Topologically Q-algebras, more

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Abstract

Topologically Q-algebras, a very convenient generalization of (topological) Q-algebras, have been recently considered by *A. Najmi* and independently by *H. Arizmendi et al.*. Here we extend certain basic results of *A. Najmi* Within the same vein of ideas, we introduce a new class of topological algebras, the *"Cauchy topologically Q-algebras"* (Cauchy tQ-algebras), that yields a new characterization of the standard advertibly complete algebras, as defined by *S. Warner*.

0 Introduction

Q-algebras possess an important place in Topological Algebras Theory, sharing several significant properties of Banach algebras. Thus, just to mention a few of them, they have equicontinuous Gel'fand space, every character is continuous, every maximal regular ideal is closed, every element has compact spectrum, while they are also advertibly complete. Many researchers have dealt with Q-algebras. *A. Mallios* proved in 1986 that *a Q-algebra is characterized by a zero neighbourhood consisting of elements normalized by means of the spectral radius* [11, p. 59, Lemma 4.2], a result known previously by *E. A. Michael* for locally

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convex (topological) algebras [12, p. 58, Proposition 13.5]. *Y. Tsertos* discovered in 1994 that *a Q-algebra is characterized by the fact that its spectral radius is bounded by the gauge function (: "Minkowski functional") of a zero neighbourhood* [14, p. 550, Theorem 4.1]. In 1999 *H. Arizmendi and V. Valov* gave some *characterizations of Q-algebras, intimating* a relation of great importance possessed by commutative locally m-convex advertibly complete ones, let alone by Q-algebras. See also *A. Mallios* [11, p. 75, Theorem 7.2 and p. 104, Theorem 6.2 or Corollary 6.4]. More precisely, the aforesaid condition reads as follows for a topological algebra E,

$$Sp_E(x) = \hat{x}(\mathfrak{M}(E)), \quad x \in E. \tag{s}$$

The same property characterizes the *topologically spectral algebras*, see *R. I. Hadjigeorgiou* [8] and/or [9]. The previous relation implies what we may call *(A-V) condition* [2, p. 12, Theorem 1]; that is, one has

$$r_E(x) = \sup_{f \in \mathfrak{M}(E)} |f(x)|, \quad x \in E. \tag{A-V}$$

Based on the condition (s), the same authors consider some generalizations of Q-algebras; namely the notions of $Q_{\mathcal{M}}$ and $Q_{\mathcal{M}^\sharp}$-algebras [2, p. 17, §4] and/or [3]. In fact, the previous notions are related with the openness of *two sorts of invertible elements*, depending on the topological $\mathfrak{M}(E) \equiv \mathcal{M}(E)$ or algebraic $M(E) \equiv \mathcal{M}^\sharp(E)$ spectrum of E, respectively. Namely, an element x of a topological algebra E satisfying (s) is called $\mathcal{M}$*-invertible* (resp. $\mathcal{M}^\sharp(E)$*-invertible*, when E satisfies (s), for $M(E) \equiv \mathcal{M}^\sharp(E)$, instead of $\mathfrak{M}(E)$) if $0 \notin \hat{x}(\mathcal{M})(E)$ (resp. $0 \notin \hat{x}(\mathcal{M}^\sharp(E))$). The present author considered in 1995 the same sort of invertible elements [8], calling then the topological algebra at issue, *inverse closed*, as well as the condition (s), calling the respective algebra *topologically spectral*. In fact, it was proved that the previous two types of algebras coincide [8] and/or [9]. Research around the (A-V) condition providing further characterizations of Q-algebras has been still continued in [9].

On the other hand, in 2002 *H. Arizmendi, A. Carrillo and L. Palacios determined the class of Q_t-algebras by assuming the group of topologically invertible elements to be open* and established relations between Q_t-algebras and Q, $Q_{\mathcal{M}}$, $Q_{\mathcal{M}^\sharp}$-algebras [4]. Independently of the aforesaid authors, *A. Najmi* distinguished the above class of Q_t-algebras, calling them tQ-algebras and proving that they possess most of the important properties of Q-algebras [13], along with the aforementioned two characterizations given by *A. Mallios* and *Y. Tsertos*. Furthermore, he presented a more general context than of *A. Mallios*, where *condition (s), therefore condition (A-V)* too, as above, *defined for topologically quasi-invertible elements, is fulfilled*; namely, *in a simplicial t-acceptable Gel'fand-Mazur topological algebra* [13, Proposition 2.21]. It should still be noted here that the set of topologically quasi-invertible elements had also been considered by *Mati Abel* [1] in 2001, however from another point of view. Specifically, the latter author distinguishes by *the coincidence of the set of topologically quasi-invertible elements with that one of quasi-invertible elements, the class of advertive algebras*, which contains Q-algebras and is contained in the class of advertibly complete algebras (ibid., p. 16, Proposition 2 and Corollary 1). He also gave a characterization of advertive algebras (ibid., p. 16, Proposition 1).

Now, motivated by the paper of *A. Najmi* [13] and following the point of view adopted in [9], we isolate the following class of algebras satisfying the condition

$$(N_s) \qquad Sp_E^t(x) = \hat{x}(\mathfrak{M}(E)), \quad x \in E.$$

More generally, we also consider those topological algebras obeying the next condition

$$(N_r) \qquad r_E^t(x) = \sup_{f \in \mathfrak{M}(E)} |f(x)|, \quad x \in E.$$

The above two classes of algebras extend the class of algebras previously introduced by *A. Najmi* (ibid.), so that we get in each particular case his results.

The question whether a tQ-algebra is advertibly complete, led us to an intermediate class of topological algebras between Q and tQ-algebras, namely, the *Cauchy tQ-algebras*, by employing the notion of topologically quasi-invertible elements for Cauchy nets; so one looks at the group of *Cauchy topologically quasi-invertible elements*. The same notion brought in light a more general class of advertive algebras, that of *Cauchy advertive algebras, giving also a new characterization of advertibly complete algebras* (Theorems 3.4, 3.5).

1 Preliminaries

In all that follows by a *topological algebra* E we mean a topological $\mathbb{C}$-vector space, which is also an algebra with separately continuous ring multiplication, having a non-empty *spectrum* or *Gel'fand space* $\mathfrak{M}(E)$ endowed with the *Gel'fand topology*. The respective *Gel'fand map* of E is given by

$$\begin{aligned} \mathcal{G} : E \longrightarrow \mathcal{C}(\mathfrak{M}(E)) : x \longmapsto \mathcal{G}(x) \equiv \hat{x} : \mathfrak{M}(E) \longrightarrow \mathbb{C} \\ : f \longmapsto \hat{x}(f) := f(x). \end{aligned}$$

The image of $\mathcal{G}$, denoted by $E^{\wedge}$, is called the *Gel'fand transform algebra* of E and is topologized as a *locally m-convex algebra* by the inclusion

$$E^{\wedge} \subseteq \mathcal{C}_c(\mathfrak{M}(E)),$$

where the algebra $\mathcal{C}(\mathfrak{M}(E))$ carries the topology "c" of compact convergence in $\mathfrak{M}(E)$ [11, p. 19, Example 3.1]. Given an algebra E, an element $x \in E$ is called *quasi-invertible*, if there exists $y \in E$ such that

$$x \circ y = 0 = y \circ x, \ \ \textit{where} \ \ x \circ y = x + y - xy. \tag{1.1}$$

The above last relation defines the so-called "*circle operation*" or else "*q-operation*". Then y is called the *quasi-inverse* of x and is unique, while the group of all quasi-invertible elements of E is denoted by E°. An element x of a topological algebra E is called *topologically quasi-invertible* (cf. [13] or [5]), if there exists a net $(x_\delta)_{\delta \in \Delta} \subseteq E$, such that

$$x \circ x_\delta \longrightarrow 0 \longleftarrow x_\delta \circ x, \tag{1.2}$$

and the set of topologically quasi-invertible elements is denoted by $(E^\circ)^t$. In the case of a unital topological algebra E, the $\circ$ operation is replaced by the ring multiplication and 0 by 1, obtaining thus the *topologically invertible elements* denoted by $(E^\bullet)^t$; namely we have

$$x \cdot x_\delta \longrightarrow 1 \longleftarrow x_\delta \cdot x. \tag{1.3}$$

If the net in relations (1.2) and (1.3) is Cauchy, then we speak about *Cauchy topologically quasi-invertible* and *Cauchy topologically invertible elements,* respectively, while the corresponding sets are denoted by $(E^\circ)^{ct}$ and $(E^\bullet)^{ct}$, respectively. We say that a topological algebra E is *advertive* (resp. invertive) if $(E^\circ)^t = E^\circ$ (resp. $(E^\bullet)^t = E^\bullet$) (cf. [1]); if $(E^\circ)^{ct} = E^\circ$ (resp. $(E^\bullet)^{ct} = E^\bullet$), then E is called *Cauchy advertive* (resp. *Cauchy invertive*).

A topological algebra E is a *Q-algebra* if E° is open. E is named a *topologically Q-algebra*, in brief *tQ-algebra*, if $(E^\circ)^t$ is open. Now, based on the preceding, we are led to the following.

Definition 1.1. A topological algebra E is called a *Cauchy topologically Q-algebra*, in brief, a *Cauchy tQ-algebra*, if $(E^\circ)^{ct}$ is open.

Therefore, one has

$$E^\circ \subseteq (E^\circ)^{ct} \subseteq (E^\circ)^t, \tag{1.4}$$

that is, equivalently,

$$Q-algebra \implies Cauchy\ \ tQ-algebra \implies tQ-algebra. \tag{1.5}$$

Moreover, E is an *advertibly complete algebra*, whenever *every advertibly null Cauchy net* $(x_\delta)_{\delta\in\Delta}$ in E, in the sense that

$$x_\delta \circ x \longrightarrow 0 \longleftarrow x \circ x_\delta,\ \ for\ some\ x \in E, \tag{1.6}$$

converges in E; its limit is obviously the quasi-inverse of x [11, p. 45, Definition 6.4]. The above more convenient terminology is still due to *A. Mallios*. *The convergence of any net* (not necessarily Cauchy) *satisfying (1.6) characterizes the algebra E as an advertive topological algebra*, according to *Mati Abel* [1, p. 16, Proposition 1]. He also proved (ibid., p. 16, Proposition 2 and Corollary 1) that

$$Q-algebra \implies advertive\ algebra \implies advertibly\ \ complete\ \ algebra. \tag{1.7}$$

E is called *simplicial* [1] or *normal* [12], if any proper closed regular (left, right, 2-sided) ideal is contained in a closed maximal regular (left, right, 2-sided) ideal, and *t-acceptable* [13], if any regular closed maximal one-sided ideal is 2-sided. If $E/M \cong \mathbb{C}$, for every (2-sided) closed maximal regular ideal M of E, then E is said to be a *Gel'fand-Mazur algebra*. We say that E is a *quasi-inverse closed algebra* if its spectrum $\mathfrak{M}(E)$ is a *quasi-inverting set*, in the sense that

$$x \in E^\circ\ if\ 1 \notin \hat{x}(\mathfrak{M}(E)) \tag{1.8}$$

[8, p. 13, Definition 2.2] and/or [9]. The converse statement is always valid, in fact, quite algebraically [11, p. 74, Lemma 7.4]. If (1.8) holds true for $(E^\circ)^t$ instead of E°, we speak for a *topologically quasi-inverse closed algebra* [13], and for a *Cauchy topologically quasi-inverse closed algebra* if (1.8) is fulfilled for $(E^\circ)^{ct}$. On the basis of (1.4), one actually has that

$$\begin{aligned} quasi-inverse\ closed &\implies Cauchy\ topologically\ quasi-inverse\ closed \\ &\implies topologically\ quasi-inverse\ closed \end{aligned} \tag{1.9}$$

Now, given an algebra E and an element $x \in E$, we denote the *spectrum*, *topological spectrum* and *Cauchy topological spectrum* of x, by

$$Sp_E(x) = \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^{-1}x \notin E^\circ\} \cup \{0\}, \tag{1.10}$$

$$Sp_E^t(x) = \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^{-1}x \notin (E^\circ)^t\} \cup \{0\}, \tag{1.11}$$

see [13], and

$$Sp_E^{ct}(x) = \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^{-1}x \notin (E^\circ)^{ct}\} \cup \{0\}, \tag{1.12}$$

respectively. Thus, based on (1.4), one gets

$$Sp_E^t(x) \subseteq Sp_E^{ct}(x) \subseteq Sp_E(x). \tag{1.13}$$

Now, the *spectral radius*, *topological spectral radius* and *Cauchy topological spectral radius* of x is defined by

$$r_E(x) = \sup_{\lambda \in Sp_E(x)} |\lambda|, \tag{1.14}$$

respectively ([13])

$$r_E^t(x) = \sup_{\lambda \in Sp_E^t(x)} |\lambda|, \tag{1.15}$$

and

$$r_E^{ct}(x) = \sup_{\lambda \in Sp_E^{ct}(x)} |\lambda|, \tag{1.16}$$

getting, in view of (1.13), the relation

$$r_E^t(x) \leq r_E^{ct}(x) \leq r_E(x). \tag{1.17}$$

When every $x \in E$ has $r_E^t(x) < +\infty$, E is called *topologically spectrally bounded*, while if $r_E^{ct}(x) < +\infty$, E is said to be *Cauchy topologically spectrally bounded*. Based on (1.17), one obtains

$$\begin{aligned} spectrally\ bounded &\implies Cauchy\ topologically\ spectrally\ bounded \\ &\implies topologically\ spectrally\ bounded. \end{aligned} \tag{1.18}$$

2 On Najmi's results

In this Section following the point of view in [9], we isolate algebras with (N_s) or (N_r) condition, obtaining thus generalizations of relevant results by *A. Najmi*

[13]. The latter author extended to tQ-algebras the characterizations given by *A. Mallios* and *Y. Tsertos* for Q-algebras; that is one has:

A topological algebra E is tQ

$$
(2.1)\qquad
\left[\begin{array}{lll}
\Longleftrightarrow & \textbf{(i)} & (E^\circ)^t \textit{ is a neighbourhood of zero} \\
& & \textit{(A. Mallios)} \\
\Longleftrightarrow & \textbf{(ii)} & T(E) := \{x \in E : r_E^t(x) \leq 1\} \textit{ is a} \\
& & \textit{neighbourhood of zero (A. Mallios)} \\
\Longleftrightarrow & \textbf{(iii)} & r_E^t \leq g_V,\ V \textit{ a neighbourhood of zero} \\
& & \textit{(Y. Tsertos)} \\
\Longrightarrow & \textbf{(iv)} & E \textit{ is topologically spectrally bounded, viz.} \\
& & E = TB(E) := \{x \in E : r_E^t(x) < +\infty\} \\
& & \textit{(A. Mallios)}
\end{array}\right.
$$

Najmi also proved that *a tQ-algebra has equicontinuous Gel'fand space and each element has compact spectrum*, properties well-known for a Q-algebra [11]. Concerning the converse of **(iv)**, he considered a simplicial t-acceptable Gel'fand-Mazur topological algebra having continuous Gel'fand map ([13, Corollary 2.23]). In fact, the latter algebra satisfies (N_s), hence (N_r) condition ([13, Proposition 2.21]), what is actually required. So one gets the following.

Proposition 2.1. *Let E be a topological algebra and consider the following assertions:*
1) E is a tQ-algebra.
2) $E = TB(E)$, that is, E is topologically spectrally bounded.
Then, 1)$\Longrightarrow$2), while 2)$\Longrightarrow$1), as well, if moreover E satisfies (N_r) condition and has continuous Gel'fand map $\mathcal{G}_E$.

Proof. 1)$\Longrightarrow$2) due to the compactness of the topological spectrum of every element [13, Lemma 2.10]. 2)$\Longrightarrow$1): By 2) and (N_r) condition, r_E^t is a semi-norm of E, and by the continuity of $\mathcal{G}_E$, r_E^t is continuous at zero. Hence, $T(E)$ is a neighbourhood of zero, thus (cf. **(ii)**) E is a tQ-algebra. ∎

Condition (N_s) provides now an extension of [13, Proposition 2.27]. That is, one has.

Proposition 2.2. *In a topological algebra E consider the following assertions:*
1) $x \in (E^\circ)^t$, $x \in E$.
2) $1 \notin \hat{x}(\mathfrak{M}(E))$, $x \in E$.
Then, 1)$\Longrightarrow$2), while 2)$\Longrightarrow$1) if E satisfies (N_s) condition.

Proof. Assuming 1), there exists a net $(x_\delta)_{\delta\in\Delta} \subseteq E$, such that $(x \circ x_\delta) \longrightarrow 0$, $x \in E$. If $1 \in \hat{x}(\mathfrak{M}(E))$, then for some $f \in \mathfrak{M}(E)$, one has $f(x) = 1$, hence $f(x \circ x_\delta) = f(x + x_\delta - xx_\delta) = f(x) + f(x_\delta) - f(x)f(x_\delta) = 1 \longrightarrow 0$, a contradiction. Conversely, considering (N_s), if $1 \notin \hat{x}(\mathfrak{M}(E)) = Sp_E^t(x)$, $x \in E$, then $x \in (E^\circ)^t$, that is the assertion. ∎

In the case of a unital topological algebra, by a similar argument, one obtains the next (cf. [13, Corollary 2.28]).

Corollary 2.3. *In a unital topological algebra E satisfying (N_s) condition, the following assertions are equivalent:*

1) $x \in (E^{\bullet})^t, \quad x \in E$.

2) $0 \notin \hat{x}(\mathfrak{M}(E)), \quad x \in E$.

On the other hand, *A. Najmi* based on our previous result [9, p. 52, Theorem 2.5], proved that (cf. [13, Proposition 2.24]):

(2.2) *a topologically quasi-inverse closed algebra is characterized by (N_s) condition .*

Based on the latter, he also gave a characterization of a tQ-algebra ([13, Theorem 2.29 and Corollary 2.33]); this can now be generalized as follows.

Theorem 2.4. *For a topological algebra E consider the two following assertions:*

1) E is a tQ-algebra.

2) $\mathfrak{M}(E)$ is equicontinuous.

Then, 1)$\Longrightarrow$2). If , in addition, E has the (N_r) condition, then 2)$\Longrightarrow$1), as well.

Proof. Assuming 1), $(E^{\circ})^t$ is open, so it contains a balanced neighbourhood of zero, say U. If we show that $U \subseteq (\mathfrak{M}(E))^{\circ}$, then $(\mathfrak{M}(E))^{\circ}$ is a neighbourhood of zero, yielding the equicontinuity of $\mathfrak{M}(E)$, since $\mathfrak{M}(E) \subseteq (\mathfrak{M}(E))^{\circ\circ}$. So, if $x \in U$, with $|f(x)| > 1$ for some $f \in \mathfrak{M}(E)$, then $f(x) = \lambda \neq 0$ and $|\frac{1}{\lambda}| < 1$. Thus, $\frac{x}{\lambda} \in U \subseteq (E^{\circ})^t$ with $f(\frac{x}{\lambda}) = 1$, a contradiction by 1)$\Longrightarrow$2) of Proposition 2.2. Therefore, $|f(x)| \leq 1$, for every $f \in \mathfrak{M}(E)$ and $x \in U$, that is $U \subseteq (\mathfrak{M}(E))^{\circ}$.

On the other hand, the equicontinuity of $\mathfrak{M}(E)$, along with (N_r) condition, implies that $T(E) = (\mathfrak{M}(E))^{\circ}$ is a neighbourhood of zero, hence by **(ii)** E is a tQ-algebra. ∎

Scholium 2.5. The previous theorem amply extends parts of the well-known classical result of *S. Warner* in [16, p. 7, Theorem 6].

From [13, the statement (3) of Corollary 2.31], one gets a sufficient condition in order a tQ-algebra to be a Q-algebra. A more general context in which that condition works is given below. In this regard, it is clear by the very definitions that:

(2.3) *a topological algebra is Q iff it is an advertive tQ-algebra.*

Theorem 2.6. *Let E be a topological algebra and consider the following assertions:*

1) E is a Q-algebra.

2) E is a tQ-algebra.

Then, 1)$\Longrightarrow$2). In particular, if E is, moreover, topologically spectral (cf. (s))*, then 2)$\Longrightarrow$1) as well, so that the previous two assertions are equivalent.*

Proof. Based on (1.5), one has to prove that 2)$\Longrightarrow$1): By hypothesis and (1.10), one gets
$$Sp^t_E(x) \subseteq Sp_E(x) = \hat{x}(\mathfrak{M}(E)) \subseteq Sp^t_E(x),$$
for every $x \in E$, hence $Sp^t_E(x) = Sp_E(x)$, for any $x \in E$. Thus, $E^{\circ} = (E^{\circ})^t$ is open, that is, E is a Q-algebra. ∎

Scholium 2.7. In the preceding proof, one remarks that *topological spectrality renders a topological algebra advertive, hence in the case of a tQ-algebra the same makes it a Q-algebra*. So, one obtains the next.

Corollary 2.8. *Every topologically spectral algebra is advertive. Therefore, if it is also a tQ-algebra, then it is, in fact, a Q-algebra.*

The fact that a topologically spectral algebra is advertive has been proved in 1999 by *Mati Abel* [1, p. 19, Proposition 6], using a different argument. Yet, a strengthening of Corollary 2.8, in terms of *Cauchy tQ-algebras* (see below), is provided by the next Section (see (3.5) in the sequel).

3 Cauchy topologically Q-algebras

The employment of arbitrary nets in the definition of invertible elements led to a broader class of Q-algebras, the *topologically Q-algebras*. In particular, the consideration instead of Cauchy nets yields the class of *Cauchy topologically Q-algebras*, quite close indeed to Q-algebras, in effect, more close than the tQ ones. Now, as a byproduct of our previous result concerning the characterization of elements of a given algebra that are quasi-invertible in its completion ([9, p. 52, Lemma 2.4]), one has.

Lemma 3.1. *For a topological algebra E, whose completion $\widetilde{E}$ is also a topological algebra, one has*

$$(E^\circ)^{ct} = (\widetilde{E})^\circ \cap E. \tag{3.1}$$

The observation that $(\widetilde{E})^\circ \cap E \subseteq (E^\circ)^t$ *becomes equality for topologically quasi-invertible elements determined by Cauchy nets*, led us to consider Cauchy topologically quasi-invertible elements.

An immediate consequence of (3.1) is that

$$(E^\circ)^{ct} \ \ is \ \ open \ \ if \ \ (\widetilde{E})^\circ \ \ is \ \ open,$$

so, one obtains the next.

Theorem 3.2. *Let E be topological algebra E whose completion $\widetilde{E}$ is a Q-algebra. Then, E is a Cauchy tQ-algebra.*

Corollary 3.3. *Every normed algebra is a Cauchy tQ-algebra.*

The above result extends an initial theorem of *Najmi* for tQ-algebras [13, Proposition 2.5].

Remarks 3.4. *i)* Theorem 3.2 strengthens a remark of *A. Mallios* that, *if the completion of a topological algebra is Q, then the initial algebra is tQ*; see *A. Najmi* [13, Remark 2.6].

ii) Replacing Sp^t_E, r^t_E, $TqinvE$ by Sp^{ct}_E, r^{ct}_E, $(E^\circ)^{ct}$, respectively, the proofs of all the results given for tQ-algebras in [13] as, well as, in Section 2 above (see e.g.

Scholium 2.5), are still valid for *Cauchy tQ-algebras*, adjusting, whenever needed, the conditions (N_s) and (N_r) respectively as follows:

$$(C_s) \qquad Sp_E^{ct}(x) = \hat{x}(\mathfrak{M}(E)), \quad x \in E,$$

and

$$(C_r) \qquad r_E^{ct}(x) = \sup_{f \in \mathfrak{M}(E)} |f(x)|, \quad x \in E$$

(cf also [11], [14]). An algebra satisfying the (C_s) condition is called a *Cauchy topologically spectral algebra.* In this context, we further remark that, according to (1.5), *any result valid for a tQ-algebra holds also true for a Cauchy tQ-algebra.*

iii) As *A. Mallios* proved [11, p. 146, Lemma 2.2], the coincidence of the Gel'fand spaces (modulo a *(continuous) bijection*) between E and its completion $\widetilde{E}$ transfers the equicontinuity (in effect, even, *local equicontinuity*) of any one of the two spaces to the other. Now, based on [11, p. 75, Proposition 7.1] and Theorems 2.4 and 3.2 above, we remark that a *Cauchy tQ-algebra* yields an example of *a topological algebra that itself and its completion have equicontinuous Gel'fand spaces, without the two spaces to coincide, up to a continuous bijection.* [The analogous case of *local equicontinuity* remains *open*.]

Now, by the very definitions of advertible completeness and Cauchy topologically quasi-invertible elements, one remarks that:

(3.2) *advertible completeness guarantees the quasi-inverse of any Cauchy topologically quasi-inverse element*, that is

(3.2.1) $\qquad (E^\circ)^{ct} \subseteq E^\circ.$

Obviously, the inverse implication is always valid. Hence, one concludes the following *characterization of advertible completeness through Cauchy advertiveness*.

Theorem 3.5. *A topological algebra is advertibly complete if, and only if, it is Cauchy advertive. That is (by definition), whenever one has,*

$$(E^\circ)^{ct} = E^\circ. \tag{3.3}$$

Thus, (1.7) is now supplemented as follows:

$$Q-algebra \Longrightarrow advertive\ algebra \Longrightarrow advertibly\ complete\ algebra \tag{3.4}$$
$$\Longleftrightarrow Cauchy\ advertive\ algebra.$$

Considering Cauchy tQ-algebras along with the relations (1.5) and (1.10), the statement (2.2) gives, in effect, a condition in order the above three types of Q-algebras coincide. That is, we have the next.

Theorem 3.6. *Let E be a topological algebra and consider the following assertions:*

1) E is a Q-algebra.

2) E is a Cauchy tQ-algebra.

3) E is a tQ-algebra.

Then, 1)$\Longrightarrow$2)$\Longrightarrow$3). In particular, if E is, moreover, advertive, then 3)$\Longrightarrow$1), as well; hence, all the previous three assertions are, in effect, equivalent.

Scholium 3.7. *The equivalence 1)$\Longleftrightarrow$2),* in the previous theorem*, can be accomplished by the weaker condition of Cauchy advertiveness. Another condition guaranteeing 1)$\Longleftrightarrow$2),* according to Theorem 2.5, *is* $Sp_E(x) = \hat{x}(\mathfrak{M}(E))$, *for every* $x \in E$; in other words, *1)$\Longleftrightarrow$2) in a topologically spectral algebra.* The proof is similar to that one given in Theorem 2.5. In fact, according to Scholium 2.7, one concludes that

(3.5) *a topologically spectral algebra is Cauchy advertive, equivalently, advertibly complete. Consequently, if the same is still Cauchy tQ, then it is actually a Q-algebra.*

The preceding are summarized in the following diagram

$$\begin{array}{ccc}
 & Q-algebra & topologically\ spectral\ algebra \\
 & \swarrow & \searrow \qquad \downarrow \\
Cauchy\, tQ-algebra & & advertive\ algebra \\
\downarrow & & \downarrow \\
tQ-algebra & & Cauchy\ advertive\ algebra \\
 & & \updownarrow \\
 & & advertibly\ complete\ algebra
\end{array}$$

The next result gives a positive answer to a question posed by *Arizmendi et al.* [4, p. 55] concerning the truth of the following statement: *In a complex unital complete locally convex algebra A every maximal ideal is of codimension 1, if and only if A is a complete Q_t-algebra, under some topology.* The same result still generalizes a relevant one given by this author for Q-algebras [9, p. 60, Theorem 3.8]. In this context, we say that a 2-sided maximal regular ideal M of E has *topological codimension one* if $E/M \cong \mathbb{C}$ within a *topological algebra isomorphism*. If $E/M \cong \mathbb{C}$, for every (2-sided) closed maximal regular ideal M of E, then E is said to be a *Gel'fand-Mazur algebra*. See [11, p. 308, Definition 9.5]; another variant of the same notion is due to *Mati Abel* [1, p. 15].

Theorem 3.8. *Let E be a unital topological algebra and consider the following assertions:*

1) Every 2-sided maximal ideal M has topological codimension 1; viz. $E/M = \mathbb{C}$, *within a topological algebra isomorphism.*

2) $Sp^t_E(x)$ *is bounded,* $x \in E$.

3) E is a tQ-algebra under some topology.

Then, one has 1)$\Longrightarrow$2). If E satisfies (N_r) condition, then 2)$\Longrightarrow$3), while if E is Mallios and Gel'fand-Mazur, then 3)$\Longrightarrow$1).

Proof. 1)$\Longrightarrow$2) follows from [15, p. 293, i)$\Longrightarrow$ii)]: indeed, the proof is independent from the definition of spectrum and according to this, assuming that $Sp^t_E(x)$ is unbounded for some $x \in E$, one finds a 2-sided maximal ideal of infinite codimension, hence of infinite topological codimension as well. 2)$\Longrightarrow$3): Using the technique of *W. Żelazko* in [15, p. 293, Theorem], let τ be the original topology of E and $\tau_{r^t_E}$ the topology that the spectral radius r^t_E defines on E, since it is a submultiplicative semi-norm under the (N_r) condition. Setting

$$\tau^* = \max(\tau, \tau_{r^t_E}),$$

then, E is endowed with a topology stronger than τ, relative to which r_E^t is continuous. Thus, (E, τ^*) is a tQ-algebra, according to the following characterization due to *A. Beddaa* [7]

$$E \text{ is a } tQ\text{-algebra iff } r_E^t \text{ is upper semi-continuous} \tag{3.6}$$

(my thanks here are due to *A. Beddaa* for communicating to me this result; see also [6, p. 13. Corollaire IV.4]). Now, 3)$\Longrightarrow$1) is immediate by the very hypothesis. ∎

Based on the previous theorem, we get at the following result.

Corollary 3.9. *Let E be a unital Mallios Gel'fand-Mazur topological algebra, satisfying the (N_r) condition. Then, all the assertions of Theorem 3.8 are equivalent.*

Remark 3.10. The previous *Theorem 3.8* has an analogous *version* for *Cauchy tQ-algebras*. In this context, we employ the analogous version of *Beddaa's result* in (3.6) *for Cauchy tQ-algebras.*

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