{d}y, \mathrm{d}y\rangle = \langle y_z, y_z\rangle \mathrm{d}z^2 + 2\langle y_z, y_{\bar z}\rangle \mathrm{d}z\mathrm{d}\bar z + \langle y_{\bar z}, y_{\bar z}\rangle \mathrm{d}\bar z^2 \\ &= 2\lambda\varphi e^{-w}\mathrm{d}z^2 + 2(\lambda^2 + |\varphi|^2)e^{-w}\mathrm{d}z\mathrm{d}\bar z + 2\lambda\bar\varphi e^{-w}\mathrm{d}\bar z^2 \\ &= 2e^{-w}(\lambda\mathrm{d}z + \bar\varphi\mathrm{d}\bar z)(\varphi\mathrm{d}z + \lambda\mathrm{d}\bar z) \\ &= 2e^{-w}(\varphi\mathrm{d}z + \lambda\mathrm{d}\bar z)\overline{(\varphi\mathrm{d}z + \lambda\mathrm{d}\bar z)}. \end{aligned} \tag{4.5}$$

Then together with (4.4) we get

$$\begin{aligned} |G_y| &= -(2\lambda\varphi e^{-w})(2\lambda\bar\varphi e^{-w}) + \{(\lambda^2 + |\varphi|^2)e^{-w}\}^2 \\ &= e^{-2w}(\lambda^2 - |\varphi|^2)^2 \\ &= e^{-2w}(h_{11}h_{22} - h_{12}^2)^2. \end{aligned} \tag{4.6}$$

Therefore, we know that G_y is nondegenerated if and only if the second fundamental form II of $x(u,v)$ is nondegenerated. ∎

In the following, we denote $f_z = f'$ and $g_{\bar z} = g'$ since (3.10). From (3.7) we have

$$\begin{aligned} x_{z\bar z} &= \rho_{z\bar z}(f+g, -i(f-g), 1-fg, 1+fg) + \rho_z g'(1, i, -f, f) \\ &\quad + \rho_{\bar z} f'(1, -i, -g, g) + \rho f'g'(0, 0, -1, 1) \\ &= \rho^{-1}\rho_{z\bar z}x + \rho_z g'(1, i, -f, f) + \rho_{\bar z} f'(1, -i, -g, g) + \rho f'g'(0, 0, -1, 1) \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \Delta x &= 2e^{-w}x_{z\bar z} = \rho^{-1}(\Delta\rho)x \\ &\quad + 2e^{-w}\left\{\rho_z g'(1, i, -f, f) + \rho_{\bar z} f'(1, -i, -g, g) + \rho f'g'(0, 0, -1, 1)\right\}. \end{aligned} \tag{4.8}$$

Then

$$\begin{aligned} \langle \Delta x, \Delta x\rangle &= 16e^{-2w}f'g'(\rho_z\rho_{\bar z} - \rho\rho_{z\bar z}) = -16e^{-2w}f'g'\rho^2(\log\rho)_{z\bar z} \\ &= -8e^{-w}f'g'\rho^2\Delta(\log\rho) = -4\Delta(\log\rho). \end{aligned}$$

Therefore

$$\begin{aligned} y(u,v) &= -\frac{1}{2}\Delta x - \frac{1}{8}\langle \Delta x, \Delta x\rangle x = -\frac{1}{2}\Delta x + \frac{1}{2}\Delta(\log\rho)x \\ &= \frac{1}{2}\left\{\Delta(\log\rho) - \rho^{-1}\Delta\rho\right\}x \\ &\quad -e^{-w}\left\{\rho_z g'(1,i,-f,f) + \rho_{\bar z} f'(1,-i,-g,g) + \rho f'g'(0,0,-1,1)\right\} \\ &= -e^{-w}\left\{\rho^{-2}\rho_z\rho_{\bar z}x + \rho_z g'(1,i,-f,f) + \rho_{\bar z} f'(1,-i,-g,g) + \right. \\ &\qquad \left. \rho f'g'(0,0,-1,1)\right\} \\ &= \frac{-1}{2\rho^2}\left\{\rho^{-2}\rho_z\rho_{\bar z}f'^{-1}g'^{-1}x + \frac{\rho_z}{f'}(1,i,-f,f) + \frac{\rho_{\bar z}}{g'}(1,-i,-g,g) + \right. \\ &\qquad \left. \rho(0,0,-1,1)\right\}. \end{aligned} \tag{4.9}$$

Then we get the following conclusion.

Proposition 4.2. *For any non constant holomorphic function $f(z)$, and real function $\rho(u,v)\neq 0$, putting $g(\bar z) = \overline{f(z)}$, the surface*

$$x(u,v) = \rho(f+g, -i(f-g), 1-fg, 1+fg) \tag{4.10}$$

is a spacelike surface in $\mathbb{Q}^3$ and (u,v) is the isothermal parameter of $x(u,v)$. Furthermore, the associate surface or duality $y = y(u,v)$ of $x(u,v)$ is given by

$$y = \frac{-1}{2\rho^2}\left\{\left(\frac{\rho_z\rho_{\bar z}}{\rho^2 f'g'}\right)x + \frac{\rho_z}{f'}(1,i,-f,f) + \frac{\rho_{\bar z}}{g'}(1,-i,-g,g) + \rho(0,0,-1,1)\right\} \tag{4.11}$$

and putting $y(u,v) = (y_1, y_2, y_3, y_4)$ we have

$$\begin{cases} y_1(u,v) = \dfrac{-\rho_z\rho_{\bar z}}{2\rho^3 f'g'}\left(f + g + \dfrac{\rho f'}{\rho_z} + \dfrac{\rho g'}{\rho_{\bar z}}\right), \\[2ex] y_2(u,v) = \dfrac{-\rho_z\rho_{\bar z}}{2\rho^3 f'g'}\left(-i(f-g) - i\dfrac{\rho f'}{\rho_z} + i\dfrac{\rho g'}{\rho_{\bar z}}\right), \\[2ex] y_3(u,v) = \dfrac{-\rho_z\rho_{\bar z}}{2\rho^3 f'g'}\left(1 - fg - \dfrac{\rho f g'}{\rho_{\bar z}} - \dfrac{\rho g f'}{\rho_z} - \dfrac{\rho^2 f'g'}{\rho_z\rho_{\bar z}}\right), \\[2ex] y_4(u,v) = \dfrac{-\rho_z\rho_{\bar z}}{2\rho^3 f'g'}\left(1 + fg + \dfrac{\rho f g'}{\rho_{\bar z}} + \dfrac{\rho g f'}{\rho_z} + \dfrac{\rho^2 f'g'}{\rho_z\rho_{\bar z}}\right). \end{cases} \tag{4.12}$$

Theorem 4.1. *Let $x = x(u,v) : \mathbf{M} \to \mathbb{Q}^3 \subset \mathbb{E}^4_1$ be a spacelike surface in $\mathbb{Q}^3$ with the isothermal parameter $z = u + iv$ and the structure functions $\{f(z), \rho(u,v)\}$ and $y = y(\tilde u, \tilde v) : \mathbf{M} \to \mathbb{Q}^3 \subset \mathbb{E}^4_1$ be the associate surface (or duality) of $x(u,v)$ with the isothermal parameter $\tau = \tilde u + i\tilde v$ and the structure functions $\{\tilde f(\tau), \tilde\rho(\tilde u, \tilde v)\}$. Then we have*

$$\mathrm{d}\tau = \varphi\mathrm{d}z + \lambda\mathrm{d}\bar z \qquad \textit{or} \qquad \mathrm{d}\tau = \overline{\varphi\mathrm{d}z + \lambda\mathrm{d}\bar z} \tag{4.13}$$

and

$$\begin{cases} \tilde{f} = f + \dfrac{\rho f'}{\rho_z} = f + \dfrac{f'}{(\log\rho)_z} = f\left(1 + \dfrac{(\log f)'}{(\log\rho)_z}\right), \\ \tilde{g} = g + \dfrac{\rho g'}{\rho_{\bar{z}}} = g + \dfrac{g'}{(\log\rho)_{\bar{z}}} = g\left(1 + \dfrac{(\log g)'}{(\log\rho)_{\bar{z}}}\right), \\ \tilde{\rho} = \dfrac{-\rho_z\rho_{\bar{z}}}{2\rho^3 f'g'} = \dfrac{-(\log\rho)_z(\log\rho)_{\bar{z}}}{2\rho f'g'} = \dfrac{-(\log\rho)_z(\log\rho)_{\bar{z}}}{2\rho fg(\log f)'(\log g)'}, \end{cases} \tag{4.14}$$

$$2\rho\tilde{\rho}(\tilde{f} - f)(\tilde{g} - g) = -1. \tag{4.15}$$

Where $g(\bar{z}) = \overline{f(z)}$ *and* $\tilde{g}(\bar{\tau}) = \overline{\tilde{f}(\tau)}$.

Proof. From (4.5) and $(\tilde{u}, \tilde{v})$ is the isothermal parameter of y we get (4.13). By (3.5) and (4.12) we obtain (4.14). From (4.14) we have (4.15). ∎

Corollary 4.2. *Let* $x = x(u,v) : \mathbf{M} \to \mathbb{Q}^3 \subset \mathbb{E}^4_1$ *be a maximal spacelike surface in* $\mathbb{Q}^3$ *with the isothermal parameter* $\{u,v\}$. *Then* $\{u,v\}$ *is also the isothermal parameter of the associate surface (or duality)* $y(u,v)$ *of* $x(u,v)$.

Proof. For the maximal spacelike surface $x(u,v)$, from (2.7) we have $\lambda \equiv 0$. Then from (4.13) we know that $\{u,v\}$ is also the isothermal parameter of $y(u,v)$. ∎

Theorem 4.3. *The associate surface (or duality)* $y(u,v)$ *of a spacelike surface* $x = x(u,v) : \mathbf{M} \to \mathbb{Q}^3 \subset \mathbb{E}^4_1$ *is maximal if and only if the spacelike surface* $x(u,v)$ *is maximal.*

Proof. From (2.4), (2.8), (3.14), (4.14) and Corollary 4.2, by a direct calculation we can get the conclusion of this theorem. ∎

Proposition 4.3. *Let* $x = x(u,v) : \mathbf{M} \to \mathbb{Q}^3 \subset \mathbb{E}^4_1$ *be a spacelike surface in* $\mathbb{Q}^3$ *with the isothermal parameter* $z = u + iv$. *Then* $x(u,v) = (x_1, x_2, x_3, x_4)$ *can be written as*

$$\begin{cases} x_1(u,v) = f(z) + \overline{f(z)}, \\ x_2(u,v) = -i(f(z) - \overline{f(z)}), \\ x_3(u,v) = 1 - f(z)\overline{f(z)}, \\ x_4(u,v) = 1 + f(z)\overline{f(z)}. \end{cases} \tag{4.16}$$

for some holomorphic function $f(z)$, *that is, the function* $\rho(u,v)$ *is constant in* (4.10), *if and only if the spacelike surface* $x(u,v)$ *is totally geodesic in* $\mathbb{Q}^3$.

Proof. From (4.2), (4.3), (3.7) and (4.9) we have

$$\begin{cases} \varphi = \langle x_{zz}, y\rangle = \rho^{-1}\rho_{zz} - 2\rho^{-2}\rho_z^2 - \rho^{-1}\rho_z f'^{-1}f'' \\ \qquad\qquad\;\; = (\log\rho)_{zz} - \{(\log\rho)_z\}^2 - (\log\rho)_z f'^{-1}f'', \\ \bar{\varphi} = \langle x_{\bar{z}\bar{z}}, y\rangle = (\log\rho)_{\bar{z}\bar{z}} - \{(\log\rho)_{\bar{z}}\}^2 - (\log\rho)_{\bar{z}} g'^{-1}g'', \\ \lambda = \langle x_{z\bar{z}}, y\rangle = \frac{1}{2}e^{w}\Delta\log\rho = \rho^2 f'g'\Delta\log\rho. \end{cases} \tag{4.17}$$

If $\rho(u,v)$ is constant, by (4.17) we get $\varphi = \lambda \equiv 0$. Then from (4.4) we know that $x(u,v)$ is totally geodesic in $\mathbb{Q}^3$. Conversely, if $\lambda \equiv 0$ we have $(\log\rho)_{z\bar{z}} = \rho\rho_{z\bar{z}} - \rho_z\rho_{\bar{z}} \equiv 0$. And $\varphi \equiv 0$ yields

$$\begin{cases} \varphi = (\log\rho)_{zz} - \{(\log\rho)_z\}^2 - (\log\rho)_z f'^{-1} f'' = 0, \\ \bar{\varphi} = (\log\rho)_{\bar{z}\bar{z}} - \{(\log\rho)_{\bar{z}}\}^2 - (\log\rho)_{\bar{z}} g'^{-1} g'' = 0. \end{cases} \tag{4.18}$$

Solving this partial differential equations, we omit the tediously process, and get the solutions $\rho_z = \rho_{\bar{z}} \equiv 0$ or

$$\rho(u,v) = \frac{1}{f(z)g(\bar{z})}.$$

Where $g(\bar{z}) = \overline{f(z)}$. ■

From (4.13) we may take

$$\mathrm{d}\tau = \varphi \mathrm{d}z + \lambda \mathrm{d}\bar{z} = \frac{\partial\tau}{\partial z}\mathrm{d}z + \frac{\partial\tau}{\partial\bar{z}}\mathrm{d}\bar{z}.$$

Then we have

$$\frac{\partial\tau}{\partial z} = \varphi, \qquad \frac{\partial\tau}{\partial\bar{z}} = \lambda. \tag{4.19}$$

Since

$$\frac{\partial}{\partial z} = \frac{\partial\tau}{\partial z}\frac{\partial}{\partial\tau} + \frac{\partial\bar{\tau}}{\partial z}\frac{\partial}{\partial\bar{\tau}}$$

we get

$$\begin{cases} (\lambda^2 - |\varphi|^2)\dfrac{\partial}{\partial\tau} = -\bar{\varphi}\dfrac{\partial}{\partial z} + \lambda\dfrac{\partial}{\partial\bar{z}}, \\ \\ (\lambda^2 - |\varphi|^2)\dfrac{\partial}{\partial\bar{\tau}} = \lambda\dfrac{\partial}{\partial z} - \varphi\dfrac{\partial}{\partial\bar{z}}. \end{cases} \tag{4.20}$$

Therefore

$$\begin{aligned} \tilde{\Delta} &= 2e^{-\tilde{w}}\partial\tau\partial\bar{\tau} = 2e^{-\tilde{w}}\frac{\partial}{\partial\tau}\frac{\partial}{\partial\bar{\tau}} \\ &= \left(\frac{2e^{-\tilde{w}}}{(\lambda^2 - |\varphi|^2)^2}\right)\left(-\bar{\varphi}\frac{\partial}{\partial z} + \lambda\frac{\partial}{\partial\bar{z}}\right)\left(\lambda\frac{\partial}{\partial z} - \varphi\frac{\partial}{\partial\bar{z}}\right), \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} e^{\tilde{w}} &= 2\tilde{\rho}^2\tilde{f}'\tilde{g}' = 2\tilde{\rho}^2\frac{\partial\tilde{f}}{\partial\tau}\frac{\partial\tilde{g}}{\partial\bar{\tau}} \\ &= 2\left(\frac{\rho_z\rho_{\bar{z}}}{2\rho^3 f'g'}\right)^2\left\{\frac{\partial}{\partial\tau}\left(f + \frac{\rho f'}{\rho_z}\right)\right\}\left\{\frac{\partial}{\partial\bar{\tau}}\left(g + \frac{\rho g'}{\rho_{\bar{z}}}\right)\right\} \end{aligned} \tag{4.22}$$

and λ, φ are given by (4.17). The Gaussian curvature of the associated surface $y(u,v)$ of spacelike surface $x(u,v)$ is given by

$$\begin{aligned}
\tilde{\kappa}(u,v) &= -\tilde{\Delta}(\log\tilde{\rho}) \qquad (4.23)\\
&= -\left(\frac{2e^{-\tilde{w}}}{(\lambda^2-|\varphi|^2)^2}\right)\left(-\bar{\varphi}\frac{\partial}{\partial z}+\lambda\frac{\partial}{\partial \bar{z}}\right)\left(\lambda\frac{\partial}{\partial z}-\varphi\frac{\partial}{\partial \bar{z}}\right)\left(\log\left(\frac{-\rho_z\rho_{\bar{z}}}{2\rho^3 f'g'}\right)\right)\\
&= -\left(\frac{2e^{-\tilde{w}}}{(\lambda^2-|\varphi|^2)^2}\right)\left(-\lambda\bar{\varphi}\frac{\partial^2}{\partial z^2}-\lambda\varphi\frac{\partial^2}{\partial \bar{z}^2}\right)\left(\log\left(\frac{-\rho_z\rho_{\bar{z}}}{2\rho^3 f'g'}\right)\right)\\
&\quad -\left(\frac{2e^{-\tilde{w}}(\lambda^2+|\varphi|^2)}{(\lambda^2-|\varphi|^2)^2}\right)\left(\frac{\partial^2}{\partial z\partial \bar{z}}\right)\left(\log\left(\frac{-\rho_z\rho_{\bar{z}}}{2\rho^3 f'g'}\right)\right)\\
&= \lambda\left(\frac{2e^{-\tilde{w}}}{(\lambda^2-|\varphi|^2)^2}\right)\left(\bar{\varphi}\frac{\partial^2}{\partial z^2}+\varphi\frac{\partial^2}{\partial \bar{z}^2}\right)\left(\log\left(\frac{-\rho_z\rho_{\bar{z}}}{2\rho^3 f'g'}\right)\right)\\
&\quad -\left(\frac{2e^{-\tilde{w}}(\lambda^2+|\varphi|^2)}{(\lambda^2-|\varphi|^2)^2}\right)\left(\frac{\partial^2}{\partial z\partial \bar{z}}\right)(\log(\rho_z\rho_{\bar{z}}))\\
&\quad +\left(\frac{6e^{-\tilde{w}}(\lambda^2+|\varphi|^2)}{(\lambda^2-|\varphi|^2)^2}\right)\left(\frac{\partial^2}{\partial z\partial \bar{z}}\right)(\log(\rho))\\
&= \lambda\left(\frac{2e^{-\tilde{w}}}{(\lambda^2-|\varphi|^2)^2}\right)\left(\bar{\varphi}\frac{\partial^2}{\partial z^2}+\varphi\frac{\partial^2}{\partial \bar{z}^2}\right)\left(\log\left(\frac{-\rho_z\rho_{\bar{z}}}{2\rho^3 f'g'}\right)\right)\\
&\quad -\left(\frac{2e^{-\tilde{w}}(\lambda^2+|\varphi|^2)}{(\lambda^2-|\varphi|^2)^2}\right)\left(\frac{\partial^2}{\partial z\partial \bar{z}}\right)(\log(\rho_z\rho_{\bar{z}}))\\
&\quad -\left(\frac{6e^{-\tilde{w}}\rho^2 f'g'(\lambda^2+|\varphi|^2)}{(\lambda^2-|\varphi|^2)^2}\right)\kappa.
\end{aligned}$$

Proposition 4.4. *The Gaussian curvatures κ and $\tilde{\kappa}$ of spacelike surface $x = x(u,v) : \mathbf{M} \to \mathbb{Q}^3 \subset \mathbb{E}^4_1$ with structure functions $\{f, g, \rho\}$ and its associate surface (or duality) $y(u,v)$ satisfy*

$$\begin{aligned}
\tilde{\kappa}(u,v) &= \lambda\left(\frac{2e^{-\tilde{w}}}{(\lambda^2-|\varphi|^2)^2}\right)\left(\bar{\varphi}\frac{\partial^2}{\partial z^2}+\varphi\frac{\partial^2}{\partial \bar{z}^2}\right)\left(\log\left(\frac{-\rho_z\rho_{\bar{z}}}{2\rho^3 f'g'}\right)\right) \qquad (4.24)\\
&\quad -\left(\frac{2e^{-\tilde{w}}(\lambda^2+|\varphi|^2)}{(\lambda^2-|\varphi|^2)^2}\right)\left(\frac{\partial^2}{\partial z\partial \bar{z}}\right)(\log(\rho_z\rho_{\bar{z}}))\\
&\quad -\left(\frac{6e^{-\tilde{w}}\rho^2 f'g'(\lambda^2+|\varphi|^2)}{(\lambda^2-|\varphi|^2)^2}\right)\kappa.
\end{aligned}$$

Where λ, φ are given by (4.17) and $e^{\tilde{w}}$ is given by (4.22).

Remark 2. With this relation between κ to $\tilde{\kappa}$ we can also easy get the conclusion of Theorem 4.3.

Remark 3. Using Theorem 3.1 and Theorem 3.2 we can easily get the examples of spacelike surfaces and maximal spacelike surfaces in 3-dimensional lightlike cone $\mathbb{Q}^3 \subset \mathbb{E}^4_1$.

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